

Regime Switching and Asset Allocation

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Abstract

This dissertation investigates the relationship between a regime-switching structure and asset allocation. In regime-switching models, conditional distributions of assets' returns are modulated by the Markov chain. Consequently, more realistic movements of assets' returns can be described. However, investors can not always exploit the regime-switching structure, which means that the informed investor who knows the regime-switching structure may underperform the uninformed investor from the views of the Sharpe ratio.

In this dissertation, I first statistically confirm the existence of regime switching in the U.S. equities market in Chapter 2. By my empirical analysis, there are two types of regimes: one is a reversible movement of regimes and the other is an irreversible structural change. The first recursive regimes are characterized by means and variances of the assets' returns and the second structural change is characterized by a change of correlations across the returns. However, in a realistic setting, my computational simulations show that the informed investor who knows a correct market structure can not outperform the uninformed investor from the views of the Sharpe ratio. Furthermore, using the other studies' settings, I also find that the informed investor underperforms the uninformed investor.

Next, in Chapter 3, I investigate values of information about regime switching. I consider three different investors' information levels and derive an optimal portfolio of a dynamic mean-variance optimization problem in each information level. Comparing means and variances of these three optimal portfolios, I show that the expected return when investing in the optimal portfolio increases as the investor's information becomes more detailed. However, I also show that the variance also increases as the investor's information becomes more detailed. This result is surprising, but it is consistent with the results of my simulation in Chapter 2 and the other empirical studies. In numerical analysis, I find a numerical example that the informed investor's Sharpe ratio is smaller than the uninformed investor's one.

In Chapter 4, I introduce a concept of multiple priors into dynamic mean-variance optimization problems. Optimization problems with multiple priors are robust to estimation errors, therefore, they can deal with the errors that estimation of more complicated models causes. I derive optimal portfolios in dynamic mean-variance optimization problems with multiple priors. I also show that without a risk-free asset, an optimal portfolio of risky assets converges to the global minimum-variance portfolio if a degree of suspicion that there are estimation errors for means becomes large, and that the optimal portfolio converges to the equally weighted portfolio if a degree of suspicion that there are estimation errors for variances becomes large. Furthermore, I find that in various data sets, the investment strategies that invests in the global minimum-variance portfolio or the equally weighted portfolio considering the market regimes often performs better than the other portfolios.

Finally, in Chapter 5, I consider optimal switching problems with multiple priors using a theory of reflected backward stochastic differential equations. I derive the system of partial differential equations that value function of the optimal switching problems with multiple priors satisfy. Furthermore, I give some financial applications of the optimal switching problems with multiple priors.

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Chapter 1

Introduction

This dissertation investigates the relationship between a regime-switching structure and asset allocation. A regime-switching structure is characterized by the Markov chain which is a random variable taking values in a discrete state space and whose conditional distribution given a current information depends only on its current value. Assuming that Markov chains affect data generating processes of asset returns, we can describe more realistic behaviors of the asset returns although the market model is more mathematically complicated. The models with Markov chains are usually called the “regime-switching models”. The regime-switching models are introduced by the seminal work of Hamilton (1989) and widely used to describe many situations in finance and economics.

In this dissertation, I have first found the existence of reversible and irreversible regimes in the United States equities market. It can be expected that this regime information is useful in asset allocation. In spite of this natural expectation, I also have empirically and theoretically shown that it is difficult that the investors efficiently exploit the regimes straightforwardly. However, I have found simple and efficient investment’s methods using the regime information. This method is that the investor invests in the typical portfolios such as the global minimum-variance portfolio and the equally weighted portfolio considering the regimes. Furthermore, I have shown that this method is a natural consequence of the mean-variance optimization problem that is extremely robust to estimation errors of the moments of the asset returns. Therefore, this method also justifies the naive investment strategies which perform well in practice.

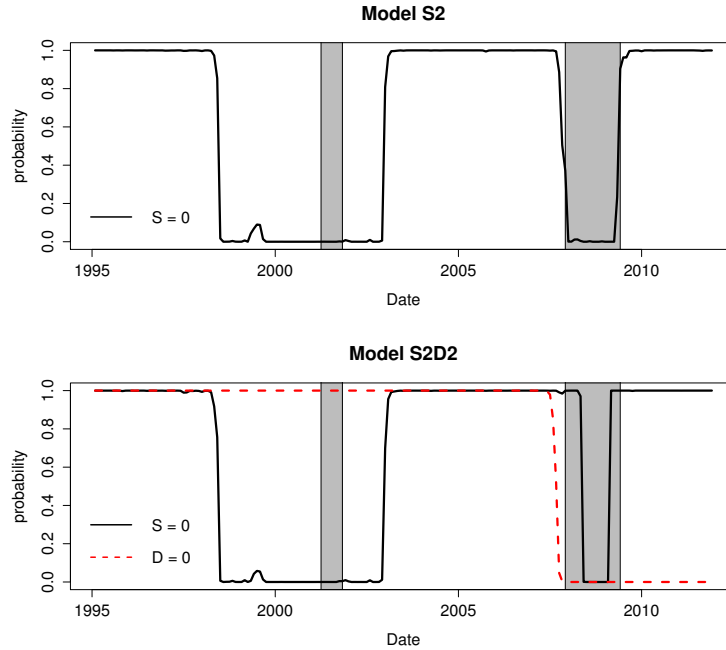
In Chapter 2, I have empirically confirmed a change of correlations across industries

in the U.S. equities market in the financial crisis of 2007-2008 using a regime-switching framework. To capture the irreversible structural change in the financial crisis and to separate it from the recurring booming-recession switches, I have introduced two Markov chains: one is reversible and the other is irreversible. The statistical tests indicate that the reversible regimes represent the differences of the expected returns and variances. In the booming regime, the expected returns are large, whereas they are small in the recession regime. Furthermore, the variances in the booming regime are smaller than those in the recession regime. On the other hand, the irreversible regime (structural change) represents the change of the correlation structure. I have statistically confirmed that the most of correlations after the structural change are higher than those before the structural change. Moreover, the timings of the changes of the recursive regimes roughly coincide with the economic booming and recession periods announced by the National Bureau of Economic Research (NBER). The timing of the structural change roughly coincides with the financial crisis (see the below figure in Figure 1.1).

As seen above, my estimation have succeeded in capturing the movements of the U.S. equities market. However, my simulations of asset allocation have revealed that investors can not always exploit information about the regimes and structure. In the simulation in which an informed investor knows an actual distribution, current regime and structure, the informed investor outperforms an uninformed investor from the view of the Sharpe ratio. However, in the other simulation in which the informed investor knows only the possibility of the structural change, the informed investor can not outperforms the uninformed investors from the view of the Sharpe ratio. Furthermore, using the settings on the other studies, Ang and Bekaert (2002) and Guidolin and Timmermann (2007), my simulation have shown similar results. These results indicate that it is difficult that we effectively use the information about the regimes from the views of the mean-variance efficiency.

In Chapter 3, I have mathematically shown that an informed investor can not always outperform an uninformed investor in the regime-switching market. I have considered three different investors' information levels. I call the first information level "partial

Figure 1.1. The smoothed probabilities and the NBER recession dates. The first figure plots the smoothed probability of Model $S2$. The second figure plots the smoothed probabilities of Model $S2D2$. In each figure, a solid line is the probability of being at $S = 0$ (that is, the probability of staying in the booming regime) and shadow areas are NBER recession dates. In the second figure, a dashed line is the probability of being at $D = 0$ (that is, the probability of the structural change not occurring).



information”, which means that the investor can not observe a movement of regimes and she only observes a history of the past returns. The second information level is “perfect information”, which means that the investor can observe both of the histories of the regimes and returns. The third information level is “prediction”, which means that the investor also can observe the histories of the regimes and returns, moreover, she knows a value of the one-period-ahead regime at each time. Comparing these three information levels, we can consider that the first information level is the most realistic and typical setting. I have considered dynamic mean-variance optimization problems in the infinite horizon in the three different information levels and provided their optimal portfolios.

In the mathematical analysis, I also have shown that the single-period unconditional expected return of the optimal portfolio increases as the investor’s information becomes more detailed. However, the single-period unconditional variance also increases as the

investor's information becomes more detailed. Therefore, the single-period Sharpe ratio of the informed investor's optimal portfolio may be worse than that of the uninformed investor's optimal portfolio. Indeed, a certain numerical example has shown the bad performance of the informed investor's optimal portfolio, in which the partial information investor underperforms the myopic investor from the views of the Sharpe ratio. This result is deep, but it indicates that investors can not always exploit the information about the regimes.

In Chapter 4, I have introduced multiple priors into dynamic mean-variance optimization problems in order to investigate how to use the information about the regimes. The utility optimization with multiple priors is introduced by Gilboa and Schmeidler (1989). They call it "max-min expected utility". An optimization with multiple priors can deal with estimation errors of distribution's parameters, in other words, it can deal with behaviors of the ambiguity aversion. Unlike the existing literature, I have introduced two types of multiple priors: the priors for expected returns and the priors for a covariance matrix. I have succeeded in deriving the optimal portfolios in the dynamic mean-variance optimization problems with multiple priors. Furthermore, my settings include the various return models such as the factor pricing models, stochastic volatility models and regime-switching models. Therefore, these optimal portfolios can be naturally applied to a regime-switching market.

To study the case when an investor considers that estimated parameters are not credible at all, I have analyzed limiting behaviors of the optimal portfolios. First, I have shown that when there exists a risk-free asset, the optimal risky assets' portfolio converges to zero as a trade-off parameter between the expected returns and associated risks or a degree of suspicion that there are estimation errors tends to infinity. This result is natural since the risk-free asset has no risk and no estimation error. However, without the risk-free asset, the optimal portfolio in limiting cases becomes the two famous portfolios: the global minimum-variance portfolio and the equally weighted portfolio. The optimal portfolio becomes the global minimum-variance portfolio as a degree of suspicion that there are estimation errors for the expected returns tends to infinity. On the other hand, the optimal portfolio becomes the equally weighted

portfolio as a degree of suspicion that there are estimation errors for the covariance matrix tends to infinity. This result is surprising. However, it can become one of the rational reasonings why an investor chooses the equally weighted portfolio.

To investigate efficiency of the optimal portfolio in my framework, I have conducted back tests in various data sets. In the back tests, the extreme strategies that invest in the global minimum-variance portfolio or the equally weighted portfolio considering the market regime often perform well in all data sets from the views of the Sharpe ratio. This implies that the investor can exploit the information about the regimes using these strategies. These strategies seem to be naive, however, as seen above, they are rationally justified by my framework; investors choose these strategies since they think that the estimated parameters are not credible at all.

In Chapter 5, changing the point of views, I have considered that investors change the regimes. This type of the problems is called an “optimal switching problem”. Furthermore, I also have introduced multiple priors into optimal switching problems. I call this optimal switching problems the “optimal switching problems under ambiguity”. Using reflected backward stochastic differential equations, I have shown that a value function of the optimal switching problem under ambiguity satisfies some system of partial differential equations. Furthermore, I have derived the conditions that the optimal switching problem under ambiguity is equivalent to a usual optimal switching problem with an a priori shift of a drift of a state variable. If these conditions hold, the results of existing literature can be applied to the optimal switching problems under ambiguity straightforwardly. However, interesting behaviors occur if these conditions do not hold. In the typical buy high and sell low problem (e.g., Zhang and Zhang (2008)), which does not satisfy these conditions, I have shown that effects of ambiguity can not be reproduced by any a priori shift of the drift.

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Chapter 2

An Irreversible Change of Correlations in the U.S. Equities Market and Difficulties in Using the Information

2.1 Introduction

The global financial crisis in 2008 shows the importance of understanding correlations of financial instruments¹. During the catastrophic shock, prices of financial assets are reported to have moved together; correlations of asset returns, including those traditionally considered weak, rapidly increased. This phenomenon causes significant impacts on derivatives prices and hence, on risk management. Accordingly, as part of post-crisis study, there has been an increasing body of literature about measuring and managing correlation risk. To name just a few, Driessen, Maenhout, and Vilkov (2009) study whether exposure to marketwide correlation shocks affects expected option returns, and Buraschi, Porchia, and Trojani (2010) propose a new optimal portfolio choice model by allowing correlation across industries to be stochastic. Driessen et al. (2009) find that assets sensitive to higher marketwide correlations earn negative excess returns. Given the above studies, this chapter has two aims: the first is to identify the irreversible structural change in correlations across industries' returns in the United States (US) equities market, and the second is to test whether the information regarding a structural change can be effective in an investment decision.

¹This chapter is a revised version of Egami, Shigeta, and Wakai (2016).

As for the first goal, we note that the abovementioned studies suggest that the correlations among the returns of the financial instruments change after the global financial crisis. Since the global financial crisis started from the US, we focus on the equities market there. To capture the correlation changes, we use the Markov regime-switching model introduced by Hamilton (1989) in financial economics. More specifically, we propose a regime-switching model with two (mutually independent) Markov chains: one reversible and the other irreversible. The latter chain is used in an attempt to separate the possibly irreversible structural change in correlations from the ordinary regime switching induced by bull and bear markets.

Our empirical findings include (1) the reversible chain captures the shifts of the mean and variance of the individual industry indexes, which alone cannot explain the change in the correlation structure among the industries, (2) the irreversible change is estimated to have occurred between August 2007 and October 2007, which roughly coincides with the period when the financial crisis was becoming obviously imminent, and (3) there is clear evidence that the correlation across industries increased after that period. These results confirm our perceptions that the financial crisis in 2007–2008 changed the market structure.

As for the second goal, we test whether investors can use information of the structural change effectively in facing a crisis of comparable magnitude to the collapse in 2007–2008. To answer this question, we conduct Monte Carlo simulations of asset allocations. In our simulations, we set investors to optimize their mean-variance criteria since the real-life measure of investment efficiency is usually based on the mean-variance preference. Our hypothetical investors are characterized by their information levels, depending on which models they believe and how much information they have about regimes and correlations. If their performances are different, then the difference represents the benefits or costs of the information regarding regimes and structure.

By the simulations, we first confirm that the information about market regimes and structure is valuable: the informed investor who knows the true market model achieves better performance than less informed investors. The sample standard deviations of the informed investor's global minimum-variance portfolio is the lowest of all the investment

strategies. The sample Sharpe ratios of the informed investor's tangency portfolio is the greatest of all the investment strategies. Next, we conduct the test in the setting in which investors do not know the distribution parameters and current regimes and structure. Thus, they need to estimate repeatedly the distribution parameters and current regimes and structure at each time period. In contrast to the first test, the portfolios of the investor who believes in both reversible and irreversible chains underperformed in terms of both the standard deviation of the minimum-variance portfolio and the Sharpe ratio of the tangency portfolio. To explain why this occurred, we measure the estimation errors of the timing of the structural change. They show that the investors do not always react correctly to the movement of structure. To investigate further whether the failure of the second test is caused by regime-switching estimation, we conduct Monte Carlo simulations based on the regime-switching models without structural changes. Nevertheless, we find that the uninformed investor achieves the best performance of all investor types. In this respect, Guidolin and Ria (2011) report that from the viewpoint of the Sharpe ratio, investors using the regime-switching models do not always outperform investors using other models. This is consistent with our results.

The rest of this section is devoted to a literature review related to the study in this chapter. There is a large number of studies on the correlation in international stock markets. For example, Berben and Jansen (2005) investigate the correlations of international stock markets by fitting smooth transition generalized autoregressive heteroskedasticity (GARCH) models to weekly return data. They report correlations among the German, United Kingdom (UK), and US stock markets doubled in the period 1980–2000. However, the correlations between Japan and other markets did not change significantly in this period. Their results indicate that the continuous change in correlations can occur whereas our result shows that a drastic change in correlations occurred in the US equities market. Other studies include Karolyi and Stulz (1996), Ramchand and Susmel (1998), Das and Uppal (2004), and Bekaert, Hodrick, and Zhang (2009).

The regime-switching model is introduced by Hamilton (1989) to capture sudden changes in economic time-series data. In the line of empirical studies, the literature

using recursive regime-switching models suggests the existence of these two kinds of sources for changes of the market conditions, namely, recursive regime shifts and an irreversible change. Ang and Bekaert (2002) use a regime-switching model to identify recursive regime shifts in the international financial market. They succeed in reproducing the asymmetric correlation patterns reported by Longin and Solnik (2001). Okimoto (2008) develops the model of Ang and Bekaert (2002) and finds non-linear and asymmetric dependence patterns of financial assets in the market. Pettenuzzo and Timmermann (2011) find irreversible structural changes in the US financial market using the irreversible regime-switching model. Our analysis integrates these two types of regime-switching models.

The literature also reports the positive values of regime information. Ang and Bekaert (2002) simulate the economic effects of observable regime switching with the CRRA (constant relative risk aversion) utility. Their study of simulation using empirically estimated parameters indicates that there is a positive economic value when an investor takes into account regime switching in the international equities market. Guidolin and Timmermann (2007) extend Ang and Bekaert (2002) under unobservable regime switching in the US financial market. They report the existence of a positive economic value in the unobservable regime-switching market. Guidolin and Timmermann (2008a) consider the optimization problem of the higher-order preference, such as skewness and kurtosis. Tu (2010) applies the Bayesian approach to the mean-variance optimization problem. Other examples include Guidolin and Timmermann (2008b), Pettenuzzo and Timmermann (2011), and Guidolin and Ria (2011).

The advantage of using the regime-switching model includes that it may capture some moment properties, frequently observed in real markets, such as auto-correlation, volatility clustering, asymmetric correlation, non-normal skewness, and kurtosis (e.g., Timmermann (2000)). To study the effects of these properties in investment, many studies consider dynamic portfolio selections with regime switching. In theoretical approaches, Yin and Zhou (2004) study the discrete-time, finite horizon mean-variance preference optimization problem under the observable regime-switching settings. Honda (2003) considers the continuous-time consumption and investment problem with the

power utility under the unobservable regime switching. However, since the settings in the abovementioned theoretical literature do not precisely match the settings in our simulations, so we use the one-period optimal portfolio.

The rest of this chapter is organized as follows: Section 2.2 introduces our regime-switching model. Section 2.3 presents estimation methodology and results. The latter part consists of the following: first, we test if the reversible Markov chain captures a stationary component of asset returns; then, we estimate whether and when the irreversible structural change occurred. Given the estimated timing of the structural change, we compare means and covariance before and after the structural change. Section 2.4 shows the results of the asset allocation problem using the Monte Carlo simulation. Section 2.5 concludes, and some technicalities about the estimation procedure, data details, and model specification are provided in the Appendix.

2.2 Models

We consider a discrete-time, finite horizon regime-switching model, where t varies from 0 to T with $T > 0$ fixed. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which hosts a coupled Markov chain $Z = (Z_t)_{t=0}^T$ that explains regime switch and some stochastic process that drives the log-return process $Y := (Y_t)_{t=0}^T$. The variable Z_t indicates a regime at time t and we assume that an evolution of regime Z is described by a couple of two independent Markov chains $(S_t)_{t=0}^T$ and $(D_t)_{t=0}^T$, so that we denote $Z = (S, D)$.

The first component $(S_t)_{t=0}^T$ captures a reversible transition in the regimes of asset returns. We assume that S is a stationary Markov chain with two regimes $\{0, 1\}$ and the time-invariant transition probability matrix

$$\mathcal{P} := \begin{pmatrix} \mathbb{P}(S_{t+1} = 0 | S_t = 0) & \mathbb{P}(S_{t+1} = 1 | S_t = 0) \\ \mathbb{P}(S_{t+1} = 0 | S_t = 1) & \mathbb{P}(S_{t+1} = 1 | S_t = 1) \end{pmatrix} = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix},$$

where p_{00} and p_{11} are constant parameters to be estimated.

The second component $(D_t)_{t=0}^T$ captures an irreversible structural change in the asset returns. In particular, we assume that the structural change can occur only once. This can be modeled via a Markov chain consisting of two regimes $\{0, 1\}$, where $D_t = 0$ represents the regime before a structural change occurs and $D_t = 1$ represents the

regime after the structural change. Accordingly, the transition probability matrix of D is time invariant and defined by

$$\mathcal{Q} := \begin{pmatrix} \mathbb{P}(D_{t+1} = 0|D_t = 0) & \mathbb{P}(D_{t+1} = 1|D_t = 0) \\ \mathbb{P}(D_{t+1} = 0|D_t = 1) & \mathbb{P}(D_{t+1} = 1|D_t = 1) \end{pmatrix} = \begin{pmatrix} q_{00} & 1 - q_{00} \\ 0 & 1 \end{pmatrix},$$

where q_{00} is a constant parameter to be estimated. By construction, $D_0 = 0$ and D_t cannot return to regime 0 once D_t moves, at some time t , to regime 1. Furthermore, for comparison, we also consider the model without any structural change, which is done by setting $q_{00} = 1$. Note that in this case, the model reduces to a conventional regime-switching model with Markov chain S . Hereafter, the model constrained with $q_{00} = 1$ is denoted by “Model $S2$ ” and the model without any constraint is denoted by “Model $S2D2$.”

The process Y is a vector of log returns of N -industry stock indexes defined by

$$Y_t = \mu_{Z_t} + \Sigma_{Z_t}^{1/2} e_t,$$

where, for each $t \geq 0$, μ_{Z_t} is the N -dimensional vector, $\Sigma_{Z_t}^{1/2}$ is the $N \times N$ matrix, which satisfies $\Sigma_{Z_t}^{1/2} (\Sigma_{Z_t}^{1/2})^\top = \Sigma_{Z_t}$, and $(e_t)_{t=0}^T$ is an N -dimensional identically and independent distributed (i.i.d.) standard normal vector. The mean and covariance of process $(Y_t)_{t=1}^T$ are affected by the regime described by $(Z_t)_{t=1}^T$ in the following way:

$$\mu_{Z_t} = \sum_{s=0}^1 \sum_{d=0}^1 \mathbb{1}\{S_t = s, D_t = d\} \mu_{s,d}, \quad \Sigma_{Z_t} = \sum_{s=0}^1 \sum_{d=0}^1 \mathbb{1}\{S_t = s, D_t = d\} \Sigma_{s,d}$$

where $\mu_{s,d}$ ($s = 0, 1, d = 0, 1$) is a constant N -dimensional vector and $\Sigma_{s,d}$ ($s = 0, 1, d = 0, 1$) is an $N \times N$ constant positive definite symmetric matrix.

By this formulation, the conditional mean and covariance of Y_t given Z_t are

$$\mathbb{E}[Y_t|Z_t] = \mu_{Z_t}, \quad \text{Var}(Y_t|Z_t) = \Sigma_{Z_t}.$$

Accordingly, the distribution of Y_t given Z_t is

$$Y_t|Z_t \sim N(\mu_{Z_t}, \Sigma_{Z_t})$$

where $N(A, B)$ is the normal distribution with mean A and covariance B .

The marginal conditional distribution is

$$Y_t^j|Z_t \sim N(\mu_{Z_t}^j, (\sigma_{Z_t}^j)^2), \quad j = 1, \dots, N$$

where μ_z^j and $(\sigma_z^j)^2$ are the j th component of μ_{Z_t} and the $j \times j$ th-element of Σ_{Z_t} , respectively.

The standard assumptions about the dependence structure of the model are as follows:

Assumption 2.2.1 Let $\mathcal{F}_t := \sigma\{Y_s : 0 \leq s \leq t\}$ be the σ -algebra generated by the log-return process Y .

- (i) S_u and D_v are independent for any pair of u and v ; $0 \leq u, v \leq T$.
- (ii) For any $t \geq 0$, given S_t , S_{t+1} is independent of $\mathcal{F}_t$, and given D_t , D_{t+1} is independent of $\mathcal{F}_t$.

We employ these assumptions in estimation; see Section 2.3 and the Appendix.

2.3 Estimation

2.3.1 Data

We estimate the models of Section 2.2 using monthly industry indexes. All the data are obtained from the ‘‘Thomson Reuters Datastream.’’ We use the indexes of Standard & Poor’s 500 sector total return indexes classified into 10 industries, following the Global Industry Classification Standard. They are the total return indexes calculated by the data source. Before estimation, we divide them to into six large groups and integrate indexes in each group². How we integrate them is outlined in Appendix 2.B. We pick out the monthly data of the integrated indexes from the daily data and compute their monthly log-returns. Consequently, we obtain the monthly data of six large sector indexes. The monthly log-returns of the integrated indexes covered from February 1995 to December 2011, so it have a total of 203 data points. Table 2.1 shows the acronyms of the industries and the correspondence of the acronyms to the group of integration.

²The estimation using the original 10 indexes is very difficult because the 10-index model has many parameters. Their number is 263 when estimating our main model, whereas the number of parameters in the 6-index model is 111; 263 is much greater than the length of available monthly data. Thus, we integrate the index to reduce the parameters.

2.3.2 Method

By using the log return process Y of the integrate sector indexes, we estimate the two unknown components of the model. The first component is an evolution of the regimes, in which we can estimate only the probability of being at particular regimes of $Z = (S, D)$. The second component is the mean vector μ_{Z_t} and covariance matrix Σ_{Z_t} , which are distinct for each regime. We jointly estimate these components based on an iteration method. In particular, we use the expectation-maximization (EM) algorithm, as introduced by Dempster, Laird, and Rubin (1977), after suitably making it fit our framework. By construction, the initial state is $Z_0 = (0, 0)$ or $Z_0 = (1, 0)$, and we denote by ρ_s the initial marginal probability of being at regime s , that is, $\mathbb{P}(S_0 = s)$.

Recall that $(\mathcal{F}_t)_{t=0}^T$ is the information of observable process $(Y_t)_{t=0}^T$ up to time t and let $\Theta^{(k)}$ be the candidate of parameters $\mathcal{P}, \mathcal{Q}, \mu_{s,d}, (s, d = 0, 1), \Sigma_{s,d}, (s, d = 0, 1)$, and $\rho_s, (s = 0, 1)$ in the k th iteration of the EM algorithm. Following Hamilton (1990), the updating formulae for our model parameters in the $(k + 1)$ th iteration are

$$\begin{aligned} \mu_{s,d}^{(k+1)} &= \frac{\sum_{t=0}^T Y_t \mathbb{P}(S_t = s, D_t = d | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=0}^T \mathbb{P}(S_t = s, D_t = d | \mathcal{F}_T; \Theta^{(k)})} \\ \Sigma_{s,d}^{(k+1)} &= \frac{\sum_{t=0}^T (Y_t - \mu_{s,d}^{(k+1)})(Y_t - \mu_{s,d}^{(k+1)})^\top \mathbb{P}(S_t = s, D_t = d | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=0}^T \mathbb{P}(S_t = s, D_t = d | \mathcal{F}_T; \Theta^{(k)})} \\ p_{ss}^{(k+1)} &= \frac{\sum_{t=1}^T \mathbb{P}(S_t = s, S_{t-1} = s | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=1}^T \mathbb{P}(S_{t-1} = s | \mathcal{F}_T; \Theta^{(k)})} \\ q_{00}^{(k+1)} &= \frac{\sum_{t=1}^T \mathbb{P}(D_t = 0, D_{t-1} = 0 | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=1}^T \mathbb{P}(D_{t-1} = 0 | \mathcal{F}_T; \Theta^{(k)})} \\ \rho_s^{(k+1)} &= \mathbb{P}(S_0 = s) = \mathbb{P}(S_0 = s | \mathcal{F}_T; \Theta^{(k)}) \quad s = 0, 1, \quad d = 0, 1, \end{aligned}$$

where $\mathbb{P}(\cdot | \cdot, \Theta^{(k)})$ is the probability calculated under the parameter set $\Theta^{(k)}$. Hamilton (1990) shows if we repeat updating the parameters using these formulae, then the sequence of the parameters obtained by this algorithm converges, as $k \rightarrow \infty$, to the maximum likelihood estimators. The EM algorithm is based on the following probabil-

ities of being at a particular regime:

$$\begin{aligned} &\mathbb{P}(S_t = s, D_t = d | \mathcal{F}_T; \Theta^{(k)}) \\ &\mathbb{P}(S_t = s, S_{t-1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}) \\ &\mathbb{P}(D_t = d, D_{t-1} = \hat{d} | \mathcal{F}_T; \Theta^{(k)}) \\ &\mathbb{P}(S_0 = s | \mathcal{F}_T; \Theta^{(k)}), \end{aligned}$$

which we estimate by the method Kim (1994) proposed. For the detail of the derivation, see the Appendix.

2.3.3 Results

The objective of this subsection is to investigate whether and when an irreversible structural change occurred and how it affected a stationary component of asset returns. In particular, we want to see how the irreversible change, if it exists, altered the correlation structure of asset returns.

Identifying the Markov Chains (S, D)

We first want to test whether a two-state reversible regime-switching model reasonably captures a stationary component of asset returns.

The first figure in Figure 2.1 shows both National Bureau of Economic Research (NBER) recession dates (shaded regions) and the probability of being at regime $S_t = 0$ estimated by Model $S2$ (without a structural change). On the other hand, the second figure in Figure 2.1 replaces the latter probability with the one estimated by Model $S2D2$ (with a structural change) and adds the probability of being at regime $D_t = 0$. Both figures indicate nearly identical probability of the recursive state variable, which supports our assumption that the irreversible structural change in asset returns occurs independently of the stationary and reversible transition in asset returns. Furthermore, we conduct the Carrasco, Hu, and Ploberger (2014) test (hereafter the CHP test) used widely for detecting the existence of Markov switching³. The result is reported in Table 2.2, where the test is performed sector by sector. The test statistics exceed 1% critical

³The CHP test is developed by Carrasco et al. (2014). The null hypothesis of this test is that a time series is not Markov switching.

value except for the energy industry (ENE), so that the data indicate the existence of Markov switching structure. Given the above results, our two-state regime-switching model seems to capture a stationary component of asset returns.

Figure 2.1 also identifies a relationship between regime S_t and the economic environment in the US. Although regime $S_t = 0$ in Model $S2$ and regime $S_t = 0$ in Model $S2D2$ do not necessarily represent a period of the economic boom, the estimated probabilities of being at these regimes and a boom period in the US economy almost coincide after 2005. This suggests that the asset returns depend on the economic environment. Therefore, for simplicity, we call $S_t = 0$ a *boom regime* and $S_t = 1$ a *recession regime*.

Next, we estimate whether and when the irreversible structural change occurred. Table 2.3 shows the Akaike information criterion (AIC) statistics for the model with and without the irreversible structural change. These statistics imply that the model with the irreversible structural change fits the data better than the model without the irreversible structural change. Moreover, the AIC of Model $S2D2$ is lowest in our examined models (see Appendix 2.C). Thus, the data suggest the existence of the irreversible structural change.

In terms of the timing of the irreversible structural change, the dashed line in the second figure of Figure 2.1 reports the estimated probability of being at regime $D_t = 0$. The result implies that the irreversible structural change occurred between August 2007 and October 2007. This roughly corresponds to a period at the start of the financial crisis. Thus, the estimated irreversible structural change corresponds to the financial crisis.

To identify the abovementioned irreversible structural change, it is crucial to estimate the model by jointly using multi-sector returns. To observe this further, we report, in Figure 2.2, the probability of being at regime $D_t = 0$ obtained by estimating the model consisting of single-sector returns.

Although all the sector returns imply that the probability of being at regime $D_t = 0$ started to decline soon after the beginning of the data, the probability declined gradually. This means that we cannot infer the exact timing of the irreversible structural change via single-sector data. On the contrary, in our multi-sector estimation (see

the second figure in Figure 2.1 again), the probability of being at regime $D_t = 0$ declined dramatically from 1 to 0 in a single period. This confirms that the multi-sector estimation is a key to infer the timing of the irreversible structural change.

Impacts of Structural Change on Means and Covariances

Table 2.4 summarizes means and volatilities of each sector returns for each regime of Models $S2$ and $S2D2$. We confirm that at the boom regime $S_t = 0$, means are relatively higher than the recession regime $S_t = 1$. Moreover, volatilities are higher in the recession regime than the boom regime. These relationships of parameters are consistent with the meanings of S_t explained in the previous subsection, that is, a business cycle.

To confirm this conjecture, we test the null hypothesis that means and volatilities in the boom regime are equal to those in the recession regime. These tests are the standard Wald-type tests for maximum likelihood estimators. As shown in Panel A of Table 2.5, we reject the null hypotheses at the 1% significant level except for a few tests. These results are statistical evidence that the reversible variable S captures booms and recessions in the US stock market. Similarly, the results of tests among different values of S in Model $S2$ (Panel C) show significant differences, although these results are weaker than in Model $S2D2$.

On the other hand, Panel B in Table 2.5 shows the results of tests whose hypotheses are that means and volatilities before the irreversible structural change are equal to those after the irreversible structural change. Contrary to the results with equality across different S , the significant difference of parameters across different D are not always found. In particular, the hypotheses that assume equality of means across different values of D in the boom regime ($S = 0$) are not rejected at the 10% significance level. Although the means in the boom regime, except for the consumer goods industry (CG), increase when the structural change occurs (see Table 2.4); however, these increases are not significant.

By contrast, the irreversible structural change has a stronger effect on the correlation structure across sector returns, which is summarized in Table 2.8. The full correlation coefficients are reported in Tables 2.6 and 2.7.

We test whether the recursive regime and structural change have increased or decreased correlations. In Model *S2D2* of Table 2.8, many pairs among the indexes after the structural change are more strongly correlated than before the change in each recursive regime. In the boom regime $S = 0$, 13 pairs (87%) after the change became more strongly correlated than that before, whereas 11 pairs (73%) after the change became more strongly correlated than that before in the recession regime $S = 1$. By contrast, a clear relationship of the correlations between the different S is not found compared to that between the different D . These results imply that the structural change variable D affects the structure of correlations.

Furthermore, we examine the hypothesis testings for whether the correlation structure changes across the two regimes. As with the tests of means and volatilities, these tests are the Wald-type test. First, we consider the hypothesis testings that test the equality of correlations before and after the structural change. In the boom regime, the significant increases of the correlations between before and after the structural change are found in 11 pairs (73%) at the critical level 10%. However, only four pairs (27%) in the recession regime increase their correlations significantly when the structural change occurs, and three pairs (20%) of correlations in the same regime after the change are significantly lower than that before the change. Therefore, the structural change affects correlations in the boom regime more strongly than correlations in the recession regime.

Second, we consider the hypothesis testings that test the equality of correlations between the boom regime and the recession regime. Before the change, the correlations of 4 out of 15 pairs among the indexes in the recession regime are significantly higher than those in the boom regime at the 10% critical level, and no pair in the boom regime is found to be more significantly correlated than in the recession regime. This result implies higher correlation in the recession regime than in the boom regime before the structural change. On the contrary, after the change, 10 pairs (67%) are more strongly correlated in the boom regime than in the recession regime, and the differences in 4 pairs (27%) of them are significant at the 10% level. However, only five pairs (33%) are more strongly correlated in the recession regime than in the boom regime and the differences in two pairs (13%) of them are significant. This result suggests the indexes

are more strongly correlated in the boom regime than in the recession regime after the structural change.

See Tables 2.9 and 2.10 for the hypothesis testing results on correlation coefficients before and after the structural change, based on which we created the summary table (i.e., Table 2.8).

2.4 Asset Allocation Test for Regime Switching

To examine the effects of the information about regime switching and structural change, we consider asset allocation problems in the regime-switching market. We consider the global minimum-variance portfolios and tangency portfolios and simulate the performances of these portfolios by Monte Carlo simulations. In the simulations, we consider two different market environments. One is the market in which investors know the market structure, namely, the distribution parameters and values of state variables. Another is the market in which investors do not know the market structure. Then, the investors need to estimate the market structure to invest rationally. In both of the simulations, we generate simulated time-series using the estimated parameters in Section 2.3. In the first simulation, we conclude that there are statistically significant advantages of the regimes and structure's information. However, the second simulation reveals that these advantages vanish in more realistic settings and indicates that there is a difficult implementation problem of regime-switching information. In subsection 2.4.1, we report the first test. The results of second simulation are shown in subsection 2.4.2. Subsection 2.4.3 checks the robustness of the results in the second subsection.

2.4.1 Values of Information

In this subsection, there are five types of investor and we examine the performance of each investor's portfolio. We assume that the true market model is Model *S2D2* in Section 2.2 with the parameters estimated in Section 2.3. The first type of investor knows the true model and true parameters, which are estimated in Section 2.3. She knows the true regime and structure of the market at every time and rebalances her portfolio in response to regime switches and structural change optimally. We call this

rebalance scheme “ $S2D2$.” The second type of investor believes that the returns are generated by the regime-switching model based on only the recursive Markov chain, S , that is, Model $S2$ in Section 2.2. She knows the true state S_t and rebalances her portfolio in response to the regime shifts. To construct her portfolio, she uses the Model $S2$ ’s distribution parameters estimated in Section 2.3. We call the second type rebalance scheme “ $S2$.” The third type of investor believes that the initial regime will not change, but the structural change will occur. Therefore, she thinks the market model is Model $D2$ (see Appendix 2.C). She knows only the current value of the irreversible Markov chain, D , at each time and uses the estimated parameters of Model $D2$ to construct her portfolios. We call the third type rebalance scheme “ $D2$.” The remaining two types of investors are used as benchmarks. The fourth investor type considers that the regimes and structure will not change, and so, she thinks that the returns of indexes are i.i.d. Therefore, in order to construct her portfolios, she uses the sample mean and variance of the actual data that we used in Section 2.3. We call her “IID.” The fifth type of investor adopts the equally weighted portfolio at every time, so her portfolio is always $\mathbf{1}_6/6$. We call this strategy “EW.”

In summary,

1. $S2D2$: The investor knows that the true market model is Model $S2D2$. She knows the current regime and structure at each time and uses the Model $S2D2$ ’s parameters estimated in Section 2.3 to construct her portfolios.
2. $S2$: The investor believes that the market model is Model $S2$ (only recursive regime shifts). She knows the current regime at each time and uses the Model $S2$ ’s parameters estimated in Section 2.3 to construct her portfolios.
3. $D2$: The investor believes that the market model is Model $D2$ (only irreversible structural change). She knows the current structure at each time and uses the Model $D2$ ’s parameters estimated in Section 2.C to construct her portfolios.
4. IID: The investor believes that the returns of vector are i.i.d. and uses the sample mean and variance of the actual data used in Section 2.3 to construct her portfolios.

5. EW: The investor always adopts the equally weighted portfolio.

Next, we consider the construction of the investors' portfolios. We should consider the dynamic portfolio selections based on the mean-variance criteria; however, they cannot be implemented easily⁴. Therefore, we assume that in every time, the investors optimize their 1-month objectives as well as Markowitz (1952) and Merton (1972). We denote the return vector of the type I investor at time t by R_t^I . Now, we consider the five types of investor $I = S2D2, S2, D2, \text{IID}$, and EW; however, the EW portfolio is $\mathbf{1}_6/6$, and so, we consider only the cases of $S2D2, S2, D2$, and IID.

Let $\phi_{I,t} = (\phi_{I,t,1}, \dots, \phi_{I,t,6})^\top$ be a portfolio of the type I investor at time t . Then, the return of the portfolio ϕ_t^I is

$$R_{t+1}^I = \sum_{i=1}^6 \phi_{I,t,i} \left(\exp \left\{ Y_{t+1}^{(i)} \right\} - 1 \right),$$

where $Y_{t+1}^{(i)}$ is the i th component of the log-return vector Y_{t+1} . The investors' information differs. We denote by $\mathcal{F}_t^I$ the information of the type I investor. The investor chooses the two portfolios that minimize the variance of return and that maximize the Sharpe ratio at each time with a short-selling constraint.

The portfolio minimizing the variance at time t is the solution of the following minimization problem.

$$\begin{aligned} & \min_{\phi_{I,t}} \text{Var}(R_{t+1}^I | \mathcal{F}_t^I) \\ & \text{subject to } \sum_{i=1}^6 \phi_{I,t,i} = 1, \quad \phi_{I,t,i} \geq 0 \text{ for all } i = 1, \dots, 6. \end{aligned}$$

We call the solution the “global minimum-variance portfolio,” according to the literature. Furthermore, the solution of the abovementioned problem depends only on the values of S_t and D_t , does not depend on their past values before time $t-1$, and so, it is sufficient to compute the four portfolios with four different values of S and D at most in order to consider the dynamic portfolio selection.

The portfolio maximizing the Sharpe ratio at time t is the solution of the following

⁴A straightforward multi-period, mean-variance optimization problem is time inconsistent in the sense that the dynamic programming principle does not hold. See Li and Ng (2000).

maximization problem.

$$\begin{aligned} & \max_{\phi_{I,t}} \frac{\mathbb{E}[R_{t+1}^I | \mathcal{F}_t^I]}{\sqrt{\text{Var}(R_{t+1}^I | \mathcal{F}_t^I)}} \\ & \text{subject to } \sum_{i=1}^6 \phi_{I,t,i} = 1, \quad \phi_{I,t,i} \geq 0 \text{ for all } i = 1, \dots, 6, \end{aligned}$$

where we assume that the risk-free rate is 0. Similarly to the global minimum-variance portfolio, the solution of the abovementioned problem depends only on the value of S_t and D_t . Therefore, we need to compute the four portfolios maximizing the Sharpe ratios the most. We call the solution of the abovementioned problem the “tangency portfolio.”

Since the prices of indexes have the mixed log-normal distributions, the conditional expectation value of return R_{t+1}^{S2D2} of the type $S2D2$ investor given by $S_t = s, D_t = d$ is,

$$\mathbb{E}[R_{t+1}^{S2D2} | S_t = s, D_t = d] = \phi_{S2D2,t}^\top (g_{s,d}(\mu, \Sigma) - \mathbf{1}_6),$$

where $\mathbf{1}_6$ is the six-dimensional vector whose all elements are 1, and where $g_{s,d}$ is a $\mathbb{R}^6$ -valued function defined as follows,

$$\begin{aligned} g_{s,d}(\mu, \Sigma) &:= \sum_{s',d'} P_{s',d'}^{s,d} \begin{pmatrix} \exp\{\mu_{s',d'}^1 + \sigma_{s',d'}^1/2\} \\ \vdots \\ \exp\{\mu_{s',d'}^6 + \sigma_{s',d'}^6/2\} \end{pmatrix}, \\ P_{s',d'}^{s,d} &:= \mathbb{P}(S_1 = s', D_1 = d' | S_0 = s, D_0 = d). \end{aligned}$$

$\mu_{s',d'}^i$ is the i th component of the vector $\mu_{s',d'}$ and $\sigma_{s',d'}^i$ is the $i \times i$ th element of the matrix $\Sigma_{s',d'}$. $\mu_{s,d}$ and $\Sigma_{s,d}$ are defined in Section 2.2. The conditional variance of her return R_{t+1}^{S2D2} given by $S_t = s, D_t = d$ is

$$\text{Var}(R_{t+1}^{S2D2} | S_t = s, D_t = d) = \phi_{S2D2,t}^\top H_{s,d}(\mu, \Sigma) \phi_{S2D2,t},$$

where $H_{s,d}$ is the $\mathbb{R}^{6 \times 6}$ -valued function defined as follows,

$$\begin{aligned} [H_{s,d}(\mu, \Sigma)]_{i,j} &= \sum_{s',d'} P_{s',d'}^{s,d} \exp \left\{ \mu_{s',d'}^i + \mu_{s',d'}^j + \frac{1}{2}(\sigma_{s',d'}^i + \sigma_{s',d'}^j + \sigma_{s',d'}^{i,j}) \right\} \\ &\quad - \left(\sum_{s',d'} P_{s',d'}^{s,d} \exp \left\{ \mu_{s',d'}^i + \frac{1}{2}\sigma_{s',d'}^i \right\} \right) \left(\sum_{s',d'} P_{s',d'}^{s,d} \exp \left\{ \mu_{s',d'}^j + \frac{1}{2}\sigma_{s',d'}^j \right\} \right), \\ &\quad i, j = 1, \dots, 6. \end{aligned}$$

$[H_{s,d}(\mu, \Sigma)]_{i,j}$ is the $i \times j$ th element of the matrix $H_{s,d}(\mu, \Sigma)$ and $\sigma_{s',d'}^{i,j}$ is the $i \times j$ th element of the matrix $\Sigma_{s',d'}$.

The partially informed investors also consider that similar moments of returns have conditioned their information. The type *S2* investor uses the conditional mean $E[R_{t+1}^{S2}|S_t = s]$ and variance $\text{Var}(R_{t+1}^{S2}|S_t = s)$ to solve the optimization problem. These moments are computed using the estimated parameters of Model *S2* in Section 2.3. Similarly, the type *D2* investor uses the conditional mean $E[R_{t+1}^{D2}|D_t = d]$ and variance $\text{Var}(R_{t+1}^{D2}|D_t = d)$, which are computed using the estimated parameter of Model *D2*. The type IID investor uses the sample mean and variance of the actual data as the one-step-ahead mean and variance in order to solve her optimization problems.

Now, we estimate the performances of these portfolios. We generate the log-return processes of Model *S2D2* by the Monte Carlo method using the parameters estimated in Section 2.3. The rebalancing interval of the investors' portfolios is 1 month. The investment horizons are 140 months. We examine the simulations in the initial state $S_0 = 0$ or 1 and the initial structure $D_0 = 0$ or 1. The number of trials is 10,000. Therefore, we simulate 10,000 independent data series with the 140 months. In each trial, nine sequences of the returns are computed—the global minimum-variance portfolios and tangency portfolios of *S2D2*, *S2*, *D2*, and IID, and the equally weighted portfolio (EW). We compute the sample means, standard deviations, skewness, kurtosis, and Sharpe ratios of those nine return sequences during 140 months in each simulation trial and measure the performances of the investment strategies by the means of these statistics over the trials.

Tables 2.12 and 2.13 show the simulation results. The performances of the type *S2D2* investor are the best of the investors in all the initial regimes and structures. The mean of sample standard deviations of the global minimum-variance portfolio of *S2D2* at each initial regime and structure is the smallest of all the strategies and the mean of the sample Sharpe ratios of the tangency portfolio of *S2D2* is the largest of them. In each initial regime and structure, the 90% credible interval of the Sharpe ratio of the *S2D2*'s tangency portfolio does not overlap the 90% credible intervals of the Sharpe ratios of other portfolios. This implies that investing in the tangency

portfolio of $S2D2$ is statistically significantly efficient. Furthermore, when the initial regime and structure start at $S = 0$ and $D = 1$, the mean of the Sharpe ratios of the $S2D2$'s tangency portfolios is positive, however the means of the Sharpe ratios of other portfolios are negative. When the initial regime and structure start at $S = 1$ and $D = 1$, all of the means of the Sharpe ratios of the tangency portfolios are negative, and so, these Sharpe ratios may not represent the efficiency of the investment performances. However, since the mean of $S2D2$'s tangency portfolio is the highest and the standard deviation is the smallest of the tangency portfolios, we can conclude that the tangency portfolio of $S2D2$ is the most effective portfolio among them. These results indicate that the knowledge of the actual market model and the values of state variables brings a positive effect in investment.

Next, we consider whether the regimes or the structures are important for investment. Tables 2.12 and 2.13 report that the investment strategies based on $D2$ are more effective than those based on $S2$. In all the initial regimes and structures, the means of the standard deviations of the $D2$'s global minimum-variance portfolios are smaller than those of the $S2$'s portfolios and the means of the Sharpe ratios of the $D2$'s tangency portfolios are larger than those of the $S2$'s portfolios. For instance, in the case in which the initial regime is 0 and the initial structure is 0, the mean of the standard deviations of the $D2$'s global minimum-variance portfolio is 3.519, whereas the mean of the standard deviations of the $S2$'s global minimum-variance portfolio is 3.550. The mean of the Sharpe ratios of the $D2$'s tangency portfolio is 0.235, however the mean of the Sharpe ratios of the $S2$'s portfolios is 0.201. Similar results appear in the other initial regimes and structures. These results suggest that the structural information is more important than the regime information. In addition, the tangency portfolios of $S2$ and $D2$ underperform the IID tangency portfolios. For instance, in the case of the initial regime 1 and structure 1, the difference of the means of the tangency between $S2$ and IID is 0.731, and this is statistically significant in the sense that their 99% credible intervals do not overlap each other. Moreover, the mean of $S2$ (-1.857) is smaller than the IID (-1.126) and the standard deviation of $S2$ (6.178) is larger than the IID (5.187). Thus, it is clear that the tangency portfolio of $S2$ is less efficient than

the IID in the mean-variance sense when the initial regime is 1 and the initial structure is 1. $D2$'s tangency portfolio also is less efficient than IID in the mean-variance sense. These results suggest that the IID investor's tangency portfolio is more mean-variance efficient than the partially informed investors' tangency portfolios.

Figure 2.3 shows the means of the standard deviations of global minimum-variance portfolios and the Sharpe ratios of tangency portfolios. The horizontal line represents the length of months investing portfolios. In all the initial regimes and structures, the standard deviation of the global minimum-variance portfolios of $S2D2$ is the smallest of the other portfolios in the whole of the investment horizon. Similarly, the Sharpe ratio of the tangency portfolios of $S2D2$ is always the highest of the other portfolios in all the initial conditions. These imply that the advantages of full information exist at various investment horizons.

Table 2.14 shows the weights of the global minimum-variance portfolios and tangency portfolios. The weight of the consumer goods industry (CG) of the type $S2D2$ investor's global minimum-variance portfolio decreases after the structural change, but the type $D2$ investor increases the weight of CG after the structural change and this weight is very high (more than 60%). On the other hand, the type $S2D2$ and $S2$ investors have similar tangency portfolios before the structural change. In the boom regime, they invest their wealth mainly in the indexes of the energy industry (ENE), CG, and the utilities industry (U). In the recession regime, they invest almost all their wealth in ENE. However, after the structural change, the type $S2D2$ investor decreases the weight of ENE and increases CG in each regime. These portfolio differences among the investors confirm that the partially informed investors (type $S2$ and $D2$ investors) cannot react to the change of the market condition correctly.

2.4.2 Rolling Estimation: Limitation of the Regime-Switching Model with the Structural Change

Next, we simulate the performances of the investors in the market in which they cannot access the distribution parameters and the information of the market regimes and structure. However, the advantages of the knowledge of the market conditions that we

observed in subsection 2.4.1 vanish in the test of this subsection.

As well as subsection 2.4.1, we consider the four investor types, namely, type $S2D2$, $S2$, IID, and EW. However, in contrast to the settings of subsection 2.4.1, they do not know the distribution parameters and the movement of the state variables, S and D . Therefore, they apply the assumed models to the observed data repeatedly in their investment horizon to construct their portfolios. The estimation methods are the same as those in this chapter (see Section 2.3 and Appendix). The IID investor constructs her portfolios using the sample means and variances of the past simulated returns. The EW investor always invests in the equally weighted portfolio, $\mathbf{1}_6/6$. The type $S2D2$, $S2$, and IID investors construct the two portfolios, the global minimum-variance portfolio and the tangency portfolio. In subsection 2.4.1, the portfolios depend only on the values of state variables; however, the portfolios in this simulation vary through time, since the estimated parameters change when the investors observe the new return data. Finally, we consider seven investment strategies, the global minimum-variance portfolios and tangency portfolios of $S2D2$, $S2$, and IID, and the equally weighted portfolio (EW).

In summary,

1. $S2D2$: The investor knows that the true market model is Model $S2D2$. She repeatedly estimates the Model $S2D2$'s parameters to construct her portfolios.
2. $S2$: The investor believes that the market model is Model $S2$ (only recursive regime shifts). She repeatedly estimates the parameters of Model $S2$ to construct her portfolios.
3. IID: The investor believes that the returns of vector are i.i.d. and repeatedly computes the sample mean and variance of the simulated sample to construct her portfolios.
4. EW: The investor always adopts the equally weighted portfolio.

Unlike the simulation in subsection 2.4.1, we do not consider the investor who believes Model $D2$ in this simulation. This is because it is numerically difficult to compute the optimal portfolios of Model $D2$. More specifically, we fail to compute the optimal

portfolios of Model $D2$ in many trials because of the singularities of the estimated variance matrixes. One explanation is that it is rare to capture changes of market conditions in early periods since Model $D2$ needs the long-run data. Furthermore, the purpose of our analysis is to investigate the values of the information of the structural change. In this respect, the difference between $S2D2$ and $S2$ is more important than the difference between $S2D2$ and $D2$ since the difference between $S2D2$ and $S2$ represents the effect of the irreversible Markov chain D . Therefore, we omit the simulations of $D2$.

We simulate samples of the returns based on Model $S2D2$ using the parameters estimated in Section 2.3. In each simulation trial, the investors do not invest in the first 60 months and they start to invest after these months passed. However, the data in the first 60 months are used for each investor to estimate parameters. The length of the total investment horizons is 140 months, and so, we simulate the 200-month data series of the indexes in each trial. The investors have to estimate the market models and compute the portfolio repeatedly, and so, this simulation is computationally intensive. Therefore, we decrease the number of simulation trials to 2,000. As a result of the abovementioned simulation plan, we simulate 2,000 time series of 200 monthly return data. We compute the sample statistics of the portfolios' returns—means, standard deviations, skewness, kurtosis, and the Sharpe ratios in each simulation trial. Therefore, we obtain the 2,000 statistics of the portfolios' returns during 140 months and take means of these statistics over the trials. The initial regime of S is generated randomly with its steady-state probabilities. The initial structure is $D = 0$.

Panel A in Table 2.15 shows the results of the Model $S2D2$ simulation. Unlike the simulation in the case in which investors can use the information of values of the state variables, the performance of type $S2D2$ is the worst of the strategies except for the EW portfolio. The mean of the standard deviations of $S2D2$'s global minimum-variance portfolios is the largest of the global minimum-variance portfolios and the mean of the Sharpe ratios of $S2D2$'s tangency portfolios is the lowest of the tangency portfolios. Notable differences appear in their variances. In both the global minimum-variance portfolios and tangency portfolios, the differences of the means of the standard deviation over the trials between $S2D2$ and the other models are statistically significant

in the sense that the 90% credible intervals do not overlap each other. These results imply that there are not any advantages of knowledge of the market model in the optimization of the mean-variance criterion, even though Ang and Bekaert (2002) and Guidolin and Timmermann (2007) find positive economic values of knowledge in the optimization of expected utilities.

In this simulation, the major change from subsection 2.4.1 is that each informed investor estimates the parameters and states repeatedly, whereas the investors in subsection 2.4.1 believe certain parameters a priori and know the current state at each time. Therefore, the estimation errors of the parameters and the states movement exist in this simulation. It is possible that the estimation errors cannot be ignored in the asset allocation with rolling estimation. Indeed, Panel A in Table 2.15 shows that the type $S2$ investor who needs to estimate the moderately complicated model underperforms the type IID investor while she outperforms the type $S2D2$ investor who needs to estimate the most complicated model. Another explanation of the disadvantage of knowledge is that the regime-switching framework will not work under the mean-variance criterion while it works under the expected utility criterion (see Ang and Bekaert (2002) and Guidolin and Timmermann (2007)). The rest of this subsection measures the estimation errors and the next subsection tests whether the regime-switching frameworks works.

To measure the degree of the estimation errors, we compute the gaps between the actual and estimated structural changing time in the Model $S2D2$ simulation. As mentioned earlier in this subsection, we fail to compute the Model $D2$'s portfolios in this simulation. Therefore, it is natural to focus on the estimation error of the timing of the structural change. The type $S2D2$ investor estimates the parameters and smoothed probabilities at each time. In the b th trial, the estimated smoothed probability at time T with which the structural change had not occurred before the time $t \leq T$ is denoted by $p_{t,T}^b$, that is,

$$p_{t,T}^b = \mathbb{P}(D_t = 0 \mid \mathcal{F}_T^b),$$

where $\mathcal{F}_T^b$ is the type $S2D2$ investor's information at time T in the b th trial. Let τ^b be the structural change time in b th trial. We define the mean of the smoothed

probabilities around the structural change as

$$\bar{p}_{i,T} = \sum_{\{b \mid \tau^b < \min\{200, T\}\}} \frac{p_{\tau^b+i,T}^b}{\text{The number of elements of } \{b \mid \tau^b < \min\{200, T\}\}}.$$

If the type *S2D2* investor estimates the market model correctly, it can be expected that $\bar{p}_{i,T}$ decreases as i increases and a large gap between $\bar{p}_{-1,T}$ and $\bar{p}_{0,T}$ exists.

Figure 2.4 displays the means of the smoothed probabilities around the structural change in the Model *S2D2* simulation. As well as our expectation, $\bar{p}_{i,T}$ is decreasing at i and there is a large gap between $\bar{p}_{-1,T}$ and $\bar{p}_{0,T}$ for all $T = 100, 150, 199$. However, its quantity remains more than 0.5 after the structural change, that is, $\bar{p}_{i,T} > 0.5$ holds for $i > 0$. This means that the estimation procedure detects the structural change after the actual structural change occurs in most cases and that there is a limit to what the type *S2D2* investor estimates as the time of the structural change.

However, it is possible that the early structural change causes the limitation. Since the time of structural change has a geometric distribution, the structural change tends to occur at the early time. Thus, in most of the trials, it is possible that the *S2D2* investor cannot identify the structural change. Therefore, we condition the means of the smoothed probabilities around the structural change to the time of the structural change. Computing them, we use only the trials in which the structural change occurs during the periods from $t = 51$ to $t = 150$, that is, we compute

$$\bar{p}_{i,T}^c := \sum_{\{b \mid 51 \leq \tau^b \leq 150\}} \frac{p_{\tau^b+i,T}^b}{\text{The number of elements of } \{b \mid 51 \leq \tau^b \leq 150\}}.$$

Under the condition of the timing of the structural change, the type *S2D2* investor observes many returns' data before and after the structural change. In addition, Figure 2.4 shows the changes of $\bar{p}_{i,T}^c$ at $T = 199$. As with the unconditional case, $\bar{p}_{i,T}^c$ is decreasing at i and a large gap between $i = -1$ and 0 appears. Moreover, the gap of $\bar{p}_{i,T}^c$ is larger than that of the unconditional means. However, its quantity is also maintained at more than 0.5 after the structural change. Thus, limits of estimation still exist.

2.4.3 Rolling Estimation: Limitation of Regime-Switching Models without a Structural Change

To study the other explanation, we conduct three asset allocation tests of regime-switching models with a recursive Markov chain only. In subsection 2.4.2, we find that the informed investor does not have an advantage compared to the uninformed investor from the views of mean-variance efficiency and that it is difficult to identify the timing of the structural change. However, it is possible that the mean-variance optimization will not work, even if only recursive Markov chains change the market environment. Thus, we focus on the regime-switching models with a recursive Markov chain only. We first conduct the simulation in which the true market model is Model $S2$, that is, the market obeys movement of one recursive Markov chain. To confirm robustness, we conduct two simulations under the settings of the regime-switching model literature.

We conduct the simulation in which the simulated market model is Model $S2$. In the simulation of Model $S2$, we simulate the sample of return vectors using the estimated parameters of Model $S2$, but the other settings do not change from the simulation of Model $S2D2$. In the simulation of Model $S2$, the type $S2$ investor correctly understands the true market model.

Panel B in Table 2.16 reports the results of the simulation of Model $S2$. Similar to subsection 2.4.2, the type $S2$ investor underperforms the IID investor in both the global minimum-variance portfolios and tangency portfolios. The mean of the standard deviations of $S2$'s global minimum-variance portfolios is greater than the mean of the standard deviations of IID's global minimum-variance portfolios. Moreover, the mean of the Sharpe ratios of $S2$'s tangency portfolios is lower than the mean of the Sharpe ratios of IID's tangency portfolios. Furthermore, the 90% credible interval of the Sharpe ratios of $S2$'s tangency portfolios does not overlap the 90% credible interval of the Sharpe ratios of IID's tangency portfolios. It follows that the difference of the Sharpe ratios between $S2$ and IID's tangency portfolios is statistically significant at the 10% significance level. However, the type $S2$ investor outperforms the type $S2D2$ investor in this simulation.

To confirm the robustness of these results, we conduct two additional simulations

using the parameters and settings of the other literature. Using the estimation results of the regime-switching models in Ang and Bekaert (2002) and Guidolin and Timmermann (2007), we simulate the samples of the data process that they report. Ang and Bekaert (2002) estimate US, UK, and German equity indexes in various models. We choose the regime-switching model with two recursive states and simulate it. On the other hand, Guidolin and Timmermann (2007) identify three assets in the US—a small stock index, a large stock index, and 10-year T-bonds—as the four recursive states’ regime-switching model, and so, we simulate the time-series using their results. In both simulations, we assume there are four types of investors, recursive 4 states, and recursive 2 states, IID and EW. The type of 4 states (resp. 2 states) investor believes that the number of market regimes is 4 (resp. 2). The IID investor believes that the returns of assets are i.i.d. The EW investor always invests in the equally weighted portfolio $\mathbf{1}_6/6$. As well as the former simulation, they do not know the distribution parameters and movement of the regimes, and they estimate their believed models repeatedly. The initial running periods are 200 months in the simulation of Ang and Bekaert (2002) and 288 months in the simulation of Guidolin and Timmermann (2007). The investment periods are 300 months in the simulation of Ang and Bekaert (2002) and 312 months in the simulation of Guidolin and Timmermann (2007). To compute the tangency portfolios and the sample Sharpe ratios, we use a positive risk-free rate in both simulations. In the simulation of Ang and Bekaert (2002), the monthly risk-free rate is 0.0041, and the monthly risk-free rate in the simulation of Guidolin and Timmermann (2007) is 0.0044. These risk-free rates are based on these studies. In both of the simulations, there are 2,000 trials and the initial regime in each trial is generated randomly with the steady probabilities.

Table.2.16 reports the results of the simulation based on the literature. In the case of Ang and Bekaert (2002), the investment strategy of the global minimum-variance portfolios based on the true model (2 states) achieves the smallest mean of the standard deviation among all the strategies, but the difference is not statistically significant. The mean of the Sharpe ratios when the investor invests in the tangency is lower than that of the IID. In the case of Guidolin and Timmermann (2007), both of the standard deviations of the global minimum-variance portfolios and the Sharpe ratios of

the tangency portfolios are worse than the other strategies. In both simulations, the portfolios based on the market models do not outperform the IID portfolios in most cases.

2.5 Concluding Remarks

In this chapter, we identify the change in correlation structure across industries in the US market in the financial crisis of 2007–2008 and find limitations of the use of regime-switching information. To capture the change in the global financial crisis, we use the regime-switching model with both a reversible Markov chain and an irreversible Markov chain, and succeed in separating the two dynamics in the market, namely, the recursive regime shifts and the irreversible structural change.

The recursive regime shifts represent the dynamics of the marginal distribution parameters of the returns, that is, means and variances. These two recursive regimes are the boom and recession regimes. In the boom regime, the indexes have high (conditionally) expected returns and low volatilities, whereas they have low expected returns and high volatility in the recession regime. Furthermore, the probability with which the market is in the recession regime remains at a high degree at the NBER announced recession periods. By contrast, the irreversible structural change represents the dynamics of the joint distribution parameters, that is, correlations. Most pairs of correlations among the industries increase in each regime after the change. The irreversible change is not found in the estimation of the time-series of the single-industry index. Moreover, the timing of the structural change almost coincides with the financial crisis in 2007–2008.

In the simulations of asset allocation, we find positive values for use of the regime-switching information if the investor knows the true parameters and current states. The global minimum-variance portfolios of the informed investor achieve the smallest variance in the investors and their tangency portfolio also achieves the highest Sharpe ratio. Moreover, these advantages appear in the various investment horizons and initial regimes and structures. However, the investors cannot exploit the structural change from the viewpoints of mean-variance efficiency if they only know the kind of

the true market model. The performances of the global minimum-variance portfolios and tangency portfolios of the informed investor are worse than those of the uninformed investors.

Appendix 2.A The EM Algorithm

Based on Hamilton (1989) (1990) and Kim (1994), we explain the EM algorithm that we use in this chapter. For notational simplicity, we consider the case of $q_{00} = 1$, so Model S2, since the extension to the general case is straightforward. First, we introduce the *full-information* likelihood function, which means we hypothesize that we can observe unobserved variables. For the random vectors, $Y = (Y_0, \dots, Y_T)$ and $S = (S_0, \dots, S_T)$, the full-information likelihood function is

$$\begin{aligned} f(Y, S; \Theta) &= \prod_{t=0}^T \left(\sum_{s=0}^1 \mathbb{1}\{S_t = s\} f(Y_t | \mathcal{F}_{t-1}, S_t = s; \Theta) \right) \\ &\times \prod_{t=1}^T \left(\sum_{s=0}^1 \sum_{\hat{s}=0}^1 \mathbb{1}\{S_t = s, S_{t-1} = \hat{s}\} \mathbb{P}(S_t = s | S_{t-1} = \hat{s}) \right) \\ &\times \sum_{s=0}^1 \mathbb{1}\{S_1 = s\} \mathbb{P}(S_1 = s) \end{aligned}$$

where $\mathcal{F}_t$ is the σ -algebra generated by Y up to time t and Θ is the distribution parameters and the conditional density $f(Y_t | \mathcal{F}_{t-1}, S_t = s; \Theta)$ is

$$f(Y_t | \mathcal{F}_{t-1}, S_t = s; \Theta) = \frac{1}{\sqrt{(2\pi)^N \det \Sigma_s}} \exp \left\{ -\frac{1}{2} (Y_t - \mu_s)^\top \Sigma_s^{-1} (Y_t - \mu_s) \right\}.$$

For the “expectation” step, we start with the initial parameter set $\Theta^{(0)}$ to compute

$$\begin{aligned} Q(\Theta; \Theta^{(0)}) &= \mathbb{E}^{\Theta^{(0)}} [\log f(Y, S; \Theta) | \mathcal{F}_T] \\ &= \sum_{t=0}^T \sum_{s=0}^1 \mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(0)}) \log f(Y_t | \mathcal{F}_{t-1}, S_t = s; \Theta) \\ &+ \sum_{t=1}^T \sum_{s=0}^1 \sum_{\hat{s}=0}^1 \mathbb{P}(S_t = s, S_{t-1} = \hat{s} | \mathcal{F}_T; \Theta^{(0)}) \log \mathbb{P}(S_t = s | S_{t-1} = \hat{s}) \\ &+ \sum_{s=0}^1 \mathbb{P}(S_1 = s | \mathcal{F}_T; \Theta^{(0)}) \log \mathbb{P}(S_1 = s). \end{aligned}$$

Next, we search the parameters maximizing $Q(\Theta; \Theta^{(0)})$ function,

$$\Theta^{(1)} = \arg \max_{\Theta \in \Theta} Q(\Theta; \Theta^{(0)})$$

where $\bar{\Theta}$ is the parameter space of this model. This step is the “maximization” step. We then continue these steps from $k = 1, 2, \dots$.

Hamilton (1990) shows if we repeat these steps to infinity, then the sequence of the parameters obtained by this algorithm converges to the maximum likelihood estimators. According to Hamilton (1990), the updating formulae of the parameters in the $(k+1)$ th iteration are

$$\begin{aligned}\mu_s^{(k+1)} &= \frac{\sum_{t=0}^T Y_t \mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=0}^T \mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)})} \quad s = 0, 1, \\ \Sigma_s^{(k+1)} &= \frac{\sum_{t=0}^T (Y_t - \mu_s^{(k+1)})(Y_t - \mu_s^{(k+1)})^\top \mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=0}^T \mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)})} \quad s = 0, 1, \\ p_{ss}^{(k+1)} &:= \mathbb{P}(S_{t+1} = s | S_t = s; \Theta^{(k+1)}) = \frac{\sum_{t=1}^T \mathbb{P}(S_t = s, S_{t-1} = s | \mathcal{F}_T; \Theta^{(k)})}{\sum_{t=1}^T \mathbb{P}(S_{t-1} = s | \mathcal{F}_T; \Theta^{(k)})} \quad s = 0, 1, \\ \rho_s^{(k+1)} &:= \mathbb{P}(S_0 = s; \Theta^{k+1}) = \mathbb{P}(S_1 = s | \mathcal{F}_T; \Theta^{(k)}) \quad s = 0, 1.\end{aligned}\tag{2.A.1}$$

These updating formulae are obtained by the first-order conditions of the maximization of $Q(\Theta; \Theta^{(0)})$ with respect to Θ .

By considering the formulae (2.A.1), we observe that we need to compute

- (a) $\mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)}), t = 0, \dots, T$ and
- (b) $\mathbb{P}(S_t = s, S_{t-1} = s | \mathcal{F}_T; \Theta^{(k)}), t = 1, \dots, T,$

which are called *smoothed probabilities*.

Let us suppose that we have estimated up to the k th iteration and explain how to update to the $(k+1)$ th estimates. Now, we perform the following:

Forward calculation: Assume further that we have obtained $\mathbb{P}(S_{t-1} = s | \mathcal{F}_{t-1}; \Theta^{(k)})$, then we have, for the next time step t ,

$$\mathbb{P}(S_t = s | \mathcal{F}_{t-1}; \Theta^{(k)}) = \sum_{\hat{s}=0}^1 \mathbb{P}(S_t = s | S_{t-1} = \hat{s}) \mathbb{P}(S_{t-1} = \hat{s} | \mathcal{F}_{t-1}; \Theta^{(k)}), \quad s = 0, 1.\tag{2.A.2}$$

By using this, we compute $\mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)}), s = 0, 1$ in the following way: By Bayes’

rule, we obtain

$$\begin{aligned}
\mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)}) &= \mathbb{P}(S_t = s | Y_t, \mathcal{F}_{t-1}; \Theta^{(k)}) \\
&= \frac{\mathbb{P}(Y_t \in dy | S_t = s, \mathcal{F}_{t-1}; \Theta^{(k)})}{\mathbb{P}(Y_t \in dy | \mathcal{F}_{t-1}; \Theta^{(k)})} \mathbb{P}(S_t = s | \mathcal{F}_{t-1}; \Theta^{(k)}) \\
&= \frac{f(Y_t | S_t = s, \mathcal{F}_{t-1}; \Theta^{(k)}) \mathbb{P}(S_t = s | \mathcal{F}_{t-1}; \Theta^{(k)})}{\sum_{\hat{s}=0}^1 f(Y_t | S_t = \hat{s}, \mathcal{F}_{t-1}; \Theta^{(k)}) \mathbb{P}(S_t = \hat{s} | \mathcal{F}_{t-1}; \Theta^{(k)})}, \quad s = 0, 1
\end{aligned} \tag{2.A.3}$$

where $\mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)})$ is called the *filtered probability*. As a proxy for the filtered probability at $t = 0$, we use $\mathbb{P}(S_0 = s; \Theta^{(k)})$. We then repeat this procedure *forwards* up to time T . In other words, we obtain the whole set of probabilities (2.A.2) and (2.A.3) for $t = 0, \dots, T$. Recall k is still fixed.

Backward calculations: For computing (a) and (b), first we use Bayes' rule to write

$$\begin{aligned}
\mathbb{P}(S_t = s, S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}) &= \mathbb{P}(S_t = s | S_{t+1} = \hat{s}, \mathcal{F}_T; \Theta^{(k)}) \mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}) \\
&\approx \mathbb{P}(S_t = s | S_{t+1} = \hat{s}, \mathcal{F}_t; \Theta^{(k)}) \mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}) \\
&= \frac{\mathbb{P}(S_{t+1} = \hat{s} | S_t = s, \mathcal{F}_t; \Theta^{(k)}) \mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)})}{\mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_t; \Theta^{(k)})} \mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}) \\
&= \frac{\mathbb{P}(S_{t+1} = \hat{s} | S_t = s; \Theta^{(k)}) \mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)})}{\mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_t; \Theta^{(k)})} \mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)})
\end{aligned} \tag{2.A.4}$$

where in the last line we use Assumption 2.2.1-(ii). We compute *backwards* starting with $t = T - 1$ down to $t = 0$. All the probabilities in the last line are known⁵ and hence, we obtain

$$\mathbb{P}(S_t = s | \mathcal{F}_T; \Theta^{(k)}) = \sum_{\hat{s}=0}^1 \mathbb{P}(S_t = s, S_{t+1} = \hat{s} | \mathcal{F}_T; \Theta^{(k)}). \tag{2.A.5}$$

for all $t = T, \dots, 1$. The resulting probabilities in (2.A.5) and (2.A.4) for $t = 0, \dots, T$ are the smoothed probabilities (a) and (b), respectively. Plugging (a) and (b) into the recursive formulae for parameter estimation above, we have updated for the $(k + 1)$ th

⁵Except for the last one, we already have the whole set of probabilities ($t = 0 \dots T$) in (2.A.4): Namely, $\mathbb{P}(S_{t+1} = \hat{s} | S_t = s; \Theta^{(k)})$ is $p_{ss}^{(k)}$ obtained in the k th iteration, and $\mathbb{P}(S_t = s | \mathcal{F}_t; \Theta^{(k)})$ and $\mathbb{P}(S_{t+1} = \hat{s} | \mathcal{F}_t; \Theta^{(k)})$ are from (2.A.2) and (2.A.3), respectively. Finally, for the last one, if we set $t = T - 1$, then $\mathbb{P}(S_T = \hat{s} | \mathcal{F}_T; \Theta^{(k)})$ is known again by (2.A.2). With all these, we obtain (2.A.5) at $t = T - 1$, that is, $\mathbb{P}(S_{T-1} = s | \mathcal{F}_T; \Theta^{(k)})$. This is then plugged for the next time step $t = T - 2$ into (2.A.4).

iteration. Note that the smoothed probabilities, which are used in the updating formulae for parameter estimation, have rich information since they estimate the probabilities of being in a certain regime at time t by using the full observations.

Appendix 2.B Data Details and Integration

There are two steps to compute the integrated indexes: the first step is to compute price-based integrated indexes, the second step is to compute total integrated indexes, that is, the indexes, including aggregate daily dividends.

The first step is as follows. $\mathbb{J}$ denotes the set of some industries, which is the group we want to integrate. To summarize the indexes of $\mathbb{J}$ into one index, we compute a capitalization-weighted average index of $\mathbb{J}$, denoted by $P_t^{\mathbb{J}}$:

$$P_t^{\mathbb{J}} = \frac{\sum_{j \in \mathbb{J}} M_t^j}{\sum_{j \in \mathbb{J}} AM_{t-1}^j} P_{t-1}^{\mathbb{J}}, \quad t \geq 1, \quad P_0^{\mathbb{J}} = 100,$$

where M_t^j and AM_t^j are the usual and adjusted market value of j -industry index at time t ; “adjusted” means that its variable takes into account capital actions of firms. By the data source, the industry index P_t^j is constructed as follows,

$$P_t^j = \frac{\sum_k n_{k,t}^j p_{k,t}^j}{\sum_k n_{k,t-1}^j p_{k,t-1}^j} P_{t-1}^j, \quad t \geq 1, \quad P_0^j = 100,$$

where $n_{k,t}^j$ and $p_{k,t}^j$ are the number of shares in issue and unadjusted price of firm k , which constitutes the j industry’s index at time t . $Adj_{k,t}^j$ is the adjusted factor of firm k at time t . This adjusts the capital action of firm k at time t . We define the adjusted market value of j industry’s index as

$$AM_{t-1}^j := \frac{P_{t-1}^j}{P_t^j} M_t^j = \sum_k n_{k,t}^j p_{k,t-1}^j Adj_{k,t}^j.$$

Then, AM_{t-1}^j is the adjusted market value taking capital actions of firms into consideration. In this manner, we compute the integrated indexes.

In the second step, we need to compute aggregate daily dividends of the industries’ indexes. The Thomson Reuters Datastream provides a total return index in each industry. The total return index of industry j , denoted by RI^j , is defined as

$$RI_t^j = RI_{t-1}^j \frac{P_t^j}{P_{t-1}^j} \left(1 + \frac{DY_t^j}{100\Delta} \right), \quad t \geq 1, \quad RI_0^j = 100,$$

where DY_t^j is the aggregate dividend yield of industry i at time t (expressed as a percentage) and where Δ is a certain number of days in a financial year (normally 260). By the definition of the total index, we compute the aggregate dividend yield (expressed as a real number),

$$\frac{DY_t^j}{100} = \Delta \left(\frac{RI_t^j}{RI_{t-1}^j} \frac{P_{t-1}^j}{P_t^j} - 1 \right).$$

The dividend yield is defined as

$$\frac{DY_t^j}{100} = \frac{\sum_k n_{k,t}^j d_{k,t}^j}{\sum_k n_{k,t}^j p_{k,t}^j} = \frac{\sum_k n_{k,t}^j d_{k,t}^j}{M_t^j},$$

where $d_{k,t}^j$ is the dividend per share of firm k at time t . Of course, firm k constitutes j industry's index. Therefore, the aggregate dividend of industry j at time t is

$$\frac{Div_t^j}{100} = M_t^j \frac{DY_t^j}{100} = M_t^j \Delta \left(\frac{RI_t^j}{RI_{t-1}^j} \frac{P_{t-1}^j}{P_t^j} - 1 \right) = \sum_k n_{k,t}^j d_{k,t}^j.$$

The dividend yield of the industry group $\mathbb{J}$ at time t , denoted by $DY_t^{\mathbb{J}}$, is

$$\frac{DY_t^{\mathbb{J}}}{100} = \frac{\sum_{j \in \mathbb{J}} Div_t^j / 100}{\sum_{j \in \mathbb{J}} M_t^j} = \frac{\sum_{j \in \mathbb{J}} \sum_k n_{k,t}^j d_{k,t}^j}{\sum_{j \in \mathbb{J}} M_t^j}.$$

Finally, the total integrated index of the industry group $\mathbb{J}$ is

$$RI_t^{\mathbb{J}} = RI_{t-1}^{\mathbb{J}} \frac{P_t^{\mathbb{J}}}{P_{t-1}^{\mathbb{J}}} \left(1 + \frac{DY_t^{\mathbb{J}}}{100\Delta} \right), \quad t \geq 1, \quad RI_0^{\mathbb{J}} = 100.$$

To confirm the validity of the abovementioned integration method, we compute the S&P 500 indexes integrating the all-sector indexes. Following the above method, we can compute the S&P 500 index when the all-industry indexes are in one group. Our integrated indexes are adjusted as if the initial values are the same values as the S&P 500 indexes on the initial day (January 23, 1995). The root mean square of the difference between our integrated price index and the S&P 500 price index is 0.366. For the total return index, the root mean square is 0.409. This indicates that the abovementioned method is a valid method to integrate different industries' indexes.

We compute the daily data of the integrated six indexes using the original 10 daily indexes and extract the monthly data from these daily data. The daily data covered the period from January 23, 1995 to December 30, 2011. Thus, the monthly data covered the period from January 1995 to December 2011.

Appendix 2.C Specification of a Regime Number

To determine a number of regimes, we apply some regime-switching models to the integrated indexes. The models are as follows,

1. two recursive regimes without structural change model (Model $S2$),
2. two recursive regimes with once structural change model (Model $S2D2$),
3. three recursive regimes without structural change model (Model $S3$),
4. four recursive regimes without structural change model (Model $S4$),
5. once structural change without recursive regime model (Model $D2$),
6. twice structural changes without recursive regime model (Model $D3$),
7. two recursive state variables that have two regimes without structural change model (Model $S2S2$).

The first two models are those this chapter is interested in. The seventh model, Model $S2S2$, needs to be explained. It is regarded as the version of Model $S2D2$ in which D is a recursive state variable, that is, this model allows D to change regime 0 from regime

1. Table 2.17 shows the AICs of these seven models.

The AIC of Model $S2D2$ is lowest among the models, which justifies focus on Model $S2D2$. Comparing Models $S2D2$ and $S2S2$, the difference of AIC between these models is 2, and thus, depends only on the parameter penalties. Indeed, these two models are not different except for numerical errors. Figure 2.5 shows the smoothed probabilities in each model.

The probabilities of the two models are almost the same in Figure 2.5. Moreover, the root mean squares of the difference of probabilities between the two models are 1.636×10^{-6} with the probability being $S = 0$ and 8.648×10^{-8} with the probability being $D = 0$. It is interesting that Model $S2S2$ captures the irreversible structural change although all the state variables in the model are reversible. This result is evidence that the irreversible structural change occurs.

Appendix 2.D Tables and Figures

Table 2.1. The Industry Codes

Integrate Group	Sector Name
ENE	Energy
MAT/IND	Materials
	Industrials
CG	Consumer Discretionary
	Consumer Staples
	Health Care
FIN	Financials
IT/Tel	Information and Technology
	Telecoms services
U	Utilities

Table 2.2. The Results of the CHP Test. We try 3000 times parametric bootstraps. The null hypothesis is that the mean and variance don't change with the progress of time. The alternative is that they are driven by a recursive Markov chain.

	Statistics	10% Critical Values	5% Critical Values	1% Critical Values
ENE	0.015	0.015	0.019	0.028
MAT/IND	0.035	0.015	0.018	0.028
CG	0.035	0.015	0.018	0.028
FIN	0.058	0.015	0.018	0.028
IT/Tel	0.055	0.015	0.019	0.028
U	0.039	0.015	0.019	0.028

Table 2.3. The AICs of Model $S2$ and Model $S2D2$. The AICs of the other models are in Table 2.17.

	Model $S2$ ($q_{00} = 1$)	Model $S2D2$ ($0 < q_{00} < 1$)
AIC	6684.234	6619.640

Table 2.4. The Estimated Means and Volatilities of Each Sector Returns.

The first table is the estimation result of Model $S2D2$. The second is the result of Model $S2$. μ_{sd} and σ_{sd} are the conditional mean and standard deviation of the return of the individual industry index in the regime s and the structure d , respectively. $p_{ss'}$ is the transition probability of S_t from the regime s to the regime s' and $q_{dd'}$ is also the transition probability of D_t from the structure d to d' . Numbers in parenthesis are standard errors.

Model $S2D2$								
Regimes	$D = 0$				$D = 1$			
	$S = 0$		$S = 1$		$S = 0$		$S = 1$	
	μ_{00}	σ_{00}	μ_{10}	σ_{10}	μ_{01}	σ_{01}	μ_{11}	σ_{11}
ENE	2.034 (0.452)	4.368 (0.395)	0.067 (0.140)	5.800 (0.620)	1.564 (1.018)	6.327 (0.690)	-7.187 (2.566)	7.496 (1.127)
MAT/IND	1.577 (0.335)	3.255 (0.226)	-0.229 (0.453)	6.038 (0.547)	1.680 (1.039)	6.162 (0.631)	-9.091 (2.540)	7.754 (0.140)
CG	1.447 (0.293)	2.792 (0.214)	-0.160 (0.299)	4.476 (0.452)	1.303 (0.581)	3.477 (0.317)	-4.218 (1.962)	6.084 (0.227)
FIN	1.811 (0.416)	3.860 (0.306)	-0.030 (0.083)	7.181 (0.945)	0.711 (1.268)	7.525 (0.793)	-12.853 (3.980)	12.364 (1.378)
IT/Tel	1.719 (0.430)	4.159 (0.266)	-0.853 (1.457)	11.019 (0.858)	1.404 (0.891)	5.548 (0.529)	-6.226 (2.051)	6.424 (0.852)
U	1.492 (0.345)	3.325 (0.252)	-0.534 (0.702)	6.380 (0.542)	1.301 (0.487)	3.177 (0.366)	-5.074 (2.038)	5.770 (0.727)
Probability Parameters								
	p_{00}	$1 - p_{00}$		q_{00}	$1 - q_{00}$	Log Likelihood		
	0.984 (0.001)	0.016		0.993 (0.004)	0.007	-3198.820		
	$1 - p_{11}$	p_{11}		$1 - q_{11}$	q_{11}			
	0.034	0.966 (0.004)		0	1			

Model $S2$						
Regimes	$S = 0$		$S = 1$		Probability Parameters	
	μ_0	σ_0	μ_1	σ_1		
ENE	1.823 (0.426)	4.919 (0.379)	-0.500 (0.691)	6.708 (0.624)	p_{00}	$1 - p_{00}$
MAT/IND	1.499 (0.373)	4.194 (0.298)	-0.919 (0.837)	7.089 (0.656)	0.983 (0.012)	0.017
CG	1.424 (0.278)	2.985 (0.183)	-0.517 (0.579)	4.829 (0.417)	$1 - p_{11}$	p_{11}
FIN	1.384 (0.451)	4.679 (0.323)	-1.291 (1.165)	9.595 (0.988)	0.030	0.970 (0.020)
IT/Tel	1.617 (0.398)	4.385 (0.246)	-1.217 (1.189)	10.374 (0.739)	Log Likelihood	
U	1.487 (0.285)	3.229 (0.206)	-0.963 (0.714)	6.266 (0.482)	-3286.117	

Table 2.5. The Result of the Hypothesis Testing. The null hypotheses are the equality of means and that of standard deviations across the two regimes. Panel A and B are the results of tests in Model *S2D2* and Panel C is that in Model *S2*. The hypotheses in Panel A are same value of *S* but different value of *D*, on the other hand, the hypotheses in Panel B are same *D* and different *S*. Rows of industries are the tests of individual industries. Rows of “All” are the tests of which hypothesis is the equalities of parameters of the all industries. The degrees of freedom in each industries test are 1, in the all industries are 15. Numbers in parenthesis are probability values.

Panel A: D is the same among the hypotheses but S is different in the Model $S2D2$.						
Fixed Regime	Before Structural Change ($D = 0$)			After Structural Change ($D = 1$)		
Hypothesis	Equal Means	Equal Volatilities	Equal Both	Equal Means	Equal Volatilities	Equal Both
ENE	17.197	3.537	26.026	9.870	0.753	10.056
	(0.000)	(0.060)	(0.000)	(0.002)	(0.386)	(0.007)
MAT/IND	10.220	21.811	31.722	15.045	6.084	24.347
	(0.001)	(0.000)	(0.000)	(0.000)	(0.014)	(0.000)
CG	15.027	10.938	22.348	7.232	45.327	46.964
	(0.000)	(0.001)	(0.000)	(0.007)	(0.000)	(0.000)
FIN	18.693	11.090	27.653	10.480	9.315	13.924
	(0.000)	(0.001)	(0.000)	(0.001)	(0.002)	(0.001)
IT/Tel	2.827	58.729	59.086	11.284	0.732	11.481
	(0.093)	(0.000)	(0.000)	(0.001)	(0.392)	(0.003)
U	6.750	26.474	34.418	9.197	9.463	23.935
	(0.009)	(0.000)	(0.000)	(0.002)	(0.002)	(0.000)
All	37.346	96.440	138.942	50.815	74.492	140.987
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

Panel B: S is the same among the hypotheses but D is different in the Model $S2D2$.						
Fixed Regime	Booming Regime ($S = 0$)			Recession Regime ($S = 1$)		
Hypothesis	Equal Means	Equal Volatilities	Equal Both	Equal Means	Equal Volatilities	Equal Both
ENE	0.178	6.074	6.087	8.003	1.832	9.597
	(0.673)	(0.014)	(0.048)	(0.005)	(0.176)	(0.008)
MAT/IND	0.009	18.724	18.743	11.746	9.408	26.341
	(0.926)	(0.000)	(0.000)	(0.001)	(0.002)	(0.000)
CG	0.048	3.178	3.181	4.130	10.248	12.522
	(0.826)	(0.075)	(0.204)	(0.042)	(0.001)	(0.002)
FIN	0.669	18.501	24.050	10.282	9.952	13.869
	(0.414)	(0.000)	(0.000)	(0.001)	(0.002)	(0.001)
IT/Tel	0.100	5.429	5.449	4.560	13.943	24.289
	(0.752)	(0.020)	(0.066)	(0.033)	(0.000)	(0.000)
U	0.100	0.113	0.337	4.545	0.501	4.547
	(0.751)	(0.737)	(0.845)	(0.033)	(0.479)	(0.103)
All	4.333	26.082	41.806	51.338	50.097	131.850
	(0.632)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

Panel C: $S = 0$ vs. $S = 1$ in the Model $S2$.							
Hypothesis	Equal Means	Equal Volatilities	Equal Both		Equal Means	Equal Volatilities	Equal Both
ENE	8.066	5.766	12.485	FIN	4.437	22.895	23.366
	(0.005)	(0.016)	(0.002)		(0.035)	(0.000)	(0.000)
MAT/IND	6.772	16.142	19.838	IT/Tel	4.951	59.704	60.887
	(0.009)	(0.000)	(0.000)		(0.026)	(0.000)	(0.000)
CG	8.936	16.238	19.384	U	10.132	33.611	39.973
	(0.003)	(0.000)	(0.000)		(0.001)	(0.000)	(0.000)
All	15.796	114.514	125.268				
	(0.015)	(0.000)	(0.000)				

Table 2.6. The Estimated Correlation Coefficients in Model $S2$. Numbers in parenthesis are standard errors. The summary is shown in Table 2.8.

Booming Regime ($S = 0$)						
	ENE	MAT/IND	CG	FIN	IT/Tel	U
ENE	1 -					
MAT/IND	0.608 (0.068)	1 -				
CG	0.420 (0.079)	0.787 (0.031)	1 -			
FIN	0.461 (0.086)	0.788 (0.036)	0.791 (0.032)	1 -		
IT/Tel	0.445 (0.072)	0.753 (0.036)	0.726 (0.042)	0.662 (0.050)	1 -	
U	0.334 (0.074)	0.326 (0.072)	0.491 (0.062)	0.397 (0.067)	0.314 (0.080)	1 -
Recession Regime ($S = 1$)						
	ENE	MAT/IND	CG	FIN	IT/Tel	U
ENE	1 -					
MAT/IND	0.668 (0.072)	1 -				
CG	0.524 (0.098)	0.800 (0.043)	1 -			
FIN	0.491 (0.101)	0.804 (0.040)	0.805 (0.044)	1 -		
IT/Tel	0.365 (0.088)	0.622 (0.061)	0.560 (0.076)	0.471 (0.068)	1 -	
U	0.652 (0.068)	0.458 (0.100)	0.361 (0.100)	0.359 (0.093)	0.143 (0.119)	1 -

Table 2.7. The Estimated Correlation Coefficients in Model $S2D2$. Numbers in parenthesis are standard errors. The summary is shown in Table 2.8.

Before Structural Change ($D = 0$)						
	Booming Regime ($S = 0$)			Recession Regime ($S = 1$)		
	ENE	MAT/IND	CG	FIN	IT/Tel	U
ENE	1					
	-					
MAT/IND	0.389 (0.081)	1				
	-					
CG	0.232 (0.086)	0.721 (0.047)	1			
	-					
FIN	0.196 (0.086)	0.685 (0.052)	0.800 (0.040)	1		
	-					
IT/Tel	0.262 (0.087)	0.718 (0.050)	0.670 (0.053)	0.610 (0.071)	1	
	-					
U	0.341 (0.091)	0.240 (0.101)	0.448 (0.084)	0.420 (0.090)	0.186 (0.102)	1
	-					
After Structural Change ($D = 1$)						
	Booming Regime ($S = 0$)			Recession Regime ($S = 1$)		
	ENE	MAT/IND	CG	FIN	IT/Tel	U
ENE	1					
	-					
MAT/IND	0.754 (0.066)	1				
	-					
CG	0.671 (0.084)	0.885 (0.029)	1			
	-					
FIN	0.599 (0.102)	0.856 (0.038)	0.780 (0.044)	1		
	-					
IT/Tel	0.709 (0.072)	0.825 (0.039)	0.821 (0.039)	0.710 (0.076)	1	
	-					
U	0.391 (0.142)	0.480 (0.096)	0.625 (0.067)	0.377 (0.102)	0.701 (0.074)	1
	-					

Table 2.8. The Summary Table for Changes of Correlations at 10% Significant Level. Numbers in parentheses are percentages of all pairs.

Model $S2$		
	Correlation in $S = 0 < S = 1$	Correlation in $S = 0 > S = 1$
Number of Changes	7 (47%)	8 (53%)
Number of Significant Changes	3 (20%)	1 (7%)

Model $S2D2$		
Fix S .		
$S = 0$	Correlation in $D = 0 > D = 1$	Correlation in $D = 0 < D = 1$
Number of Changes	2 (13%)	13 (87%)
Number of Significant Changes	0 (0%)	11 (73%)
$S = 1$	Correlation in $D = 0 > D = 1$	Correlation in $D = 0 < D = 1$
Number of Changes	4 (27%)	11 (73%)
Number of Significant Changes	3 (20%)	4 (27%)
Fix D .		
$D = 0$	Correlation in $S = 0 > S = 1$	Correlation in $S = 0 < S = 1$
Number of Changes	8 (53%)	7 (47%)
Number of Significant Changes	0 (0%)	4 (27%)
$D = 1$	Correlation in $S = 0 > S = 1$	Correlation in $S = 0 < S = 1$
Number of Changes	10 (67%)	5 (33%)
Number of Significant Changes	4 (27%)	2 (13%)

Table 2.9. The Results of the Hypothesis Testings in Model $S2D2$. The null hypothesis is the equality of correlations between two regimes $S = 0$ and $S = 1$. Marks *, **, and * * * indicate rejecting the null at significant level 10%, 5%, and 1%, respectively. Numbers in parentheses are P-values. The summary is shown in Table 2.8.

Null Hypothesis: Correlations are equal between $S = 0$ and $S = 1$:					
Before Structural Break ($D = 0$)					
	ENE	MAT/IND	CG	FIN	IT/Tel
MAT/IND	4.458 0.035				
CG	3.162* (0.075)	8.015*** (0.005)			
FIN	6.114** (0.013)	0.639 (0.424)	0.267 (0.606)		
IT/Tel	2.033 (0.154)	0.001 (0.978)	0.017 (0.897)	1.174 (0.279)	
U	11.134*** (0.001)	0.415 (0.520)	1.309 (0.253)	2.743* (0.098)	1.502 (0.220)
All correlations are equal (df = 15): 70.494*** (0.000)					
After Structural Break ($D = 1$)					
	ENE	MAT/IND	CG	FIN	IT/Tel
MAT/IND	4.852** (0.028)				
CG	3.217* (0.073)	0.000 (0.987)			
FIN	9.131*** (0.003)	0.465 (0.495)	0.009 (0.923)		
IT/Tel	0.008 (0.928)	1.096 (0.295)	1.611 (0.204)	1.072 (0.300)	
U	3.249* (0.071)	0.331 (0.565)	0.905 (0.341)	0.081 (0.775)	0.195 (0.659)
All correlations are equal (df = 15): 29.976** (0.012)					

Table 2.10. The Results of the Hypothesis Testing in Model $S2D2$. The null hypothesis is the equality of correlations between two regimes $D = 0$ and $D = 1$. Marks *, **, and * * * indicate rejecting the null at significant level 10%, 5%, and 1%, respectively. Numbers in parentheses are P-values. The summary is shown in Table 2.8.

Null Hypothesis: Correlations are equal between $D = 0$ and $D = 1$:					
Booming Regime ($S = 0$)					
	ENE	MAT/IND	CG	FIN	IT/Tel
MAT/IND	12.323*** (0.000)				
CG	14.004*** (0.000)	8.852*** (0.003)			
FIN	9.152*** (0.002)	6.941*** (0.008)	0.118 (0.731)		
IT/Tel	15.775*** (0.000)	2.863* (0.091)	5.434** (0.020)	0.888 (0.346)	
U	0.090 (0.764)	2.994* (0.084)	2.785* (0.095)	0.097 (0.756)	16.936*** (0.000)
All correlations are equal (df = 15):			54.622***	(0.000)	
Recession Regime ($S = 1$)					
	ENE	MAT/IND	CG	FIN	IT/Tel
MAT/IND	2.729* (0.099)				
CG	0.848 (0.357)	15.766*** (0.000)			
FIN	17.21*** (0.000)	0.100 (0.752)	0.161 (0.688)		
IT/Tel	0.324 (0.569)	3.504* (0.061)	3.982** (0.046)	0.152 (0.697)	
U	11.275*** (0.001)	1.224 (0.269)	0.673 (0.412)	3.137* (0.077)	2.483 (0.115)
All correlations are equal (df = 15):			123.992***	(0.000)	

Table 2.11. The Results of the Hypothesis Testings in Model S2. The null hypothesis is the equality of correlations between two regimes $S = 0$ and $S = 1$. Marks *, **, and *** indicate rejecting the null at significant level 10%, 5%, and 1%, respectively. Numbers in parentheses are P-values. The summary is shown in Table 2.8.

Null Hypothesis: Correlations are equal between $S = 0$ and $S = 1$:					
	ENE	MAT/IND	CG	FIN	IT/Tel
MAT/IND	12.323*** (0.000)				
CG	14.004*** (0.000)	8.852*** (0.003)			
FIN	9.152*** (0.002)	6.941*** (0.008)	0.118 (0.731)		
IT/Tel	15.775*** (0.000)	2.863* (0.091)	5.434** (0.020)	0.888 (0.346)	
U	0.090 (0.764)	2.994* (0.084)	2.785* (0.095)	0.097 (0.756)	16.936*** (0.000)
All correlations are equal (df = 15):					
			54.622***	(0.000)	

Table 2.12. The Simulation Results of the Performances with Various Investment Strategies. Numbers in parentheses are the standard deviations of them. All of the means and the standard deviations, and their standard errors in this table are multiplied by 100. Number of trials is 10000. Each trial has 140 observations.

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Initial State $S = 0, D = 0$					
Global Minimum-Variance Portfolios					
$S2D2$	0.616 (0.010)	3.501 (0.008)	-0.329 (0.004)	3.559 (0.009)	0.237 (0.003)
$S2$	0.631 (0.010)	3.550 (0.008)	-0.304 (0.004)	3.536 (0.008)	0.233 (0.003)
$D2$	0.658 (0.010)	3.519 (0.008)	-0.314 (0.004)	3.558 (0.009)	0.245 (0.003)
IID	0.607 (0.010)	3.557 (0.008)	-0.295 (0.004)	3.514 (0.008)	0.225 (0.003)
Tangency Portfolios					
$S2D2$	0.798 (0.010)	3.962 (0.009)	-0.328 (0.005)	4.115 (0.013)	0.258 (0.003)
$S2$	0.495 (0.014)	4.386 (0.013)	-0.421 (0.006)	4.378 (0.016)	0.201 (0.003)
$D2$	0.652 (0.011)	3.857 (0.009)	-0.333 (0.004)	3.651 (0.010)	0.235 (0.003)
IID	0.681 (0.011)	3.832 (0.009)	-0.241 (0.003)	3.463 (0.008)	0.239 (0.003)
EW	0.491 (0.014)	4.458 (0.012)	-0.239 (0.004)	3.627 (0.009)	0.184 (0.003)
Initial State $S = 1, D = 0$					
Global Minimum-Variance Portfolios					
$S2D2$	0.313 (0.011)	3.717 (0.007)	-0.302 (0.003)	3.464 (0.008)	0.136 (0.003)
$S2$	0.335 (0.010)	3.762 (0.007)	-0.280 (0.003)	3.447 (0.008)	0.136 (0.003)
$D2$	0.363 (0.011)	3.737 (0.007)	-0.287 (0.003)	3.468 (0.008)	0.147 (0.003)
IID	0.315 (0.011)	3.767 (0.007)	-0.271 (0.003)	3.429 (0.007)	0.130 (0.003)
Tangency Portfolios					
$S2D2$	0.532 (0.011)	4.402 (0.009)	-0.264 (0.004)	4.045 (0.011)	0.163 (0.003)
$S2$	0.205 (0.015)	4.846 (0.012)	-0.342 (0.005)	4.197 (0.013)	0.111 (0.003)
$D2$	0.347 (0.012)	4.100 (0.009)	-0.298 (0.004)	3.535 (0.008)	0.141 (0.003)
IID	0.373 (0.012)	4.035 (0.008)	-0.210 (0.003)	3.368 (0.007)	0.145 (0.003)
EW	0.145 (0.016)	4.743 (0.011)	-0.197 (0.003)	3.510 (0.008)	0.092 (0.003)

Table 2.13. The Simulation Results of the Performances with Various Investment Strategies. Numbers in parentheses are the standard deviations of them. All of the means and the standard deviations, and their standard errors in this table are multiplied by 100. Number of trials is 10000. Each trial has 140 observations.

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Initial State $S = 0, D = 1$					
Global Minimum-Variance Portfolios					
$S2D2$	-0.266 (0.013)	4.195 (0.007)	-0.449 (0.004)	3.456 (0.009)	-0.014 (0.003)
$S2$	-0.141 (0.012)	4.273 (0.007)	-0.417 (0.003)	3.462 (0.008)	0.011 (0.003)
$D2$	-0.170 (0.012)	4.227 (0.007)	-0.422 (0.003)	3.458 (0.008)	0.005 (0.003)
IID	-0.172 (0.013)	4.279 (0.007)	-0.408 (0.003)	3.434 (0.008)	0.004 (0.003)
Tangency Portfolios					
$S2D2$	-0.090 (0.012)	4.404 (0.009)	-0.527 (0.004)	3.844 (0.011)	0.031 (0.003)
$S2$	-0.752 (0.018)	5.453 (0.013)	-0.665 (0.005)	4.045 (0.015)	-0.069 (0.003)
$D2$	-0.353 (0.015)	4.712 (0.009)	-0.473 (0.004)	3.530 (0.009)	-0.024 (0.003)
IID	-0.254 (0.014)	4.882 (0.006)	-0.214 (0.003)	3.129 (0.006)	-0.022 (0.003)
EW	-0.787 (0.018)	5.857 (0.007)	-0.142 (0.003)	3.012 (0.005)	-0.106 (0.003)
Initial State $S = 1, D = 1$					
Global Minimum-Variance Portfolios					
$S2D2$	-1.079 (0.014)	4.582 (0.006)	-0.448 (0.003)	3.250 (0.008)	-0.207 (0.003)
$S2$	-0.903 (0.013)	4.659 (0.006)	-0.423 (0.003)	3.285 (0.007)	-0.166 (0.003)
$D2$	-0.935 (0.013)	4.609 (0.006)	-0.426 (0.003)	3.276 (0.008)	-0.175 (0.003)
IID	-0.943 (0.013)	4.658 (0.006)	-0.413 (0.003)	3.257 (0.007)	-0.176 (0.003)
Tangency Portfolios					
$S2D2$	-0.820 (0.013)	4.910 (0.008)	-0.533 (0.004)	3.649 (0.010)	-0.133 (0.003)
$S2$	-1.857 (0.019)	6.178 (0.010)	-0.656 (0.004)	3.751 (0.013)	-0.261 (0.003)
$D2$	-1.252 (0.015)	5.167 (0.007)	-0.478 (0.003)	3.329 (0.009)	-0.212 (0.003)
IID	-1.126 (0.015)	5.187 (0.005)	-0.216 (0.002)	3.007 (0.005)	-0.200 (0.003)
EW	-1.922 (0.019)	6.181 (0.006)	-0.122 (0.003)	2.896 (0.005)	-0.298 (0.003)

Table 2.14. The Global Minimum-Variance Portfolios and Tangency Portfolios. Panel A shows the portfolio weight of global minimum-variance portfolio for each investor type. On the other hands, Panel A shows the portfolio weight of tangency portfolio for each investor type.

Panel A: Global Minimum-Variance Portfolios

	<i>S2D2</i>			
	$S = 0, D = 0$	$S = 1, D = 0$	$S = 0, D = 1$	$S = 1, D = 1$
ENE	0.128	0.056	0.000	0.000
MAT/IND	0.126	0.000	0.000	0.000
CG	0.472	0.694	0.401	0.357
FIN	0.000	0.000	0.000	0.000
IT/Tel	0.000	0.000	0.000	0.115
U	0.274	0.250	0.599	0.528

	<i>S2</i>		<i>D2</i>		IID
	$S = 0$	$S = 1$	$D = 0$	$D = 1$	
ENE	0.063	0.000	0.159	0.000	0.028
MAT/IND	0.000	0.000	0.000	0.000	0.000
CG	0.547	0.694	0.600	0.635	0.654
FIN	0.000	0.000	0.000	0.000	0.000
IT/Tel	0.000	0.000	0.000	0.000	0.000
U	0.390	0.306	0.241	0.365	0.319

Panel B: Tangency Portfolios

	<i>S2D2</i>			
	$S = 0, D = 0$	$S = 1, D = 0$	$S = 0, D = 1$	$S = 1, D = 1$
ENE	0.268	0.916	0.025	0.000
MAT/IND	0.073	0.000	0.000	0.000
CG	0.308	0.000	0.396	0.992
FIN	0.066	0.084	0.000	0.000
IT/Tel	0.061	0.000	0.000	0.008
U	0.225	0.000	0.580	0.000

	<i>S2</i>		<i>D2</i>		IID
	$S = 0$	$S = 1$	$D = 0$	$D = 1$	
ENE	0.170	1.000	0.270	0.231	0.318
MAT/IND	0.000	0.000	0.000	0.000	0.000
CG	0.407	0.000	0.631	0.071	0.568
FIN	0.000	0.000	0.099	0.000	0.000
IT/Tel	0.035	0.000	0.000	0.000	0.000
U	0.387	0.000	0.000	0.697	0.114

Table 2.15. The Simulation Results of the Performances with Various Types of Investment Strategies. 2000-time series of the indices with 200 months are simulated. Each type investor start to invest after 60 months passed from the starting point. The data of the first 60 months are only used for them to estimate the distribution parameters in their considered model. After the first 60 months, the past data-windows for the investors are expanded as time passes. This table reports the means of the statistics. All of the statistic are computed using the data of returns in each trial. Numbers in parentheses are the standard errors. All of the means and standard deviations and their standard errors in this table are multiplied by 100. We assume that the risk-free rate is 0.

Panel A: The Actual Market Model is Model *S2D2*

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Global Minimum-Variance Portfolios					
<i>S2D2</i>	0.047 (0.034)	4.424 (0.022)	-0.341 (0.010)	3.788 (0.027)	0.077 (0.007)
<i>S2</i>	0.102 (0.032)	3.998 (0.020)	-0.357 (0.009)	3.539 (0.021)	0.096 (0.008)
IID	0.224 (0.028)	3.882 (0.018)	-0.341 (0.008)	3.484 (0.019)	0.118 (0.007)
Tangency Portfolios					
<i>S2D2</i>	0.082 (0.035)	5.269 (0.026)	-0.263 (0.012)	4.464 (0.042)	0.070 (0.007)
<i>S2</i>	0.083 (0.035)	4.920 (0.028)	-0.349 (0.011)	4.167 (0.038)	0.082 (0.007)
IID	0.172 (0.034)	4.717 (0.028)	-0.235 (0.008)	3.513 (0.019)	0.103 (0.007)
EW	-0.116 (0.149)	5.032 (0.102)	-0.194 (0.028)	3.389 (0.071)	0.049 (0.029)

Panel B: The Actual Market Model is Model *S2*

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Global Minimum-Variance Portfolios					
<i>S2D2</i>	0.835 (0.014)	4.038 (0.015)	-0.137 (0.008)	3.794 (0.023)	0.230 (0.004)
<i>S2</i>	0.813 (0.013)	3.554 (0.012)	-0.192 (0.007)	3.546 (0.023)	0.252 (0.004)
IID/Myopic	0.801 (0.013)	3.503 (0.011)	-0.184 (0.006)	3.429 (0.016)	0.252 (0.004)
Tangency Portfolios					
<i>S2D2</i>	0.852 (0.016)	5.101 (0.025)	-0.086 (0.011)	4.647 (0.041)	0.197 (0.004)
<i>S2</i>	0.855 (0.015)	4.901 (0.032)	-0.146 (0.011)	4.716 (0.043)	0.215 (0.004)
IID	0.892 (0.014)	4.324 (0.026)	-0.111 (0.007)	3.635 (0.021)	0.241 (0.004)
EW	0.847 (0.054)	4.375 (0.060)	-0.143 (0.025)	3.568 (0.067)	0.217 (0.015)

Table 2.16. The Simulation Results of the Performances Based on the Results of the Literature. 2000-time series of the indices are simulated. Each type investor start to invest after 200 months (resp. 288 months) passed from the starting point in the simulation of Ang and Bekaert (2002) (resp. Guidolin and Timmermann (2007)). The data of the initial runnings are only used for them to estimate the distribution parameters in their considered model. After the initial running months, the past data-windows for the investors are expanded as time passes. This table reports the means of the statistics. All of the statistic are computed using the data of returns in each trial. Numbers in parentheses are the standard errors. All of the means and standard deviations and their standard errors in this table are multiplied by 100.

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Simulation of Ang and Bekaert (2002)					
Only 2-state recursive regime (S) model					
The risk-free rate is 0.0041 (monthly rate).					
Global Minimum-Variance Portfolios					
4 states	0.618 (0.015)	4.080 (0.017)	-0.106 (0.016)	4.264 (0.052)	0.154 (0.004)
2 states	0.621 (0.014)	4.040 (0.017)	-0.110 (0.015)	4.182 (0.044)	0.156 (0.004)
IID	0.623 (0.014)	4.042 (0.017)	-0.109 (0.015)	4.150 (0.042)	0.156 (0.004)
Tangency Portfolios					
4 states	0.602 (0.017)	4.809 (0.032)	-0.034 (0.024)	5.444 (0.101)	0.129 (0.004)
2 states	0.612 (0.017)	4.654 (0.034)	-0.052 (0.023)	5.260 (0.102)	0.136 (0.004)
IID	0.621 (0.015)	4.307 (0.026)	-0.072 (0.016)	4.203 (0.050)	0.147 (0.004)
EW	0.392 (0.016)	4.518 (0.022)	-0.345 (0.017)	4.705 (0.052)	0.089 (0.004)
Simulation of Guidolin and Timmermann (2007)					
Only 4-state recursive regime (S) model					
The risk-free rate is 0.0044 (monthly rate).					
Global Minimum-Variance Portfolios					
4 states	0.357 (0.024)	6.728 (0.031)	0.550 (0.017)	4.710 (0.069)	0.052 (0.003)
2 states	0.369 (0.024)	6.718 (0.030)	0.525 (0.016)	4.600 (0.063)	0.054 (0.003)
IID	0.368 (0.024)	6.718 (0.030)	0.512 (0.016)	4.553 (0.061)	0.053 (0.003)
Tangency Portfolios					
4 states	1.853 (0.054)	13.756 (0.111)	0.689 (0.023)	5.085 (0.135)	0.133 (0.004)
2 states	1.876 (0.055)	13.622 (0.134)	0.643 (0.020)	4.678 (0.102)	0.136 (0.004)
IID	1.922 (0.056)	13.591 (0.155)	0.629 (0.018)	4.433 (0.080)	0.139 (0.003)
EW	0.503 (0.035)	9.871 (0.033)	0.070 (0.012)	3.572 (0.028)	0.051 (0.004)

Table 2.17. The AICs of Various Models. The first row displays the AICs, which are defined as $-2(\log \text{likelihood} - \# \text{ of parameters})$, where $\#$ of parameters means the number of parameters.

Model	<i>S2</i>	<i>S2D2</i>	<i>S3</i>	<i>S4</i>	<i>D2</i>	<i>D3</i>	<i>S2S2</i>
AIC	6684.234	6619.640	6662.203	6634.194	6778.565	6717.901	6621.640
# of Parameters	56	111	87	120	55	83	112

Figure 2.1. The Smoothed Probabilities and the NBER Recession Dates.

The first figure plots the smoothed probability of Model $S2$. The second figure plots the smoothed probabilities of Model $S2D2$. In each figure, a solid line is the probability of being at $S = 0$ and shadow areas are NBER recession dates. In the second figure, a dashed line is the probability of being at $D = 0$.

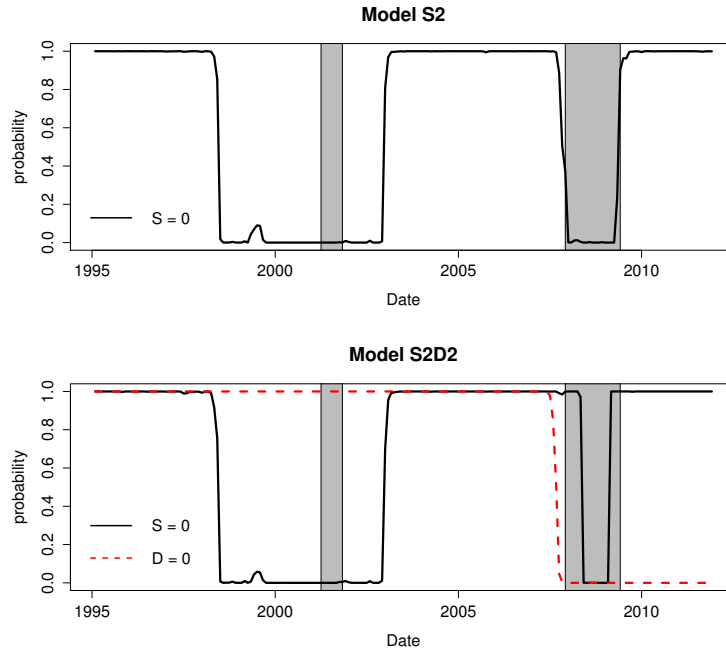


Figure 2.2. The Smoothed Probability of Being at $D_t = 0$ Computed by Using Individual Sector Returns.

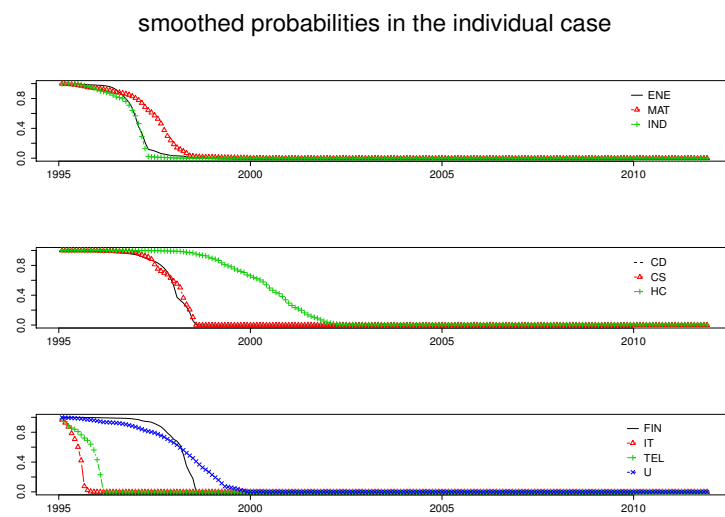
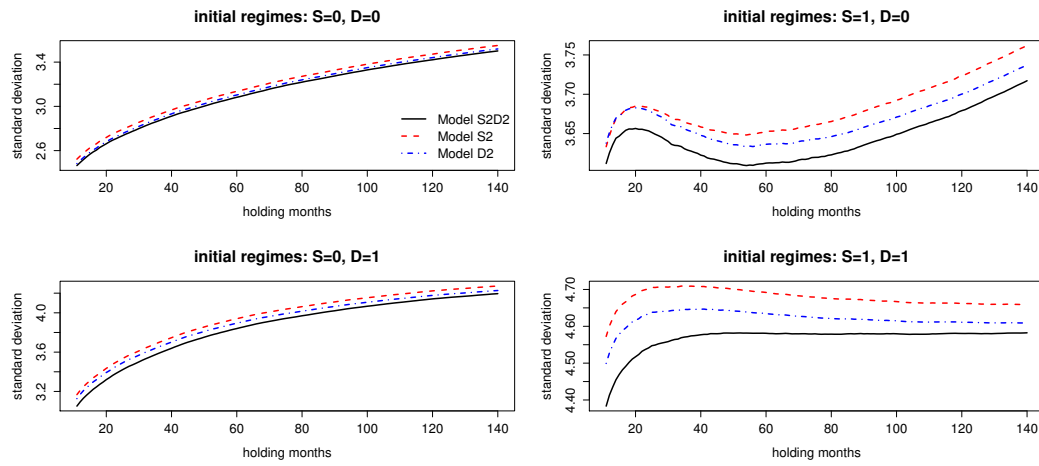


Figure 2.3. The Standard Deviations and Sharpe Ratios. In computing the Sharpe ratios, we assume that the risk-free rate equal 0.

The changes of simulated standard deviations



The changes of simulated Sharpe ratios

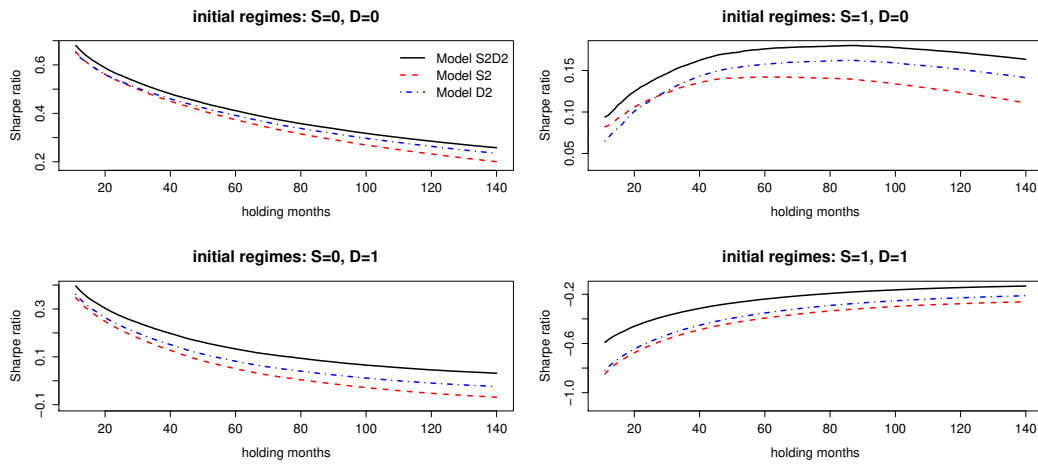


Figure 2.4. The Means of the Smoothed Probabilities around the Structural Change. Lines of the 100th, 150th and 199th month represent the changes of the means of the smoothed probabilities around the structural change for $T = 100, 150$ and 199 , respectively. A line of the 199th month conditioned by change time represents the means of the smoothed probabilities around the structural change with the condition that the structural change happens during the periods from $t = 51$ to $t = 50$.

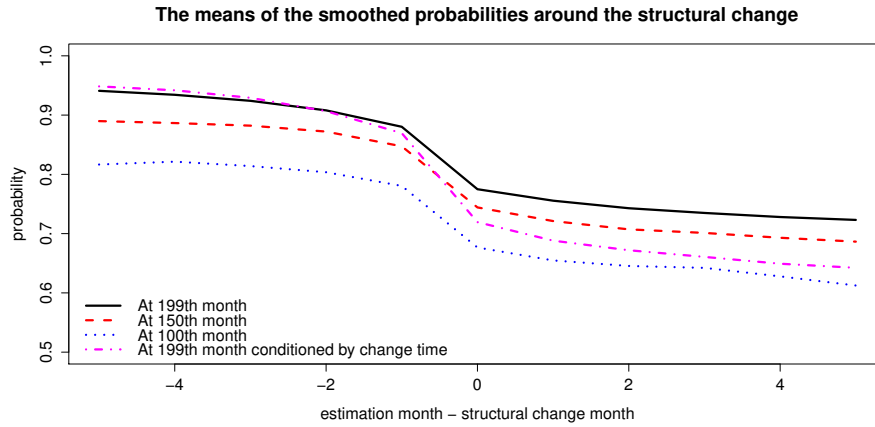
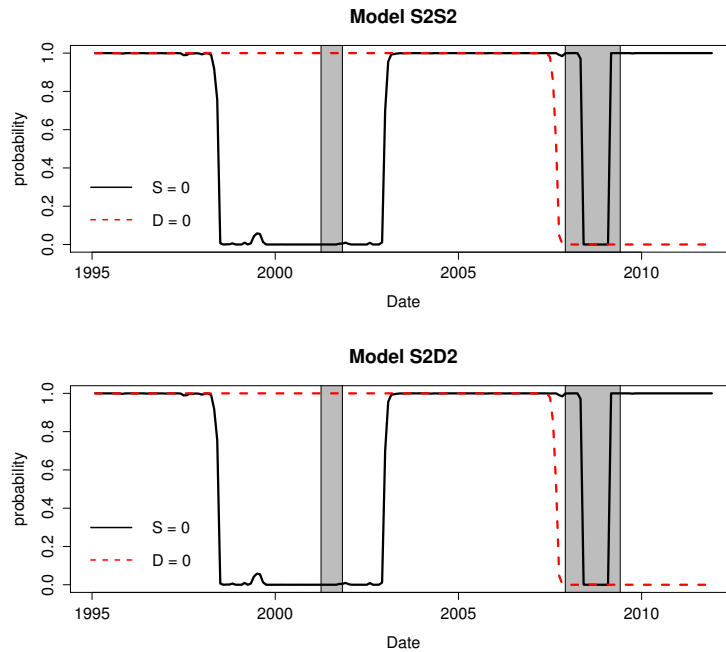


Figure 2.5. The Smoothed Probabilities and the NBER Recession Dates. The first figure plots the smoothed probabilities of Model $S2S2$. The second figure plots the smoothed probabilities of Model $S2D2$. In each figure, a solid line is the probability of being at $S = 0$, shadow areas are NBER recession dates and a dashed line is the probability of being at $D = 0$.



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Chapter 3

Mean-Variance Efficiency of Discrete-Time, Infinite Horizon Mean-Variance Portfolio Selections in Regime Switching Financial Markets

3.1 Introduction

Since the original work of Markowitz (1952), the mean-variance analysis (hereafter MV analysis) has been one of the fundamental methods for making investment decisions. The MV analysis can deal with the trade-off between the expected returns and associated risks. Markowitz (1952) studies the single-period MV analysis, however, many papers after Markowitz (1952) take up the multi-period MV analysis. An important problem when dealing with the multi-period MV analysis is the absence of time consistency, which means that it is difficult to use the Bellman optimality of the dynamic programming. That is because the variance operator does not have monotonicity, which means that even if a random variable X is always larger than another random variable Y , it is possible that the variance of X is smaller than the variance of Y . Furthermore, the variance operator does not have separability, that is, any two random variables X and Y do not always satisfy $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. Because of the lack of monotonicity and separability of the variance operator, we can not apply the standard procedure of the dynamic programming to the multi-period MV analysis.

One of the methods to avoid the absence of time consistency is changing the original

mean-variance problem into the stochastic linear quadratic (LQ) problem. In line of the stochastic LQ problem, Li and Ng (2000) study the multi-period MV analysis in discrete time and Zhou and Li (2000) consider the continuous-time MV analysis. There are many extensions of the multi-period MV analysis that is transformed into the stochastic LQ problem. Li, Zhou, and Lim (2002) provide the multi-period MV analysis with a short-selling constraint. Bielecki, Jin, Pliska, and Zhou (2005) consider the case of bankruptcy prohibition. Lim and Zhou (2002) allow parameters to vary randomly over time. Dai, Xu, and Zhou (2010) take into account the effects of transaction costs. Another method changes the objective function into the form that its solution is time consistent. In Chen, Li, and Guo (2013), the objective function of the discrete-time, finite horizon MV analysis up to time T is expressed as

$$\mathbb{E} \left[\sum_{t=1}^T \left(\text{Var}(X_t | \mathcal{F}_{t-1}) - \lambda \mathbb{E}[X_t | \mathcal{F}_{t-1}] \right) \right],$$

where X_t is the investor's wealth at time t , and $\mathcal{F}_{t-1}$ is the information σ -algebra of the investor. Under a self-financing constraint, Chen et al. (2013) have proved that the minimizing problem of the above objective function is time-consistent.

In this chapter, we aim to provide the time-consistent optimal policies of the multi-period MV analysis in the unobservable regime switching financial market and to assess the value of information about regime switching. Hamilton (1989) has proposed a Markov chain to express dependent structure of GDP over time. Financial and economic models using Markov chains are called regime switching models. These models allow a dependent structure of asset returns over time. First, we consider discrete-time, infinite horizon MV analysis, that is similar to Chen et al. (2013) and Chen, Li, and Zhao (2014), under three different investor's information levels. The frameworks of Chen et al. (2013) and Chen et al. (2014) can avoid the absence of time-consistency. The first level is the most realistic setting, where the investor can not observe the market regimes and only observes the asset returns. In the second information level, the investor observes both the market regimes and the asset returns. The informed investor of the third level can predict future regimes correctly. We derive the time-consistent optimal policies of the multi-period MV analysis in the three information levels.

The results concerning the value of information are very surprising. The expected returns of the informed investor's optimal policies are higher than those of the uninformed investor. However, the variances of returns also increase as the information increases. These results can be theoretical explanations of the limited advantage of information reported by Guidolin and Ria (2011).

Regime-switching models are useful to describe dependent structures of assets' returns. They can identify economic regimes – for example, economic booming and recession, so we can understand the conditions of the financial market more directly. After the influential work of Hamilton (1989), many papers have adopted regime switching frameworks to analyze financial markets in both empirically and theoretically. However, there are some debates as to whether regime-switching models have advantage in empirical studies or not. Ang and Bekaert (2002) apply a finitely dimensional, discrete-time regime switching model to the international equity market of 1972-1997. They identify two economic regimes, booming and recession, and reproduce the correlation pattern reported by Longin and Solnik (2001). Moreover, they test the advantage of the knowledge of regimes by conducting simulations. They report that the informed investor who knows the current economic regime has advantage over the myopic investor in the case of the constant relative risk aversion (CRRA) expected utility optimization. Guidolin and Timmermann (2007) extend the arguments of Ang and Bekaert (2002). They study the U.S. financial market empirically using a regime switching model. They identify four economic regimes using a discrete-time regime switching model and find that the informed investor who can estimate the current regime has better performance than the myopic investor, when optimizing the CRRA expected utility. However, in the MV analysis, these advantages are very limited in empirically realistic settings. Guidolin and Ria (2011) report that in their back test, using mean-variance optimal portfolios (the tangency portfolios), the informed investor's Sharpe ratio is not always higher than that of the myopic investor. Since the Sharpe ratio is one of the famous measures of the investment efficiency, this practical pitfall is an important issue. We discuss the above problem of the mean-variance efficiency in the regime switching markets, and provide a mathematical explanation of these empirical studies.

Theoretically, the regime switching MV analysis can be conducted under many different settings. Yin and Zhou (2004) study the discrete-time MV analysis in the regime switching market, where investors can observe the current values of the Markov chain driving the financial market. Costa and Araujo (2008) provide optimal policies with a bankruptcy constraint. Based on the work of Chen et al. (2013), Chen et al. (2014) derive time-consistent optimal portfolios using the regime switching MV analysis. These analyses assume that the investor knows the current or future values of the Markov chain, however, the investor does not know these values in practice. An example of a study concerning portfolio selections in the unobservable regime switching models is Honda (2003), that considers the CRRA utility optimization. Another example of an optimization problem under unobservable regimes is a study by Sass and Haussmann (2004). However, there are only few studies of portfolio selections that use unobservable regime-switching models. That is because, in the case of portfolio selections in unobservable regime-switching models, there are two state variables: investor's wealth and the probability with which the current market regime emerges. These variables are more interactive than the ones in observable regimes. Therefore, the portfolio selections in the unobservable case are more complicated mathematically than those in the observable case. In this chapter, we formulate and solve portfolio selections under unobservable regimes by considering the extensions of Chen et al. (2014).

The rest of this chapter is organized as follows. In Section 3.2, we formulate discrete time, infinite horizon MV optimizations and derive their optimal policies. In Section 3.3, we assess the values of regime information. We compute the expected returns and the standard deviations of the three different information levels. In Section 3.4 we conduct numerical analysis. Section 3.5 concludes.

3.2 Mean-Variance Analysis with Regime Switching

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In the financial market, there are one risk-free and d risky assets. The risk-free rate is constant over time, denoted by r_f . Let $(R_t)_{t=1}^\infty$ be a return vector process of the risky assets such that

$$R_t = \mu(S_t) + \sigma(S_t)\epsilon_t,$$

where $(S_t)_{t=0}^\infty$ is a Markov chain taking values in a space $\mathcal{D} = \{1, \dots, K\}$ and $(\epsilon_t)_{t=1}^\infty$ is a sequence of d -dimensional, square integrable, i.i.d. random variables. We assume that

$$\mathbb{E}[\epsilon_t] = 0, \quad \text{Var}(\epsilon_t) = I_d, \quad \text{for all } t = 1, 2, \dots,$$

where I_d is a d -dimensional identity matrix. $\mu : \mathcal{D} \rightarrow \mathbb{R}^d$ and $\sigma : \mathcal{D} \rightarrow \mathbb{R}^{d \times d}$ are measurable functions. The transition probability matrix of the Markov chain $(S_t)_{t=0}^\infty$ is constant over time and denoted by Q . We assume that $(S_t)_{t=0}^\infty$ and $(\epsilon_t)_{t=1}^\infty$ are mutually independent. The Markov chain $(S_t)_{t=0}^\infty$ represents a state of the market. Therefore, we call it the state variable.

To construct dynamic mean-variance optimization problems, we consider the following three levels of investor's information:

- 1. Partial information:** At each time t , the investor does not know the history of the state variable values $S_0, S_1, \dots, S_{t-1}, S_t$. She only observes the history of the return values, $R_1, \dots, R_{t-1}, R_t$.
- 2. Perfect information:** At each time t , the investor knows the histories of the state variable values $S_0, S_1, \dots, S_{t-1}, S_t$ and returns $R_1, \dots, R_{t-1}, R_t$.
- 3. Prediction:** At each time t , the investor knows the history of the state variable values and returns. Moreover, the investor knows the one-period-ahead value of the state variable S_{t+1} .

The first partial information case is a typical setting of information in the financial market. The partially informed investor makes investment decisions while estimating

and predicting movement of the state variable. The second perfect information case is more informative for the investors. The perfectly informed investor makes investment decisions while only predicting movement of the state variable. The third prediction case is an artificial setting. The third type investor can take positions predicting the conditions of the future market correctly.

In the partial information case, the investor's information at time t , denoted by $\mathcal{F}_t^R$, coincides with the σ -algebra generated by $R_1, \dots, R_t$, that is

$$\mathcal{F}_t^R = \sigma\{R_s, s \leq t\}, \quad t \geq 1, \quad \mathcal{F}_0^R = \{\emptyset, \Omega\}.$$

In the perfect information case, the investor's information at time t is more informative than in the partial information case. The perfectly informed investor's information at time t is

$$\mathcal{F}_t^{R,S} = \sigma(\mathcal{F}_t^R \cup \sigma\{S_s, s \leq t\}), \quad t \geq 1, \quad \mathcal{F}_0^{R,S} = \sigma\{S_0\}.$$

In the prediction case, the investor's information at time t is

$$\mathcal{F}_t^{R,S,P} = \sigma(\mathcal{F}_t^R \cup \sigma\{S_s, s \leq t+1\}), \quad t \geq 1, \quad \mathcal{F}_0^{R,S,P} = \sigma\{S_0, S_1\}.$$

Then, it is clear that $\mathcal{F}_t^R \subset \mathcal{F}_t^{R,S} \subset \mathcal{F}_t^{R,S,P}$ holds for all $t \geq 0$. The three filtrations, $(\mathcal{F}_t^R)_{t=0}^\infty$, $(\mathcal{F}_t^{R,S})_{t=0}^\infty$, and $(\mathcal{F}_t^{R,S,P})_{t=0}^\infty$ represent flows of the investors' information.

Let $(X_t)_{t=0}^\infty$ be a sequence of the investor's wealth. At each time t , the investor constructs portfolios by investing wealth with the self-financing constraint. Let $u_t = (u_{1,t}, \dots, u_{d,t})' \in \mathbb{R}^d$ be a portfolio at time $t \geq 0$ whose j th element represents cash invested in the j th risky asset at time $t \geq 0$ ¹. The investor invests $X_t - \sum_{j=1}^d u_{j,t}$ dollars in the risk-free asset at time $t \geq 0$. We call the portfolio process $u = (u_t)_{t=0}^\infty$ the *control process*. We assume that any control process is an adapted process with respect to the investor's information filtration. The wealth dynamics at time t is expressed as

$$X_{t+1} = (1 + r_f)X_t + (R_{t+1}^e)'u_t,$$

where R_{t+1}^e is the excess return vector $R_{t+1}^e = R_{t+1} - r_f \mathbf{1}_d$, and $\mathbf{1}_d = (1, \dots, 1)' \in \mathbb{R}^d$.

¹For any $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$, x' and A' mean the transposes of x and A , respectively.

The investor faces a trade-off between the expected returns and associated risks of her wealth at each time. We define the investor I 's single-period risk measure at time $t - 1$ as

$$\rho_{t-1}(X_t) := \mathbb{E}[X_t | \mathcal{F}_{t-1}^I] - \frac{\gamma_{t-1}^I}{2} \text{Var}(X_t | \mathcal{F}_{t-1}^I),$$

where $\mathcal{F}_{t-1}^I$ is the investor I 's information σ -algebra at time t , $\mathcal{F}_t^R, \mathcal{F}_t^{R,S}$ or $\mathcal{F}_t^{R,S,P}$, and γ_t^I is the investor I 's trade-off parameter at time t . We assume that γ_t^I is an $\mathcal{F}_t^I$ -measurable random variable, and that it is strictly positive for all $t \geq 0$. The objective function of the investor I is

$$\mathbb{E} \left[\sum_{t=1}^{\infty} \delta^t \left(\mathbb{E}[X_t | \mathcal{F}_{t-1}^I] - \frac{\gamma_{t-1}^I}{2} \text{Var}(X_t | \mathcal{F}_{t-1}^I) \right) \right],$$

where δ is a constant discount parameter.

When the initial wealth is x , the optimization problem can be expressed as

$$\begin{aligned} & \sup_{u \in \mathcal{A}_I(x)} \mathbb{E} \left[\sum_{t=1}^{\infty} \delta^t \left(\mathbb{E}[X_t | \mathcal{F}_{t-1}^I] - \frac{\gamma_{t-1}^I}{2} \text{Var}(X_t | \mathcal{F}_{t-1}^I) \right) \right] \\ & \text{subject to} \quad (1 + r_f)X_{t-1} + (R_t^e)'u_{t-1} = X_t, \quad t \geq 1, \\ & \quad X_0 = x, \end{aligned} \tag{3.2.1}$$

where $\mathcal{A}_I(x)$ is a set of admissible control processes such that

$$\mathbb{E} \left[\sum_{t=1}^{\infty} \delta^t \left(\left| \mathbb{E}[X_t | \mathcal{F}_{t-1}^I] \right| + \frac{\gamma_{t-1}^I}{2} \text{Var}(X_t | \mathcal{F}_{t-1}^I) \right) \right] < \infty.$$

In the following subsections, we study the above optimization problems under the three different investor's information levels.

3.2.1 Partial Information

First, we consider the partially informed investor's optimization problem. We define the following probabilities:

$$\begin{aligned} \pi_t(i) &= \mathbb{P}(S_t = i | \mathcal{F}_t^R), \quad t \geq 0, \quad i \in \mathcal{D}, \\ p_t(i) &= \mathbb{P}(S_{t+1} = i | \mathcal{F}_t^R), \quad t \geq 0, \quad i \in \mathcal{D}. \end{aligned}$$

$\pi_t(i)$ means the probability of $S_t = i$ conditioned by the partially informed investor's information at time t . On the other hand, $p_t(i)$ represents the probability of $S_{t+1} = i$

conditioned by the investor's information at time t . Let $\mathcal{D}_\pi$ be a set of K dimensional probability vectors such that

$$\mathcal{D}_\pi = \left\{ \pi \in [0, 1]^K, \left| \sum_{i=1}^K \pi(i) = 1 \right. \right\},$$

Then, the probability vectors $\pi_t = (\pi_t(1), \dots, \pi_t(K))' \in \mathbb{R}^K$

and $p_t = (p_t(1), \dots, p_t(K))' \in \mathbb{R}^K$ are in $\mathcal{D}_\pi$ for all t . Additionally, we specify the form of the trade-off parameter γ_t^R . We assume that γ_t^R has the following form at each time $t \geq 0$,

$$\gamma_t^R = \gamma^R(\pi_t),$$

where γ^R is a bounded measurable function from $\mathcal{D}_\pi$ to $\mathbb{R}_{++}$. It is clear that γ_t^R is $\mathcal{F}_t^R$ -measurable for all $t \geq 0$.

In Theorem 3.1, we study verification conditions of the Bellman equation for the dynamic optimization. Here, we first discuss the probability space of the mean-variance optimization. Since $(S_t)_{t=0}^\infty$ is the Markov chain, the transition probability matrix Q is a Markov kernel $\mathcal{Q}$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$, where $\mathcal{B}(\mathcal{D})$ is the Borel σ -algebra on $\mathcal{D}$. This means that for every $t \geq 1$,

$$\mathbb{P}(S_t = i \mid S_{t-1} = j) = \mathcal{Q}(i, j) = q_{j,i},$$

for all $i, j \in \mathcal{D}$, where $q_{j,i}$ is the $j \times i$ th element of Q . We also consider a probability measure μ_ϵ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\mu_\epsilon(B) = \mathbb{P}(\epsilon_1 \in B), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

By definition, μ_ϵ is the distribution of ϵ_1 . On a measurable space $(\mathbb{R} \times \mathcal{D} \times (\mathbb{R}^d)^\infty \times \mathcal{D}^\infty, \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{D}) \times \mathcal{B}((\mathbb{R}^d)^\infty \times \mathcal{D}^\infty))$, we introduce a probability measure conditioned by the initial condition $x \in \mathbb{R}$, $\pi \in \mathcal{D}_\pi$, denoted by $\mathbb{P}_{x,\pi}$, such that for all $B \in \mathcal{B}(\mathbb{R}^d)$ and $i \in \mathcal{D}$,

$$\mathbb{P}_{x,\pi}(\epsilon_t \in B) = \mu_\epsilon(B), \quad \mathbb{P}_{x,\pi}(S_t = i \mid S_{t-1}) = \mathcal{Q}(i, S_{t-1}), \quad t \geq 1,$$

$$\mathbb{P}_{x,\pi}(S_0 = i) = \pi(i), \quad \mathbb{P}_{x,\pi}(X_0 = x) = 1.$$

Furthermore, $X_0, (S_t)_{t=0}^\infty$, and $(\epsilon_t)_{t=1}^\infty$ are mutually independent under $\mathbb{P}_{x,\pi}$. The Ionescu-Tulcea theorem (cf. Theorem IV.4.7 in Çinlar (2011)) guarantees the existence of the

above probability measure $\mathbb{P}_{x,\pi}$ for all $x \in \mathbb{R}$, $\pi \in \mathcal{D}_\pi$. $\mathbb{P}_{x,\pi}$ represents the probability distribution of $(S_t)_{t=0}^\infty$ and $(\epsilon_t)_{t=1}^\infty$ given the initial condition that $X_0 = x$ and $\mathbb{P}(S_0 = i) = \pi(i)$, $i \in \mathcal{D}$. We have considered the abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. However, hereafter, we consider the above probability space $(\mathbb{R} \times \mathcal{D} \times (\mathbb{R}^d)^\infty \times \mathcal{D}^\infty, \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{D}) \times \mathcal{B}((\mathbb{R}^d)^\infty \times \mathcal{D}^\infty), \mathbb{P}_{x,\pi})$. For convenience of the notations, we write the measurable space $(\mathbb{R} \times \mathcal{D} \times (\mathbb{R}^d)^\infty \times \mathcal{D}^\infty, \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{D}) \times \mathcal{B}((\mathbb{R}^d)^\infty \times \mathcal{D}^\infty))$ as $(\Omega, \mathcal{F})$. The investors' information σ -algebras can be defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{x,\pi})$ again. We denote by $E_{x,\pi}$ the expectation operator of $\mathbb{P}_{x,\pi}$.

For any control process u , it follows that

$$\begin{aligned} E_{x,\pi}[X_t | \mathcal{F}_{t-1}^R] &= (1 + r_f)X_{t-1} + \left(\sum_{i=1}^K \mu^e(i) p_{t-1}(i) \right)' u_{t-1}, \\ \mu^e(i) &= \mu(i) - r_f \mathbf{1}_d, \\ \text{Var}_{x,\pi}(X_t | \mathcal{F}_{t-1}^R) &= u_{t-1}' \left(\sum_{i=1}^K \left(\Sigma(i) + \mu^e(i)(\mu^e(i))' \right) p_{t-1}(i) \right) u_{t-1} \\ &\quad - u_{t-1}' \left(\sum_{i=1}^K \mu^e(i) p_{t-1}(i) \right) \left(\sum_{i=1}^K \mu^e(i) p_{t-1}(i) \right)' u_{t-1}, \\ \Sigma(i) &= \sigma(i)(\sigma(i))', \end{aligned}$$

for all $t \geq 0$. By the definition, $p_t = Q' \pi_t$ holds for all $t \geq 0$. Therefore, p_t is a linear function of π_t . We define a function $p : \mathcal{D}_\pi \rightarrow \mathcal{D}_\pi$ such that

$$p(\pi) := (p(1, \pi), \dots, p(K, \pi))' = Q' \pi, \quad \pi \in \mathcal{D}_\pi \quad (3.2.2)$$

Then, it follows that $p_t = p(\pi_t) = (p(1, \pi_t), \dots, p(K, \pi_t))' \in \mathcal{D}_\pi$. Furthermore, we introduce the following functions on $\mathcal{D}_\pi$,

$$\begin{aligned} m^e(\pi) &:= \sum_{i=1}^K \mu^e(i) p(i, \pi), \\ A(\pi) &:= \sum_{i=1}^K \left(\Sigma(i) + \mu^e(i)(\mu^e(i))' \right) p(i, \pi) - \left(\sum_{i=1}^K \mu^e(i) p(i, \pi) \right) \left(\sum_{i=1}^K \mu^e(i) p(i, \pi) \right)'. \end{aligned}$$

Then, the conditional expectation and variance of X can be expressed as

$$\begin{aligned} E_{x,\pi}[X_t | \mathcal{F}_{t-1}^R] &= (1 + r_f)X_{t-1} + (m^e(\pi_{t-1}))' u_{t-1}, \\ \text{Var}_{x,\pi}(X_t | \mathcal{F}_{t-1}^R) &= u_{t-1}' A(\pi_{t-1}) u_{t-1}. \end{aligned}$$

Let $\pi^\pi = (\pi_t^\pi)_{t=0}^\infty$ be a $(\pi_t)_{t=0}^\infty$ starting from $\pi_0 = \pi \in \mathcal{D}_\pi$, and let $X^{x,u} = (X_t^{x,u})_{t=0}^\infty$ be a wealth process, starting from x , driven by a control process u . We need to rewrite the generic optimization problem (3.2.1) as a specific optimization problem of the partially informed investor. The partially informed investor's optimization problem with the initial wealth x and probability π is

$$\begin{aligned} V_{par}(x, \pi) = & \sup_{u \in \mathcal{A}_{par}(x, \pi)} \mathbb{E}_{x, \pi} \left[\sum_{t=1}^{\infty} \delta^t \left(\mathbb{E}_{x, \pi} [X_t^{x,u} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x, \pi}(X_t^{x,u} | \mathcal{F}_{t-1}^R) \right) \right] \\ \text{subject to } & (1 + r_f)X_{t-1}^{x,u} + (R_t^e)'u_{t-1} = X_t^{x,u}, \quad t \geq 1, \\ & X_0^{x,u} = x, \\ & \pi_0^\pi = \pi, \end{aligned} \quad (3.2.3)$$

where $\mathcal{A}_{par}(x, \pi)$ is a set of admissible control processes such that any $u \in \mathcal{A}_{par}(x, \pi)$ satisfies

$$\mathbb{E}_{x, \pi} \left[\sum_{t=1}^{\infty} \delta^t \left(\left| \mathbb{E}_{x, \pi} [X_t^{x,u} | \mathcal{F}_{t-1}^R] \right| + \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x, \pi}(X_t^{x,u} | \mathcal{F}_{t-1}^R) \right) \right] < \infty,$$

and

$$\lim_{T \rightarrow \infty} \delta^T \mathbb{E}_{x, \pi} [|X_T^{x,u}|] = 0.$$

Theorem 3.1 (verification theorem and time-consistency) Suppose that U_{par} is a solution of the following functional equation,

$$\begin{aligned} U_{par}(x, \pi) = & \max_u \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x, \pi} [U_{par}(X, \pi_1^\pi)] \right\} \\ \text{subject to } & X = (1 + r_f)x + (R_1^e)'u, \end{aligned} \quad (3.2.4)$$

Let u be a mapping from $\mathbb{R} \times \mathcal{D}_\pi$ onto $\mathbb{R}^d$ such that

$$\begin{aligned} u(x, \pi) \in & \arg \max_u \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x, \pi} [U_{par}(X, \pi_1^\pi)] \right\} \\ \text{subject to } & X = (1 + r_f)x + (R_1^e)'u, \end{aligned}$$

and suppose that u is a measurable mapping. Moreover, we suppose that for all $(x, \pi) \in \mathbb{R}_+ \times \mathcal{D}_\pi$, $|U_{par}(x, \pi)| \leq C(1 + |x|)$ and $\|u(x, \pi)\|^2 \leq C$ for some positive constant C and that $\delta(1 + r_f) < 1$. Then, $\delta U_{par} = V_{par}$. In addition, for all $(x, \pi) \in \mathbb{R}_+ \times \mathcal{D}_\pi$, if a wealth process $X^{x, \pi, u} = (X_t^{x, \pi, u})_{t=0}^\infty$ is defined as

$$X_t^{x, \pi, u} = (1 + r_f)X_{t-1}^{x, \pi, u} + (R_t^e)'u(X_{t-1}^{x, \pi, u}, \pi_{t-1}^\pi), \quad t \geq 1, \quad X_0^{x, \pi, u} = x,$$

then $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$ is a time-consistent optimal Markov control of the problem (3.2.3) at (x, π) .

Proof of Theorem 3.1. Fix any admissible control $u = (u_t)_{t=0}^\infty$. Since u is $\mathcal{F}_t^R$ -adapted, there exists a sequence of measurable functions $(f_t)_{t=0}^\infty$ such that

$$u_t = f_t(x, \pi_0, R_1, \dots, R_t), \quad t \geq 1,$$

and $u_0 = f_0(x, \pi_0)$. Moreover, since the sequence of probability vectors $(\pi_t)_{t=0}^\infty$ is a Markov process, there exists a measurable function $g : \mathbb{R}^d \times \mathcal{D}_\pi \rightarrow \mathcal{D}_\pi$ such that

$$\pi_t = g(R_t, \pi_{t-1}), \quad t \geq 1.$$

The investor's wealth process $X^{x,u}$ starting from $x \in \mathbb{R}$ and driven by u satisfies the following update formula,

$$X_t^{x,u} = (1 + r_f)X_{t-1}^{x,u} + (R_t^e)'u_{t-1}, \quad t \geq 1.$$

It follows that the value of $X_t^{x,u}$ is determined by the values of $X_{t-1}^{x,u}$, R_t^e , and u_{t-1} for all $t \geq 1$. Furthermore, u_{t-1} and $X_{t-1}^{x,u}$ are $\mathcal{F}_{t-1}^R$ measurable and the distribution of (ϵ_t, S_t) given by $\mathcal{F}_{t-1}^R$ is the same as that of (ϵ_t, S_t) given by π_{t-1}^π . This means that for any Borel sets $B \in \mathcal{B}(\mathbb{R})$, and $D \in \mathcal{B}(\mathcal{D}_\pi)$,

$$\begin{aligned} & \mathbb{P}_{x,\pi}(X_t^{x,u} \in B, \pi_t^\pi \in D \mid \mathcal{F}_{t-1}^R) \\ &= \mathbb{P}_{x,\pi}(X_t^{x,u} \in B, g(R_t, \pi_{t-1}^\pi) \in D \mid u_{t-1} = f_{t-1}(x, \pi_0, R_1, \dots, R_{t-1}), X_{t-1}^{x,u}, \pi_{t-1}^\pi) \\ &= \mathbb{P}_{X_{t-1}^{x,u}, \pi_{t-1}^\pi}(X_1^{x,u} \in B, g(R_1, \pi_{t-1}^\pi) \in D \mid u_1 = f_{t-1}(x, \pi_0, R_1, \dots, R_{t-1})). \end{aligned}$$

Therefore, by the definition of U_{par} , the following inequality holds.

$$\begin{aligned} & \delta^t \mathbb{E}_{x,\pi} \left[U_{par}(X_t^{x,u}, \pi_t^\pi) \mid \mathcal{F}_{t-1}^R \right] \\ &= \delta^t \mathbb{E}_{X_{t-1}^{x,u}, \pi_{t-1}^\pi} \left[U_{par}(X_1^{x,u}, g(R_1, \pi_{t-1}^\pi)) \mid u_1 = f_{t-1}(x, \pi_0, R_1, \dots, R_{t-1}) \right] \\ &\leq \delta^{t-1} \left(U_{par}(X_{t-1}^{x,u}, \pi_{t-1}^\pi) - (1 + r_f)X_{t-1}^{x,u} - (m^e(\pi_{t-1}^\pi))'u_{t-1} + \frac{\gamma^R(\pi_{t-1}^\pi)}{2} u_{t-1}' A(\pi_{t-1}^\pi) u_{t-1} \right) \\ &= \delta^{t-1} \left(U_{par}(X_{t-1}^{x,u}, \pi_{t-1}^\pi) - \mathbb{E}_{x,\pi}[X_t^{x,u} \mid \mathcal{F}_{t-1}^R] + \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi}(X_t^{x,u} \mid \mathcal{F}_{t-1}^R) \right) \end{aligned}$$

for all $t \geq 1$, where we have used $u_{t-1} = f_{t-1}(x_0, \pi_0, R_1, \dots, R_{t-1})$. Summing up the above inequality up to $T \geq 1$ and taking the expectation $E_{t,x}$, we obtain

$$U_{par}(x, \pi) \geq E_{t,x} \left[\sum_{t=1}^{T-1} \delta^{t-1} \left(E_{x,\pi} [X_t^{x,u} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} (X_t^{x,u} | \mathcal{F}_{t-1}^R) \right) \right] + \delta^T E_{x,\pi} [U_{par}(X_T^{x,u}, \pi_T^\pi)]. \quad (3.2.5)$$

Since u is the admissible control, the dominated convergence theorem leads to

$$\begin{aligned} & \lim_{T \rightarrow \infty} E_{x,\pi} \left[\sum_{t=1}^{T-1} \delta^{t-1} \left(E_{x,\pi} [X_t^{x,\pi} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} (X_t^{x,\pi} | \mathcal{F}_{t-1}^R) \right) \right] \\ &= E_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^{t-1} \left(E_{x,\pi} [X_t^{x,\pi} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} (X_t^{x,\pi} | \mathcal{F}_{t-1}^R) \right) \right]. \end{aligned}$$

By the linear growth condition of U_{par} , we have

$$\delta^T E_{x,\pi} [U_{par}(X_T^{x,u}, \pi_T^\pi)] \leq \delta^T C \left(1 + E_{x,\pi} [|X_T^{x,u}|] \right) \rightarrow 0.$$

when T goes to infinity. Therefore, taking the limit in (3.2.5), we have

$$U_{par}(x, \pi) \geq E_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^{t-1} \left(E_{x,\pi} [X_t^{x,\pi} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} (X_t^{x,\pi} | \mathcal{F}_{t-1}^R) \right) \right].$$

This implies that $\delta U_{par}(x, \pi) \geq V_{par}(x, \pi)$ for all (x, π) .

Let $X^{x,\pi,u} = (X_t^{x,\pi,u})_{t=0}^\infty$ be a wealth process, starting from x , whose update formula is

$$X_t^{x,\pi,u} = (1 + r_f)X_{t-1}^{x,\pi,u} + (R_t^e)'u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi), \quad t \geq 1.$$

Then, from the inequality (3.2.5), the following equality holds,

$$\begin{aligned} U_{par}(x, \pi) &= E_{x,\pi} \left[\sum_{t=1}^{T-1} \delta^{t-1} \left(E_{x,\pi} [X_t^{x,\pi,u} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} (X_t^{x,\pi,u} | \mathcal{F}_{t-1}^R) \right) \right] \\ &\quad + \delta^T E_{x,\pi} [U_{par}(X_T^{x,\pi,u}, \pi_T^\pi)]. \end{aligned}$$

Therefore, it suffices to show the admissibility of $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$. For all $T \geq 1$, we have

$$\begin{aligned} X_T^{x,\pi,u} &= (1 + r_f)X_{T-1}^{x,\pi,u} + (R_T^e)'u(X_{T-1}^{x,\pi,u}, \pi_{T-1}^\pi) \\ &= (1 + r_f)^2 X_{T-2}^{x,\pi,u} + (R_T^e)'u(X_{T-1}^{x,\pi,u}, \pi_{T-1}^\pi) + (1 + r_f)(R_{T-1}^e)'u(X_{T-2}^{x,\pi,u}, \pi_{T-2}^\pi) \\ &= \dots \\ &= \sum_{t=1}^T (1 + r_f)^{T-t} (R_t^e)'u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi) + (1 + r_f)^T x. \end{aligned}$$

By the bounded condition on u , there exists $C_u \geq 0$ such that

$$\mathbb{E} \left[\left| (R_t^e)' u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi) \right| \right] < C_u,$$

for all $t \geq 1$. Therefore, we have

$$\begin{aligned} \delta^T E \left[\left| X_T^{x,\pi,u} \right| \right] &\leq \delta^T C_u \sum_{t=1}^T (1 + r_f)^{T-t} + \delta^T (1 + r_f)^T x \\ &= C_u \frac{\delta^T - \delta^T (1 + r_f)^T}{r_f} + \delta^T (1 + r_f)^T x \rightarrow 0, \end{aligned}$$

as $T \rightarrow \infty$ since $\delta(1 + r_f) < 1$. It can be easily shown that

$$\mathbb{E}_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^t (1 + r_f) |X_{t-1}^{x,\pi,u}| \right] < \infty.$$

Moreover, by the bounded condition on u , there exists a positive constants C such that

$$\begin{aligned} \left| \mathbb{E}_{x,\pi} \left[X_t^{x,\pi,u} \middle| \mathcal{F}_{t-1}^R \right] \right| &\leq (1 + r_f) |X_{t-1}^{x,\pi,u}| + |(m^e(\pi_{t-1}^\pi))' u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi)| \\ &\leq (1 + r_f) |X_{t-1}^{x,\pi,u}| + C, \\ \text{Var}_{x,\pi} \left(X_t^{x,\pi,u} \middle| \mathcal{F}_{t-1}^R \right) &= (u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi))' A(\pi_{t-1}^\pi) u(X_{t-1}^{x,\pi,u}, \pi_{t-1}^\pi) \leq C, \end{aligned}$$

for all $t \geq 1$. Hence we conclude that

$$\mathbb{E}_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^t \left(\left| \mathbb{E}_{x,\pi} [X_t^{x,\pi,u} | \mathcal{F}_{t-1}^R] \right| + \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi}(X_t^{x,\pi,u} | \mathcal{F}_{t-1}^R) \right) \right] < \infty.$$

These inequalities imply that $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$ is admissible. Therefore, it follows that

$$U_{par}(x, \pi) = \mathbb{E}_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^{t-1} \left(\mathbb{E}_{x,\pi} \left[X_t^{x,\pi,u} \middle| \mathcal{F}_{t-1}^R \right] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi} \left(X_t^{x,\pi,u} \middle| \mathcal{F}_{t-1}^R \right) \right) \right].$$

Then $\delta U_{par}(x, \pi) = V_{par}(x, \pi)$ for all $x \in \mathbb{R}$, $\pi \in \mathcal{D}_\pi$, and $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$ is an optimal Markov control. The time consistency of $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$ is obvious since $(u(X_t^{x,\pi,u}, \pi_t^\pi))_{t=0}^\infty$ is a Markov control. $\square$

The assumptions in Theorem 3.1 are the boundedness of the optimal Markov policy function and the linear growth assumption of the solution of the equation (3.2.4) with respect to the wealth x . These assumptions play an important role in the proof of Theorem 3.1.

Next, we will find the solution of (3.2.4).

Proposition 3.2 Let $a(\pi) = (m^e(\pi))'(A(\pi))^{-1}m^e(\pi)$. Then, the value function is

$$V_{par}(x, \pi) = \frac{\delta}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x, \pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_t^\pi)}{\gamma^R(\pi_t^\pi)} \right] + \frac{\delta(1 + r_f)}{1 - \delta(1 + r_f)} x, \quad (3.2.6)$$

and the optimal Markov control is

$$u_{par}(\pi) = \frac{1}{\gamma^R(\pi)(1 - \delta(1 + r_f))} (A(\pi))^{-1} m^e(\pi).$$

Proof of Proposition 3.2. Suppose that the solution of the equation (3.2.4) has the following form,

$$U^*(x, \pi) = \alpha(\pi) + \beta(\pi)x, \quad x \in \mathbb{R}, \pi \in \mathcal{D}_\pi,$$

where α and β are measurable functions on $\mathcal{D}_\pi$. For all $x \in \mathbb{R}$, $\pi \in \mathcal{D}_\pi$ and $u \in \mathbb{R}^d$, it holds that

$$\begin{aligned} & \delta \mathbb{E}_{x, \pi} [U^*((1 + r_f)x + (R_1^e)'u, \pi_1^\pi)] \\ &= \delta \left(\mathbb{E}_{x, \pi} [\alpha(\pi_1^\pi)] + \mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)](1 + r_f)x + (\mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)R_1^e])'u \right). \end{aligned}$$

The optimization problem of the right hand side of the equation (3.2.4) can be written as

$$\begin{aligned} & \max_u \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x, \pi} [U^*(X, \pi_1^\pi)] \right\} \\ &= \max_u \left\{ \left(1 + \delta \mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)] \right) (1 + r_f)x + \left(m^e(\pi) + \delta \mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)R_1^e] \right)'u \right. \\ & \quad \left. - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x, \pi} [\alpha(\pi_1^\pi)] \right\}. \end{aligned}$$

The objective function of the above optimization problem is concave in u , so there exists the unique solution u^* that satisfies the following first order condition,

$$m^e(\pi) + \delta \mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)R_1^e] - \gamma^R(\pi)A(\pi)u^* = \mathbf{0}_d,$$

where $\mathbf{0}_d$ is a zero vector of $\mathbb{R}^d$. Solving this for u^* , we obtain the optimal policy.

$$u^*(\pi) := u^* = \frac{1}{\gamma^R(\pi)} (A(\pi))^{-1} \left(m^e(\pi) + \delta \mathbb{E}_{x, \pi} [\beta(\pi_1^\pi)R_1^e] \right).$$

Then, $u^*(\pi)$ is independent of the value of wealth x . Substituting u^* into the objective in the right hand side of the equation (3.2.4), we have

$$\begin{aligned} \alpha(\pi) + \beta(\pi)x = & \left(m^e(\pi) + \delta \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi) R_1^e] \right)' u^*(\pi) - \frac{\gamma^R(\pi)}{2} (u^*(\pi))' A(\pi) u^*(\pi) \\ & + \left(1 + \delta \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)] \right) (1 + r_f)x + \delta \mathbb{E}_{x,\pi}[\alpha(\pi_1^\pi)]. \end{aligned} \quad (3.2.7)$$

Since the equation (3.2.7) holds for all $x \in \mathbb{R}$ and $\pi \in \mathcal{D}_\pi$, α and β satisfy the following equations,

$$\alpha(\pi) = \left(m^e(\pi) + \delta \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi) R_1^e] \right)' u^*(\pi) - \frac{\gamma^R(\pi)}{2} (u^*(\pi))' A(\pi) u^*(\pi) + \delta \mathbb{E}_{x,\pi}[\alpha(\pi_1^\pi)] \quad (3.2.8)$$

$$\beta(\pi) = \left(1 + \delta \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)] \right) (1 + r_f) \quad (3.2.9)$$

Assume that β is a constant function. Then, we can solve the equation (3.2.9) and

$$\beta = \frac{1 + r_f}{1 - \delta(1 + r_f)} \quad (3.2.10)$$

Substituting (3.2.10) into the equation (3.2.8) and the optimal policy $u^*(\pi)$ give us

$$\begin{aligned} u^*(\pi) &= \frac{1}{\gamma^R(\pi)(1 - \delta(1 + r_f))} (A(\pi))^{-1} m^e(\pi), \\ \alpha(\pi) &= \frac{1}{2(1 - \delta(1 + r_f))^2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha(\pi_1^\pi)], \end{aligned} \quad (3.2.11)$$

where $a(\pi) = (m^e(\pi))'(A(\pi))^{-1} m^e(\pi)$. Since m^e and A are continuous functions on a compact set $\mathcal{D}_\pi$, a and u^* are bounded on $\mathcal{D}_\pi$. Therefore, u^* satisfies the bounded condition of the optimal policy in Theorem 3.1. Suppose that α has the following form,

$$\alpha(\pi) = \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_t^\pi)}{\gamma^R(\pi_t^\pi)} \right]. \quad (3.2.12)$$

Then, α is a positive bounded function, since $a(\pi)/\gamma^R(\pi)$ is positive and bounded. Moreover, α defined in (3.2.12) satisfies the equation (3.2.11). Indeed, α defined in

(3.2.12) can be expressed as

$$\begin{aligned}
\alpha(\pi) &= \frac{a(\pi)}{2\gamma^R(\pi)(1 - \delta(1 + r_f))^2} + \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{t=1}^{\infty} \delta^t \frac{a(\pi_t^\pi)}{\gamma^R(\pi_t^\pi)} \right] \\
&= \frac{a(\pi)}{2\gamma^R(\pi)(1 - \delta(1 + r_f))^2} + \delta \mathbb{E}_{x,\pi} \left[\frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_{t+1}^\pi)}{\gamma^R(\pi_{t+1}^\pi)} \mid \mathcal{F}_1^R \right] \right] \\
&= \frac{a(\pi)}{2\gamma^R(\pi)(1 - \delta(1 + r_f))^2} + \delta \mathbb{E}_{x,\pi} \left[\frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{X_1^{x,\pi}, \pi_1^\pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_t^{\pi_1^\pi})}{\gamma^R(\pi_t^{\pi_1^\pi})} \right] \right] \\
&= \frac{1}{2(1 - \delta(1 + r_f))^2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi} [\alpha(\pi_1^\pi)],
\end{aligned}$$

where we have used the Markov property of $(\pi_t)_{t=1}^\infty$. Hence, a solution of Bellman equation (3.2.4) is

$$U^*(x, \pi) = \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_t^\pi)}{\gamma^R(\pi_t^\pi)} \right] + \frac{1 + r_f}{1 - \delta(1 + r_f)} x.$$

By the boundedness of α , U^* satisfies the linear growth assumption. Since U^* and u^* satisfy the assumptions of Theorem 3.1, we conclude that the value function is

$$\begin{aligned}
V_{par}(x, \pi) &= \delta U^*(x, \pi) \\
&= \frac{\delta}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\pi_t^\pi)}{\gamma^R(\pi_t^\pi)} \right] + \frac{\delta(1 + r_f)}{1 - \delta(1 + r_f)} x, \quad x \in \mathbb{R}, \pi \in \mathcal{D}_\pi,
\end{aligned}$$

and the optimal Markov control is

$$u_{par}(\pi) = u^*(\pi) = \frac{1}{\gamma^R(\pi)(1 - \delta(1 + r_f))} (A(\pi))^{-1} m^e(\pi), \quad \pi \in \mathcal{D}_\pi.$$

□

The readers may be concerned with uniqueness of the solution of the equation (3.2.4). We need additional assumptions to prove the uniqueness. Let $C_0(A)$ be a set of continuous functions from A onto $\mathbb{R}$. We assume that

1. $\gamma^R(\pi)$ is continuous on $\mathcal{D}_\pi$,
2. $\sigma(i)$ is invertible for all $i \in \mathcal{D}$,
3. ϵ_t has a probability density function f_ϵ .

Then, the equation (3.2.4) has the unique solution if it has the following form,

$$U(x, \pi) = \alpha(\pi) + \beta(\pi)x, \quad x \in \mathbb{R}, \quad \pi \in \mathcal{D}_\pi, \quad (3.2.13)$$

where α and β are continuous functions on $\mathcal{D}_\pi$. If U solves the equation (3.2.4), α and β satisfy the equations (3.2.8) and (3.2.9), respectively. Let T_β be a functional operator such that

$$T_\beta \beta(\pi) = \left(1 + \delta \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)]\right)(1 + r_f).$$

Then, T_β is a contraction mapping from $C_0(\mathcal{D}_\pi)$ onto $C_0(\mathcal{D}_\pi)$. It is obvious that T_β has the contraction property, since $\delta < 1$ and T_β is monotone in the sense that $T_\beta f \leq T_\beta f'$ if $f \leq f'$. By the additional assumptions 2 and 3, π_1^π can be expressed as

$$\pi_1^\pi = \left(\frac{f_\epsilon((\sigma(1))^{-1}(R_1^e - \mu^e(1)))p(1, \pi)}{\sum_{i=1}^K f_\epsilon((\sigma(i))^{-1}(R_1^e - \mu^e(i)))p(i, \pi)}, \dots, \frac{f_\epsilon((\sigma(K))^{-1}(R_1^e - \mu^e(K)))p(K, \pi)}{\sum_{i=1}^K f_\epsilon((\sigma(i))^{-1}(R_1^e - \mu^e(i)))p(i, \pi)} \right)' \in \mathcal{D}_\pi$$

Therefore, π_1^π is continuous in $\pi \in \mathcal{D}_\pi$ for all outcomes of R_1^e since $p(i, \pi)$, $i = 1, \dots, K$ is also continuous in $\pi \in \mathcal{D}_\pi$. The expectation $\mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)]$ can be expressed as

$$\mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)] = \sum_{i=1}^K \pi(i) \mathbb{E}_{x,\pi}[\beta(\pi_1^\pi) \mid S_1 = i].$$

By the bounded convergence theorem and the continuity of π_1^π , $\mathbb{E}_{x,\pi}[\beta(\pi_1^\pi) \mid S_1 = i]$ is continuous in $\pi \in \mathcal{D}_\pi$ for all $i \in \mathcal{D}$, and therefore $\mathbb{E}_{x,\pi}[\beta(\pi_1^\pi)]$ is also continuous in $\pi \in \mathcal{D}_\pi$. This yields the continuity of $T_\beta \beta$ on $\mathcal{D}_\pi$. Then, by the Banach fixed point theorem, T_β has a unique fixed point on $C_0(\mathcal{D}_\pi)$. It is (3.2.10). We have seen that substituting β in (3.2.10) into the equation (3.2.9) leads to the equation (3.2.11). Let T_α be a functional operator such that

$$T_\alpha \alpha(\pi) = \frac{1}{2(1 - \delta(1 + r_f))^2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha(\pi_1^\pi)].$$

Using the similar manner as in the proof for T_β , we can prove that T_α is a contraction mapping from $C_0(\mathcal{D}_\pi)$ onto $C_0(\mathcal{D}_\pi)$. We need the continuity of γ^R for this proof and it is guaranteed by the additional assumption 1. Then, T_α has a unique fixed point on $C_0(\mathcal{D}_\pi)$, and it is given by (3.2.12). Since α and β are uniquely determined in the space of functions $C_0(\mathcal{D}_\pi)$, the solution of the equation (3.2.4) is also uniquely determined if it has the form represented in (3.2.13).

However, it is possible that the uniqueness on a set of more general functions does not hold. By the well-known dynamic programming approach (cf. Section 4 in Hernández-Lerma and Lasserre (1996)), the standard dynamic optimization problem has a unique bounded value function if its reward function is bounded. But, in our settings, the reward function,

$$(1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2}u'A(\pi)u,$$

is unbounded, so the standard approach can not be applied to our problem. However, our value function (3.2.6) can be characterized by a limit of value iteration of a finite horizon problem. We consider the following finite horizon problem.

$$\begin{aligned} V_{par,T}(x, \pi) = \sup_{u \in (\mathbb{R}^d)^T} E_{x,\pi} \left[\sum_{t=1}^T \delta^t \left(E_{x,\pi}[X_t^{x,u} | \mathcal{F}_{t-1}^R] - \frac{\gamma^R(\pi_{t-1}^\pi)}{2} \text{Var}_{x,\pi}(X_t^{x,u} | \mathcal{F}_{t-1}^R) \right) \right] \\ \text{subject to} \quad (1 + r_f)X_{t-1}^{x,u} + (R_t^e)'u_{t-1} = X_t^{x,u}, \quad t = 1, \dots, T, \\ X_0^{x,u} = x, \\ \pi_0^\pi = \pi, \end{aligned} \tag{3.2.14}$$

The problem (3.2.14) is a discounted version of the one in Chen et al. (2014), therefore we can apply the backward induction method to (3.2.14) and conclude that its optimal policy is time consistent.

Proposition 3.3 The value function of (3.2.14) can be expressed as

$$V_{par,T}(x, \pi) = \delta \left(\alpha_T(\pi) + \beta_T x \right),$$

where

$$\begin{aligned} \alpha_t(\pi) &= \frac{(1 + \delta\beta_{t-1})^2}{2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta E_{x,\pi}[\alpha_{t-1}(\pi_1^\pi)], \quad t = 1, \dots, T, \\ \beta_t &= (1 + \delta\beta_{t-1})(1 + r_f), \quad t = 1, \dots, T, \\ \alpha_0(\pi) &= \beta_0 = 0. \end{aligned}$$

The optimal policy in the problem (3.2.14) is

$$u_t(\pi) = \frac{1 + \delta\beta_{T-(t+1)}}{\gamma^R(\pi)} (A(\pi))^{-1} m^e(\pi), \quad t = 0, 1, \dots, T-1.$$

Proof of Proposition 3.3. Let $(U_t^T)_{t=0}^{T-1}$ be a sequence of value functions, where U_t^T solves the temporal optimization problem at time t . Then, $V_{par,T} = \delta U_0^T$ holds. At time $T-1$, consider the following optimization problem,

$$U_{T-1}^T(x, \pi) = \max_{u \in \mathbb{R}^d} \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u \right\}.$$

The above maximization problem is a concave maximization problem, so it has a unique solution. The optimal policy of the above problem is

$$u_{T-1}^*(\pi) = \frac{1}{\gamma^R(\pi)} (A(\pi))^{-1} m^e(\pi) = \frac{1 + \delta\beta_0}{\gamma^R(\pi)} (A(\pi))^{-1} m^e(\pi),$$

and the value function is

$$U_{T-1}^T(x, \pi) = \frac{1}{2} \frac{a(\pi)}{\gamma^R(\pi)} + (1 + r_f)x = \alpha_1(\pi) + \beta_1 x.$$

We assume that at some time $t+1$, U_{t+1}^T has the following functional form,

$$U_{t+1}^T(x, \pi) = \alpha_{T-(t+1)}(\pi) + \beta_{T-(t+1)} x.$$

Then, at time t , it holds that

$$\begin{aligned} U_t^T(x, \pi) &= \max_{u \in \mathbb{R}^d} \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x,\pi}[U_{t+1}^T(X_1^{x,u}, \pi_1^\pi)] \right\} \\ &= \max_{u \in \mathbb{R}^d} \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u \right. \\ &\quad \left. + \delta \left(\mathbb{E}_{x,\pi}[\alpha_{T-(t+1)}(\pi_1^\pi)] + \beta_{T-(t+1)} \left((1 + r_f)x + (m^e(\pi))'u \right) \right) \right\} \\ &= \max_{u \in \mathbb{R}^d} \left\{ (1 + \delta\beta_{T-(t+1)})(1 + r_f)x + (1 + \delta\beta_{T-(t+1)})(m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u \right. \\ &\quad \left. + \delta \mathbb{E}_{x,\pi}[\alpha_{T-(t+1)}(\pi_1^\pi)] \right\} \end{aligned}$$

This is also a concave maximization problem. Its optimal policy is

$$u_t^*(\pi) = \frac{1 + \delta\beta_{T-(t+1)}}{\gamma^R(\pi)} (A(\pi))^{-1} m^e(\pi),$$

and the value function is

$$\begin{aligned}
U_t^T(x, \pi) &= (1 + \delta\beta_{T-(t+1)})(1 + r_f)x + (1 + \delta\beta_{T-(t+1)})(m^e(\pi))'u_t^*(\pi) \\
&\quad - \frac{\gamma^R(\pi)}{2}(u_t^*(\pi))'A(\pi)u_t^*(\pi) + \delta \mathbb{E}_{x,\pi}[\alpha_{T-(t+1)}(\pi_1^\pi)] \\
&= \beta_{T-t}x + (1 + \delta\beta_{T-(t+1)})^2 \frac{a(\pi)}{\gamma^R(\pi)} \\
&\quad - \frac{(1 + \delta\beta_{T-(t+1)})^2}{2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha_{T-(t+1)}(\pi_1^\pi)] \\
&= \frac{(1 + \delta\beta_{T-(t+1)})^2}{2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha_{T-(t+1)}(\pi_1^\pi)] + \beta_{T-t}x \\
&= \alpha_{T-t}(\pi) + \beta_{T-t}x.
\end{aligned}$$

Then, the mathematical induction leads to the following.

$$U_t^T(x, \pi) = \alpha_{T-t}(\pi) + \beta_{T-t}x,$$

for all $t = 0, 1, \dots, T-1$. This implies that $V_{par,T}(x, \pi) = \delta U_0^T(x, \pi) = \delta(\alpha_T(\pi) + \beta_T x)$ for all $x \in \mathbb{R}$ and $\pi \in \mathcal{D}_\pi$. $\square$

β_T in Proposition 3.3 has the following explicit form,

$$\beta_T = (1 + r_f) \sum_{t=0}^{T-1} \delta^t (1 + r_f)^t = (1 + r_f) \frac{1 - \delta^T (1 + r_f)^T}{1 - \delta(1 + r_f)}, \quad T = 1, 2, \dots$$

Therefore, we have

$$\beta_\infty := \lim_{T \rightarrow \infty} \beta_T = \frac{1 + r_f}{1 - \delta(1 + r_f)}.$$

α_t in Proposition 3.3 also has the following explicit form,

$$\alpha_t(\pi) = \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{s=0}^{t-1} (1 - \delta^{t-s} (1 + r_f)^{t-s})^2 \delta^s \frac{a(\pi_s^\pi)}{\gamma^R(\pi_s^\pi)} \right],$$

where we have used the Markov property of $(\pi_t^\pi)_{t=0}^\infty$. Let $(\alpha_t^*)_{t=0}^\infty$ be a sequence of real-valued functions on $\mathcal{D}_\pi$ such that

$$\alpha_t^*(\pi) = \frac{(1 + \delta\beta_\infty)^2}{2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha_{t-1}^*(\pi_1^\pi)] = \frac{1}{2(1 - \delta(1 + r_f))^2} \frac{a(\pi)}{\gamma^R(\pi)} + \delta \mathbb{E}_{x,\pi}[\alpha_{t-1}^*(\pi_1^\pi)],$$

for all $t \geq 1$ and $\alpha_0^*(\pi) = 0$. Then, α_t^* has an explicit form such that

$$\alpha_t^*(\pi) = \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,\pi} \left[\sum_{s=0}^{t-1} \delta^s \frac{a(\pi_s^\pi)}{\gamma^R(\pi_s^\pi)} \right].$$

By the additional assumption 1, $a(\pi)/\gamma^R(\pi)$ is bounded. Let M be a positive constant such that $\sup_{\pi \in \mathcal{D}_\pi} |a(\pi)/\gamma^R(\pi)| \leq M$. Then, for all $t \geq 1$, we have

$$\begin{aligned} & \sup_{\pi \in \mathcal{D}_\pi} |\alpha_t^*(\pi) - \alpha_t(\pi)| \\ & \leq \frac{M}{2(1 - \delta(1 + r_f)^2)} \left(2\delta^t(1 + r_f)^t \sum_{s=0}^{t-1} \frac{1}{(1 + r_f)^s} + \delta^{2t}(1 + r_f)^{2t} \sum_{s=0}^{t-1} \frac{1}{\delta^s(1 + r_f)^{2s}} \right) \\ & = \frac{M}{2(1 - \delta(1 + r_f)^2)} \left(2\delta^t \frac{(1 + r_f)^t - 1}{1 - (1 + r_f)^{-1}} + \delta^t \frac{\delta^t(1 + r_f)^{2t} - 1}{1 - \delta^{-1}(1 + r_f)^{-2}} \right) \\ & \rightarrow 0, \end{aligned}$$

when t goes to infinity. Furthermore, let

$$\alpha_\infty(\pi) = \frac{1}{2(1 - \delta(1 + r_f)^2)} \mathbb{E}_{x,\pi} \left[\sum_{s=0}^{\infty} \delta^s \frac{a(\pi_s^\pi)}{\gamma^R(\pi_s^\pi)} \right].$$

Then,

$$\sup_{\pi \in \mathcal{D}_\pi} |\alpha_\infty(\pi) - \alpha_t^*(\pi)| \leq \frac{M}{(1 - \delta(1 + r_f)^2)^2} \sum_{s=t}^{\infty} \delta^s \rightarrow 0,$$

when t goes to infinity. Hence, we have

$$\lim_{t \rightarrow \infty} \sup_{\pi \in \mathcal{D}_\pi} |\alpha_\infty(\pi) - \alpha_t(\pi)| = 0.$$

By Proposition 3.2, it holds that

$$V_{par}(x, \pi) = \lim_{T \rightarrow \infty} V_{par,T}(x, \pi) = \delta \left(\alpha_\infty(\pi) + \beta_\infty x \right).$$

By the above discussion, we obtain the following corollary.

Corollary 3.4 For any compact subset $C \subset \mathbb{R} \times \mathcal{D}_\pi$, it holds that

$$\lim_{T \rightarrow \infty} \sup_{(x,\pi) \in C} |V_{par}(x, \pi) - V_{par,T}(x, \pi)| = 0.$$

Let T_U be a functional operator such that

$$T_U U(x, \pi) = \max_u \left\{ (1 + r_f)x + (m^e(\pi))'u - \frac{\gamma^R(\pi)}{2} u' A(\pi) u + \delta \mathbb{E}_{x,\pi} [U(X_1^{x,u}, \pi_1^\pi)] \right\}.$$

Then, we have

$$V_{par,T+1}(x, \pi) = T_U V_{par,T}(x, \pi), \quad x \in \mathbb{R}, \pi \in \mathcal{D}_\pi,$$

for all T . By Corollary 3.4, we obtain

$$\lim_{T \rightarrow \infty} \sup_{(x, \pi) \in C} |V_{par}(x, \pi) - (T_U)^T V_{par,1}(x, \pi)| = 0,$$

for any compact subset $C \subset \mathbb{R} \times \mathcal{D}_\pi$. This implies that V_{par} can be characterized by the value iteration method.

We investigate the properties of the Markov optimal policy.

If $c(\pi) := \mathbf{1}'_d(A(\pi))^{-1}m^e(\pi) \neq 0$ for all $\pi \in \mathcal{D}_\pi$, then the optimal Markov control can be written as

$$u_{par}(\pi) = \lambda(\pi)\phi(\pi),$$

where

$$\lambda(\pi) = \frac{c(\pi)}{\gamma^R(\pi)(1 - \delta(1 + r_f))}, \quad \phi(\pi) = \frac{1}{c(\pi)}(A(\pi))^{-1}m^e(\pi).$$

ϕ is independent of the preference parameters $\gamma^R(\pi)$ and δ , moreover, $\mathbf{1}'_d\phi(\pi) = 1$ for all $\pi \in \mathcal{D}_\pi$. Thus, we can consider ϕ as a portfolio of the risky assets. Additionally, ϕ has the same form as the tangency portfolio of Merton (1972). Therefore, we can derive the following corollary.

Corollary 3.5 (separation theorem) Every investor chooses a proportional of a specific portfolio consisting of the risky assets as the optimal portfolio: When the filtered probability is $\pi \in \mathcal{D}_\pi$, the optimal portfolio of risky assets is

$$\begin{aligned} u_{par}(\pi) &= \lambda(\pi)\phi(\pi) \\ \lambda(\pi) &= \frac{c(\pi)}{\gamma^R(\pi)(1 - \delta(1 + r_f))}, \quad \phi(\pi) = \frac{1}{c(\pi)}(A(\pi))^{-1}m^e(\pi), \\ c(\pi) &= \mathbf{1}'_d(A(\pi))^{-1}m^e(\pi). \end{aligned}$$

When the investor invests in the optimal portfolio, the conditional expected return at the current wealth x (> 0) and probability π is

$$\mu^*(x, \pi) := E_{x, \pi} \left[\frac{X_t^{x,u} - X_{t-1}^{x,u}}{X_t^{x,u}} \middle| \mathcal{F}_{t-1}^R, X_{t-1}^{x,u} = x, \pi_{t-1}^\pi = \pi \right] = r_f + \frac{a(\pi)}{x\gamma^R(\pi)(1 - \delta(1 + r_f))},$$

and the conditional standard deviation is

$$\sigma^*(x, \pi) := \sqrt{\text{Var} \left(\frac{X_t^{x,u} - X_{t-1}^{x,u}}{X_t^{x,u}} \middle| \mathcal{F}_{t-1}^R, X_{t-1}^{x,u} = x, \pi_{t-1}^\pi = \pi \right)} = \frac{\sqrt{a(\pi)}}{x\gamma^R(\pi)(1 - \delta(1 + r_f))}.$$

Therefore,

$$\mu^*(x, \pi) = r_f + \sigma^*(x, \pi) \sqrt{a(\pi)},$$

for all $x \in \mathbb{R}_+$ and $\pi \in \mathcal{D}_\pi$. If x is small, σ^* is large. This means that the partially informed investor takes more risks to get large excess return when she possesses a small amount of wealth.

3.2.2 Perfect Information and Prediction

In the previous subsection, we considered the partial information case. We can use the same argument in the perfect information case as well. First, we assume that the trade-off parameter is

$$\gamma_t^{R,S} = \gamma^{R,S}(S_t), \quad \text{for all } t \geq 0,$$

where $\gamma^{R,S}$ is a bounded measurable function from $\mathcal{D}$ to $\mathbb{R}_{++}$. In the perfect information case, the prediction probability can be expressed as

$$p_t^{R,S}(i) := \mathbb{P}_{x,\pi}(S_{t+1} = i | \mathcal{F}_t^{R,S}) = \mathbb{P}_{x,\pi}(S_{t+1} = i | S_t), \quad i \in \mathcal{D}.$$

Therefore, the prediction probability at time t satisfies

$$p_t^{R,S} = (p_t^{R,S}(1), \dots, p_t^{R,S}(K))' = p(\iota(S_t)),$$

where the function p is defined in (3.2.2) and $\iota : \mathcal{D} \rightarrow \mathcal{D}_\pi$ is a vector-valued function such that the i th element of $\iota(i)$ is 1 and the other elements are 0.

Let $(S_t^i)_{t=0}^\infty$ be the state variable starting from $S_0 = i \in \mathcal{D}$. Then, the perfectly informed investor's optimization problem can be written as

$$\begin{aligned} V_{per}(x, i) = & \sup_{u \in \mathcal{A}_{per}(x, i)} \mathbb{E}_{x, \iota(i)} \left[\sum_{t=1}^{\infty} \delta^t \left(\mathbb{E}_{x, \iota(i)}[X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}] - \frac{\gamma^{R,S}(S_{t-1}^i)}{2} \text{Var}_{x, \iota(i)}(X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}) \right) \right] \\ \text{subject to } & (1 + r_f)X_{t-1}^{x,u} + (R_t^e)' u_{t-1} = X_t^{x,u}, \quad t \geq 1, \\ & X_0^{x,u} = x, \\ & S_0^i = i, \end{aligned} \tag{3.2.15}$$

where $\mathcal{A}_{per}(x, i)$ is a set of admissible control processes such that any $u \in \mathcal{A}_{per}(x, i)$ satisfies

$$\mathbb{E}_{x, \iota(i)} \left[\sum_{t=1}^{\infty} \delta^t \left(\left| \mathbb{E}_{x, \iota(i)}[X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}] \right| + \frac{\gamma^{R,S}(S_{t-1}^i)}{2} \text{Var}_{x, \iota(i)}(X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}) \right) \right] < \infty.$$

and

$$\lim_{T \rightarrow \infty} \delta^T E_{x, \iota(i)}[|X_T^{x,u}|] = 0.$$

The conditional expectation and variance of wealth $X_t^{x,u}$, given the information $\mathcal{F}_{t-1}^{R,S}$, are

$$\begin{aligned} E_{x, \iota(i)}[X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}] &= (1 + r_f)X_{t-1}^{x,u} + \left(\sum_{i=1}^K \mu^e(i) p(i, \iota(S_{t-1})) \right)' u_{t-1} \\ &= (1 + r_f)X_{t-1}^{x,u} + (m^e(\iota(S_{t-1})))' u_{t-1}, \\ \text{Var}_{x, \iota(i)}(X_t^{x,u} | \mathcal{F}_{t-1}^{R,S}) &= u_{t-1}' \left(\sum_{i=1}^K \left(\Sigma(i) + \mu^e(i)(\mu^e(i))' \right) p(i, \iota(S_{t-1})) \right) u_{t-1} \\ &\quad - u_{t-1}' \left(\sum_{i=1}^K \mu^e(i) p(i, \iota(S_{t-1})) \right) \left(\sum_{i=1}^K \mu^e(i) p(i, \iota(S_{t-1})) \right)' u_{t-1} \\ &= u_{t-1}' A(\iota(S_{t-1})) u_{t-1}. \end{aligned}$$

Consider the following functional equation,

$$\begin{aligned} U_{per}(x, i) &= \max_u \left\{ (1 + r_f)x + (m^e(\iota(i)))' u - \frac{\gamma^{R,S}(i)}{2} u' A(\iota(i)) u + \delta E_{x, \iota(i)} [U_{per}(X, S_1^i)] \right\} \\ \text{subject to } X &= (1 + r_f)x + (R_1^e)' u, \end{aligned} \quad (3.2.16)$$

In the same manner as in Theorem 3.1, we can prove that the solution to (3.2.16), denoted by U_{per}^* , satisfies $\delta U_{per}^* = V_{per}$ under appropriate assumptions, which are the linear growth assumption of U_{per}^* and the boundedness assumption of the optimal Markov policy. We can guess that the solution to (3.2.16) can be written as,

$$U_{per}^*(x, i) = \tilde{\alpha}(i) + \tilde{\beta}x, \quad (x, i) \in \mathbb{R} \times \mathcal{D},$$

where $\tilde{\alpha} : \mathcal{D} \rightarrow \mathbb{R}$ is a measurable function, and $\tilde{\beta}$ is a constant. Then, for any $u \in \mathbb{R}^d$, it follows that

$$E_{x, \iota(i)}[U_{per}^*(X, S_1^i)] = E_{x, \iota(i)}[\tilde{\alpha}(S_1^i)] + \tilde{\beta}((1 + r_f)x + (m^e(\iota(i)))' u).$$

The optimization problem in (3.2.16) becomes

$$\begin{aligned} \max_u \left\{ (1 + r_f)x + (m^e(\iota(i)))' u - \frac{\gamma^{R,S}(i)}{2} u' A(\iota(i)) u \right. \\ \left. + \delta (E_{x, \iota(i)}[\tilde{\alpha}(S_1^i)] + \tilde{\beta}((1 + r_f)x + (m^e(\iota(i)))' u)) \right\}. \end{aligned}$$

It is a concave maximization problem, therefore, its solution is

$$u_{per}(i) := \frac{1 + \delta\tilde{\beta}}{\gamma_{R,S}(i)} (A(\iota(i)))^{-1} m^e(\iota(i)).$$

Hence, the following equation holds.

$$U_{per}^*(x, i) = \delta \mathbb{E}_{x, \iota(i)}[\tilde{\alpha}(S_1^i)] + \frac{(1 + \delta\tilde{\beta})^2}{2\gamma_{R,S}(i)} a(\iota(i)) + (1 + \delta\tilde{\beta})(1 + r_f)x.$$

The above equation holds for all $(x, i) \in \mathbb{R} \times \mathcal{D}$ if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy

$$\begin{aligned} \tilde{\alpha}(i) &= \frac{(1 + \delta\tilde{\beta})^2}{2\gamma_{R,S}(i)} a(\iota(i)) + \delta \mathbb{E}_{x, \iota(i)}[\tilde{\alpha}(S_1^i)], \\ \tilde{\beta} &= (1 + \delta\tilde{\beta})(1 + r_f). \end{aligned}$$

Therefore, the followings hold:

$$\begin{aligned} \tilde{\alpha}(i) &= \frac{1}{(1 - \delta(1 + r_f))^2} \mathbb{E}_{x, \iota(i)} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\iota(S_t^i))}{\gamma_{R,S}(S_t^i)} \right], \\ \tilde{\beta} &= \frac{1 + r_f}{1 - \delta(1 + r_f)}. \end{aligned}$$

By the monotone convergence theorem, we have

$$\mathbb{E}_{x, \iota(i)} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\iota(S_t^i))}{\gamma_{R,S}(S_t^i)} \right] = \sum_{t=0}^{\infty} \delta^t \mathbb{E}_{x, \iota(i)} \left[\frac{a(\iota(S_t^i))}{\gamma_{R,S}(S_t^i)} \right].$$

Let $\tilde{\kappa}$ be a K -dimensional constant vector such that

$$\tilde{\kappa} = \begin{pmatrix} \frac{a(\iota(1))}{\gamma_{R,S}(1)} \\ \vdots \\ \frac{a(\iota(K))}{\gamma_{R,S}(K)} \end{pmatrix}.$$

Then, it holds that

$$\mathbb{E}_{x, \iota(i)} \left[\frac{a(\iota(S_t^i))}{\gamma_{R,S}(S_t^i)} \right] = (\iota(i))' Q^t \tilde{\kappa}$$

for all $t \geq 0$. Hence we have

$$\mathbb{E}_{x, \iota(i)} \left[\sum_{t=0}^{\infty} \delta^t \frac{a(\iota(S_t^i))}{\gamma_{R,S}(S_t^i)} \right] = \sum_{t=0}^{\infty} \delta^t (\iota(i))' Q^t \tilde{\kappa} = (\iota(i))' \left(\sum_{t=0}^{\infty} (\delta Q)^t \right) \tilde{\kappa} = (\iota(i))' (I_K - \delta Q)^{-1} \tilde{\kappa},$$

where I_K is a K -dimensional identity matrix. Finally, we conclude that

$$U_{per}^*(x, i) = \frac{(\iota(i))' (I_K - \delta Q)^{-1} \tilde{\kappa}}{2(1 - \delta(1 + r_f))^2} + \frac{1 + r_f}{1 - \delta(1 + r_f)} x,$$

and the optimal Markov control is

$$u_{per}(i) = \frac{1}{\gamma_{R,S}(i)(1 - \delta(1 + r_f))} (A(\iota(i)))^{-1} m^e(\iota(i)).$$

We summarize these results in the following proposition.

Proposition 3.6 Let $\tilde{\kappa}$ be a K -dimensional constant vector whose i th element is $a(\iota(i))/\gamma^{R,S}(i)$. Then, the value function of the problem (3.2.15) is

$$V_{per}(x, i) = \frac{\delta(\iota(i))'(I_K - \delta Q)^{-1}\tilde{\kappa}}{2(1 - \delta(1 + r_f))^2} + \frac{\delta(1 + r_f)}{1 - \delta(1 + r_f)}x,$$

and the optimal Markov control is

$$u_{per}(i) = \frac{1}{\gamma^{R,S}(i)(1 - \delta(1 + r_f))} (A(\iota(i)))^{-1} m^e(\iota(i)).$$

As in the partial information case, the separation result also holds in this case. The perfectly informed investor invests in the risk-free asset and in some portfolio of risky assets. The risky assets' portfolio is proportional to the following portfolio

$$\phi(i) = \frac{1}{c(\iota(i))} (A(\iota(i)))^{-1} m^e(\iota(i)), \quad c(\iota(i)) = \mathbf{1}'_d (A(\iota(i)))^{-1} m^e(\iota(i)),$$

if $c(\iota(i)) \neq 0$.

Next, we consider the prediction case. We assume that the trade-off parameter in the prediction case is

$$\gamma_t^{R,S,P} = \gamma^{R,S,P}(S_{t+1}), \quad \text{for all } t \geq 0,$$

where $\gamma^{R,S,P}$ is a bounded measurable function from $\mathcal{D}$ to $\mathbb{R}_{++}$.

The investor with the prediction ability does not need to estimate the one-period-ahead movement of the state variable. For convenience of the notations, we introduce a probability measure $\mathbb{P}_{x,i}^1$, $x \in \mathbb{R}$, $i \in \mathcal{D}$ such that

$$\mathbb{P}_{x,i}^1(A) := \mathbb{P}_{x,\pi}(A \mid S_1 = i), \quad A \in \mathcal{F}.$$

We denote by $E_{x,i}^1$ the expectation operator of $\mathbb{P}_{x,i}^1$. Then, the optimization problem of the investor with the prediction ability is

$$\begin{aligned} V_{pre}(x, i) = & \sup_{u \in \mathcal{A}_{pre}(x, i)} E_{x,i}^1 \left[\sum_{t=1}^{\infty} \delta^t \left(E_{x,i}^1[X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}] - \frac{\gamma^{R,S,P}(S_t)}{2} \text{Var}_{x,i}^1(X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}) \right) \right] \\ \text{subject to } & (1 + r_f)X_{t-1}^{x,u} + (R_t^e)'u_{t-1} = X_t^{x,u}, \quad t \geq 1, \\ & X_0^{x,u} = x, \\ & S_1 = i, \end{aligned} \tag{3.2.17}$$

where $\mathcal{A}_{pre}(x, i)$ is a set of admissible control processes such that any $u \in \mathcal{A}_{pre}(x, i)$ satisfies

$$\mathbb{E}_{x,i}^1 \left[\sum_{t=1}^{\infty} \delta^t \left(\left| \mathbb{E}_{x,i}^1 [X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}] \right| + \frac{\gamma^{R,S,P}(S_t)}{2} \text{Var}_{x,i}^1(X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}) \right) \right] < \infty,$$

and

$$\lim_{T \rightarrow \infty} \delta^T \mathbb{E}_{x,i}^1 [|X_T^{x,u}|] = 0.$$

The expectation and variance of $X_t^{x,u}$ conditioned by $\mathcal{F}_{t-1}^{R,S,P}$ are

$$\begin{aligned} \mathbb{E}_{x,i}^1 [X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}] &= (1 + r_f) X_{t-1}^{x,u} + (\mu^e(S_t))' u_{t-1}, \\ \text{Var}_{x,i}^1 (X_t^{x,u} | \mathcal{F}_{t-1}^{R,S,P}) &= u_{t-1}' \Sigma(S_t) u_{t-1}. \end{aligned}$$

Let $(S_t^{1,i})_{t=1}^{\infty}$ be a state variable process such that $S_1^{1,i} = i$. We consider the following functional equation,

$$\begin{aligned} U_{pre}(x, i) &= \max_u \left\{ (1 + r_f)x + (\mu^e(i))' u - \frac{\gamma^{R,S,P}(i)}{2} u' \Sigma(i) u + \delta \mathbb{E}_{x,i}^1 [U_{pre}(X, S_2^{1,i})] \right\} \\ \text{subject to } X &= (1 + r_f)x + (R_1^e)' u. \end{aligned} \quad (3.2.18)$$

Using the argument similar to the one used in the partial information and perfect information cases, we can prove that the solution of the above equation (3.2.18), denoted by U_{pre}^* , satisfies $\delta U_{pre}^* = V_{pre}$ under the linear growth assumption of U_{pre}^* and the boundedness assumption of the optimal Markov policy.

We guess the solution as follows,

$$U_{pre}^*(x, i) = \hat{\alpha}(i) + \hat{\beta}x,$$

where $\hat{\alpha} : \mathcal{D} \rightarrow \mathbb{R}$ is a measurable function and $\hat{\beta}$ is a constant. Then, for any u , we have

$$\mathbb{E}_{x,i}^1 [U_{pre}^*(X, S_2^{1,i})] = \mathbb{E}_{x,i}^1 [\hat{\alpha}(S_2^{1,i})] + \hat{\beta}((1 + r_f)x + (\mu^e(i))' u).$$

Therefore, the optimization problem of our interest is

$$\begin{aligned} \max_u \left\{ (1 + r_f)x + (\mu^e(i))' u - \frac{\gamma^{R,S,P}(i)}{2} u' \Sigma(i) u \right. \\ \left. + \delta \mathbb{E}_{x,i}^1 [\hat{\alpha}(S_2^{1,i})] + \delta \hat{\beta}((1 + r_f)x + (\mu^e(i))' u) \right\}. \end{aligned}$$

It is a concave maximization problem, so its first order condition is

$$(1 + \delta\widehat{\beta})\mu^e(i) - \gamma^{R,S,P}(i)\Sigma(i)u^* = \mathbf{0}_d.$$

Solving the first order condition for u^* , we obtain

$$u_{pre}(i) := u^* = \frac{1 + \delta\widehat{\beta}}{\gamma^{R,S,P}(i)}(\Sigma(i))^{-1}\mu^e(i).$$

Therefore,

$$U_{pre}^*(x, i) = \delta \mathbb{E}_{x,i}^1 [\widehat{\alpha}(S_2^{1,i})] + \frac{(1 + \delta\widehat{\beta})^2}{2\gamma^{R,S,P}(i)}\widehat{a}(i) + (1 + \delta\widehat{\beta})(1 + r_f)x,$$

where

$$\widehat{a}(i) = (\mu^e(i))'(\Sigma(i))^{-1}\mu^e(i).$$

Hence, α and β need to satisfy the following equations:

$$\begin{aligned}\widehat{\alpha}(i) &= \frac{(1 + \delta\widehat{\beta})^2}{2\gamma^{R,S,P}(i)}\widehat{a}(i) + \delta \mathbb{E}_{x,i}^1 [\widehat{\alpha}(S_2^{1,i})], \quad i \in \mathcal{D}, \\ \widehat{\beta} &= (1 + \delta\widehat{\beta})(1 + r_f).\end{aligned}$$

The solutions of the above equations for $\widehat{\alpha}$ and $\widehat{\beta}$ are

$$\begin{aligned}\widehat{\alpha}(i) &= \frac{1}{2(1 - \delta(1 + r_f))^2} \mathbb{E}_{x,i}^1 \left[\sum_{t=0}^{\infty} \delta^t \frac{\widehat{a}(S_{t+1}^{1,i})}{\gamma^{R,S,P}(S_{t+1}^{1,i})} \right], \quad i \in \mathcal{D}, \\ \widehat{\beta} &= \frac{1 + r_f}{1 - \delta(1 + r_f)}.\end{aligned}$$

Moreover, from the similar argument as in the perfect information case, we deduce that

$$\mathbb{E}_{x,i}^1 \left[\sum_{t=0}^{\infty} \delta^t \frac{\widehat{a}(S_{t+1}^{1,i})}{\gamma^{R,S,P}(S_{t+1}^{1,i})} \right] = (\iota(i))'(I_K - \delta Q)^{-1}\widehat{\kappa},$$

where

$$\widehat{\kappa} = \begin{pmatrix} \frac{\widehat{a}(1)}{\gamma^{R,S,P}(1)} \\ \vdots \\ \frac{\widehat{a}(K)}{\gamma^{R,S,P}(K)} \end{pmatrix}.$$

Then, the solution of (3.2.18) is

$$U_{pre}^*(x, i) = \frac{(\iota(i))'(I_K - \delta Q)^{-1}\widehat{\kappa}}{2(1 - \delta(1 + r_f))^2} + \frac{1 + r_f}{1 - \delta(1 + r_f)}x$$

and the optimal Markov control is

$$u_{pre}(i) = \frac{1}{\gamma^{R,S,P}(i)(1 - \delta(1 + r_f))}(\Sigma(i))^{-1}\mu^e(i).$$

Proposition 3.7 Let $\widehat{a}(i) = (\mu^e(i))'(\Sigma(i))^{-1}\mu^e(i)$. Let $\widehat{\kappa}$ be a K -dimensional constant vector whose i th element is $\widehat{a}(i)/\gamma^{R,S,P}(i)$. Then, the value function of the problem (3.2.18) is

$$V_{pre}(x, i) = \frac{\delta(\iota(i))'(I_K - \delta Q)^{-1}\widehat{\kappa}}{2(1 - \delta(1 + r_f))^2} + \frac{\delta(1 + r_f)}{1 - \delta(1 + r_f)}x,$$

and the optimal Markov control is

$$u_{pre}(i) = \frac{1}{\gamma^{R,S,P}(i)(1 - \delta(1 + r_f))}(\Sigma(i))^{-1}\mu^e(i).$$

The separation result also holds in the prediction case. The prediction investor invests in the risk-free asset and in some portfolio of risky assets. This risky portfolio is proportional to the following portfolio

$$\phi(i) = \frac{1}{c(i)}(\Sigma(i))^{-1}\mu^e(i), \quad c(i) = \mathbf{1}_d'(\Sigma(i))^{-1}\mu^e(i),$$

for all $i \in \mathcal{D}$ if $c(i) \neq 0$.

3.3 Benefits of the Investor's Information

In the previous section, we derived the optimal Markov controls under three different information levels.

1. Partial information

$$u_{par}(\pi) = \frac{1}{\gamma^R(\pi)(1 - \delta(1 + r_f))}(A(\pi))^{-1}m^e(\pi);$$

2. Perfect information

$$u_{per}(i) = \frac{1}{\gamma^{R,S}(i)(1 - \delta(1 + r_f))}(A(\iota(i)))^{-1}m^e(\iota(i));$$

3. Prediction

$$u_{pre}(i) = \frac{1}{\gamma^{R,S,P}(i)(1 - \delta(1 + r_f))}(\Sigma(i))^{-1}\mu^e(i).$$

In this section, we consider the profit from the investor's information. First, we assume that the trade-off parameters are the same constants for all information levels.

$$\gamma := \gamma^R(\pi) = \gamma^{R,S}(i) = \gamma^{R,S,P}(i) > 0, \quad \text{for all } \pi \in \mathcal{D}_\pi, i \in \mathcal{D}.$$

Moreover, we assume that the state variable $(S_t)_{t=0}^\infty$ has a steady state, that is, there exists a K -dimensional vector $\bar{\pi} \in \mathcal{D}_\pi$ such that $\bar{\pi} = Q'\bar{\pi}$. From the assumption concerning the trade-off parameters, it follows that for all $\pi \in \mathcal{D}_\pi$ and $i \in \mathcal{D}$,

$$u_{par}(\pi) = \lambda(A(\pi))^{-1}m^e(\pi), \quad u_{per}(i) = \lambda(A(\iota(i)))^{-1}m^e(\iota(i)), \quad u_{pre}(i) = \lambda(\Sigma(i))^{-1}\mu^e(i),$$

where $\lambda = (\gamma(1 - \delta(1 + r_f)))^{-1}$.

Let X_t^I and R_t^I be the wealth and return of the investor I at time t . Then, we have

$$\begin{aligned} R_t^{par} &= r_f + \frac{\lambda}{X_{t-1}^{par}}(R_t^e)'(A(\pi_{t-1}))^{-1}m^e(\pi_{t-1}), \\ R_t^{per} &= r_f + \frac{\lambda}{X_{t-1}^{per}}(R_t^e)'(A(\iota(S_{t-1})))^{-1}m^e(\iota(S_{t-1})), \\ R_t^{pre} &= r_f + \frac{\lambda}{X_{t-1}^{pre}}(R_t^e)'(\Sigma(S_t))^{-1}\mu^e(S_t). \end{aligned}$$

To compare the benefits of information, we take the mean and variance conditioned only on the wealth; so, we assume that the current state is the steady state. The one-period-ahead expected returns are

$$\begin{aligned} E_{x,\bar{\pi}}[R_1^{par}] &= r_f + \frac{\lambda}{x} E_{x,\bar{\pi}} \left[(R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0) \right], \\ E_{x,\bar{\pi}}[R_1^{per}] &= r_f + \frac{\lambda}{x} E_{x,\bar{\pi}} \left[(R_1^e)'(A(\iota(S_0)))^{-1}m^e(\iota(S_0)) \right], \\ E_{x,\bar{\pi}}[R_1^{pre}] &= r_f + \frac{\lambda}{x} E_{x,\bar{\pi}} \left[(R_1^e)'(\Sigma(S_1))^{-1}\mu^e(S_1) \right], \end{aligned}$$

The one-period-ahead variance of returns are

$$\begin{aligned} \text{Var}_{x,\bar{\pi}}(R_1^{par}) &= \left(\frac{\lambda}{x} \right)^2 \text{Var}_{x,\bar{\pi}} \left((R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0) \right), \\ \text{Var}_{x,\bar{\pi}}(R_1^{per}) &= \left(\frac{\lambda}{x} \right)^2 \text{Var}_{x,\bar{\pi}} \left((R_1^e)'(A(\iota(S_0)))^{-1}m^e(\iota(S_0)) \right), \\ \text{Var}_{x,\bar{\pi}}(R_1^{pre}) &= \left(\frac{\lambda}{x} \right)^2 \text{Var}_{x,\bar{\pi}} \left((R_1^e)'(\Sigma(S_1))^{-1}\mu^e(S_1) \right). \end{aligned}$$

Moreover, we have

$$\begin{aligned} E_{x,\bar{\pi}}[(R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0)] &= E_{x,\bar{\pi}} \left[E_{x,\bar{\pi}} \left[(R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0) \middle| \mathcal{F}_0^R \right] \right] \\ &= E_{x,\bar{\pi}} \left[(E_{x,\bar{\pi}} [R_1^e | \mathcal{F}_0^R])'(A(\pi_0))^{-1}m^e(\pi_0) \right] \\ &= E_{x,\bar{\pi}} \left[(m^e(\pi_0))'(A(\pi_0))^{-1}m^e(\pi_0) \right] \\ &= E_{x,\bar{\pi}}[a(\pi_0)], \end{aligned}$$

and

$$\begin{aligned}
& \text{Var}_{x,\bar{\pi}}((R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0)) \\
&= \text{E}_{x,\bar{\pi}} \left[\left((R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0) \right)^2 \right] - \left(\text{E}_{x,\bar{\pi}}[(R_1^e)'(A(\pi_0))^{-1}m^e(\pi_0)] \right)^2 \\
&= \text{E}_{x,\bar{\pi}} \left[(m^e(\pi_0))'(A(\pi_0))^{-1} \text{E}_{x,\bar{\pi}} \left[R_1^e(R_1^e)' \middle| \mathcal{F}_0^R \right] (A(\pi_0))^{-1}m^e(\pi_0) \right] - (\text{E}_{x,\bar{\pi}}[a(\pi_0)])^2 \\
&= \text{E}_{x,\bar{\pi}} \left[(m^e(\pi_0))'(A(\pi_0))^{-1} \left(A(\pi_0) + m^e(\pi_0)(m^e(\pi_0))' \right) (A(\pi_0))^{-1}m^e(\pi_0) \right] - (\text{E}_{x,\bar{\pi}}[a(\pi_0)])^2 \\
&= \text{E}_{x,\bar{\pi}} \left[a(\pi_0) + (a(\pi_0))^2 \right] - (\text{E}_{x,\bar{\pi}}[a(\pi_0)])^2 \\
&= \text{E}_{x,\bar{\pi}}[a(\pi_0)] + \text{Var}_{x,\bar{\pi}}(a(\pi_0)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \text{E}_{x,\bar{\pi}} \left[(R_1^e)'(A(\iota(S_0)))^{-1}m^e(\iota(S_0)) \right] = \text{E}_{x,\bar{\pi}} \left[a(\iota(S_0)) \right], \\
& \text{E}_{x,\bar{\pi}} \left[(R_1^e)'(\Sigma(S_1))^{-1}\mu^e(S_1) \right] = \text{E}_{x,\bar{\pi}} \left[\hat{a}(S_1) \right], \\
& \text{Var}_{x,\bar{\pi}} \left((R_1^e)'(A(\iota(S_0)))^{-1}m^e(\iota(S_0)) \right) = \text{E}_{x,\bar{\pi}} \left[a(\iota(S_0)) \right] + \text{Var}_{x,\bar{\pi}}(a(\iota(S_0))), \\
& \text{Var}_{x,\bar{\pi}} \left((R_1^e)'(\Sigma(S_1))^{-1}\mu^e(S_1) \right) = \text{E}_{x,\bar{\pi}} \left[\hat{a}(S_1) \right] + \text{Var}_{x,\bar{\pi}}(\hat{a}(S_1))
\end{aligned}$$

Therefore, the single-period Sharpe ratios are

$$\begin{aligned}
SR^{par} &= \frac{\text{E}_{x,\bar{\pi}}[R_1^{par}] - r_f}{\sqrt{\text{Var}_{x,\bar{\pi}}(R_1^{par})}} = \frac{\text{E}_{x,\bar{\pi}}[a(\pi_0)]}{\sqrt{\text{E}_{x,\bar{\pi}}[a(\pi_0)] + \text{Var}_{x,\bar{\pi}}(a(\pi_0))}}, \\
SR^{per} &= \frac{\text{E}_{x,\bar{\pi}}[R_1^{per}] - r_f}{\sqrt{\text{Var}_{x,\bar{\pi}}(R_1^{per})}} = \frac{\text{E}_{x,\bar{\pi}}[a(\iota(S_0))]}{\sqrt{\text{E}_{x,\bar{\pi}}[a(\iota(S_0)) + \text{Var}_{x,\bar{\pi}}(a(\iota(S_0)))]}}, \\
SR^{pre} &= \frac{\text{E}_{x,\bar{\pi}}[R_1^{pre}] - r_f}{\sqrt{\text{Var}_{x,\bar{\pi}}(R_1^{pre})}} = \frac{\text{E}_{x,\bar{\pi}}[\hat{a}(S_1)]}{\sqrt{\text{E}_{x,\bar{\pi}}[\hat{a}(S_1)] + \text{Var}_{x,\bar{\pi}}(\hat{a}(S_1))}}.
\end{aligned}$$

So, the single-period Shape ratio of each information level is independent of the initial wealth.

Next, we consider a myopic optimization problem as a benchmark. Since $\bar{\pi} = Q'\bar{\pi} = p(\bar{\pi})$, the unconditional expected return vector of the risky assets is

$$\text{E}_{x,\bar{\pi}}[R_t^e] = \sum_{i=1}^K \mu^e(i)\bar{\pi}(i) = \sum_{i=1}^K \mu^e(i)p(i, \bar{\pi}) = m^e(\bar{\pi}).$$

Similarly, the unconditional variance matrix of the risky assets' returns is

$$\text{Var}_{x,\bar{\pi}}(R_t^e) = A(\bar{\pi}).$$

The single-period MV problem of the myopic investor with the initial wealth x is

$$\begin{aligned} & \max_u \left\{ E_{x,\bar{\pi}}[X] - \frac{\gamma^m}{2} \text{Var}_{x,\bar{\pi}}(X) \right\} \\ & \text{subject to } X = (1 + r_f)x + (R_1^e)'u, \end{aligned}$$

where γ^m is the trade-off parameter of the myopic investor. The solution to the above problem is

$$u^{myo} = \frac{1}{\gamma^m} (A(\bar{\pi}))^{-1} m^e(\bar{\pi}).$$

If $\gamma^m = \gamma(1 - \delta(1 + r_f))$, the myopic investor's return is

$$R_1^{myo} = r_f + \frac{\lambda}{x} (R_1^e)' (A(\bar{\pi}))^{-1} m^e(\bar{\pi}),$$

and the single-period Sharpe ratio of the myopic investor is

$$SR^{myo} = \frac{E_{x,\bar{\pi}}[(R_1^e)'(A(\bar{\pi}))^{-1}m^e(\bar{\pi})]}{\sqrt{\text{Var}_{x,\bar{\pi}}((R_1^e)'(A(\bar{\pi}))^{-1}m^e(\bar{\pi}))}} = \frac{a(\bar{\pi})}{\sqrt{a(\bar{\pi})}} = \sqrt{a(\bar{\pi})}.$$

The following lemmas are important when discussing the benefits of information concerning regimes. Let $\mathcal{S}_d$ be a set of d -dimensional, symmetric, and positive-definite matrices.

Lemma 3.8 $a(\pi)$ is convex in π .

Proof of Lemma 3.8. $A(\pi)$ can be expressed as

$$A(\pi) = G(\pi) - m^e(\pi)(m^e(\pi))',$$

where

$$G(\pi) = \sum_{i=1}^K \left(\Sigma(i) + \mu^e(i)(\mu^e(i))' \right) p(i, \pi).$$

Note that G is a symmetric and positive-definite matrix for all $\pi \in \mathcal{D}_\pi$. First, we prove that $(m^e(\pi))(G(\pi))^{-1}m^e(\pi) < 1$ for all $\pi \in \mathcal{D}_\pi$. Fix an arbitrary $\pi \in \mathcal{D}_\pi$. By the Sherman-Morrison formula, we have

$$(G(\pi))^{-1} = \left(A(\pi) + m^e(\pi)(m^e(\pi))' \right)^{-1} = (A(\pi))^{-1} - \frac{(A(\pi))^{-1}m^e(\pi)(m^e(\pi))'(A(\pi))^{-1}}{1 + (m^e(\pi))'(A(\pi))^{-1}m^e(\pi)},$$

It follows that

$$(m^e(\pi))(G(\pi))^{-1}m^e(\pi) = \frac{(m^e(\pi))(A(\pi))^{-1}m^e(\pi)}{1 + (m^e(\pi))'(A(\pi))^{-1}m^e(\pi)} = \frac{a(\pi)}{1 + a(\pi)}.$$

This implies that $(m^e(\pi))(G(\pi))^{-1}m^e(\pi)$ is in $[0, 1)$, since $a(\pi)$ is non-negative. By the Sherman-Morrison formula again, we have

$$(A(\pi))^{-1} = \left(G(\pi) - m^e(\pi)(m^e(\pi))' \right)^{-1} = (G(\pi))^{-1} + \frac{(G(\pi))^{-1}m^e(\pi)(m^e(\pi))'(G(\pi))^{-1}}{1 - (m^e(\pi))'(G(\pi))^{-1}m^e(\pi)}.$$

Hence

$$a(\pi) = (m^e(\pi))'(A(\pi))^{-1}m^e(\pi) = \frac{(m^e(\pi))'(G(\pi))^{-1}m^e(\pi)}{1 - (m^e(\pi))'(G(\pi))^{-1}m^e(\pi)}.$$

Therefore, $a(\pi)$ can be expressed as

$$a(\pi) = g\left((m^e(\pi))'(G(\pi))^{-1}m^e(\pi)\right), \quad \pi \in \mathcal{D}_\pi, \quad g(x) = \frac{x}{1-x}, \quad x \in \mathbb{R}.$$

Since g is convex and increasing on $[0, 1)$ and since $(m^e(\pi))'(G(\pi))^{-1}m^e(\pi) \in [0, 1)$ for all $\pi \in \mathcal{D}_\pi$, $a(\pi)$ is convex in π if $(m^e(\pi))'(G(\pi))^{-1}m^e(\pi)$ is convex in π . For any $\pi^1, \pi^2 \in \mathcal{D}_\pi$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} & (m^e(\lambda\pi^1 + (1-\lambda)\pi^2))'(G(\lambda\pi^1 + (1-\lambda)\pi^2))^{-1}m^e(\lambda\pi^1 + (1-\lambda)\pi^2) \\ &= (\lambda m^e(\pi^1) + (1-\lambda)m^e(\pi^2))'(\lambda G(\pi^1) + (1-\lambda)G(\pi^2))^{-1}(\lambda m^e(\pi^1) + (1-\lambda)m^e(\pi^2)), \end{aligned}$$

since m^e and G are linear in π . Therefore, if the following mapping from $\mathbb{R}^d \times \mathcal{S}_d$ onto $\mathbb{R}_+$,

$$f(x, A) = x'A^{-1}x,$$

is convex in $(x, A) \in \mathbb{R}^d \times \mathcal{S}_d$, then $(m^e(\pi))'(G(\pi))^{-1}m^e(\pi)$ is convex in π . However, since f is a matrix fractional function, it is convex on $\mathbb{R}^d \times \mathcal{S}_d$ (cf. Example 3.4 in Boyd and Vandenberghe (2004)). Therefore, $(m^e(\pi))'(G(\pi))^{-1}m^e(\pi)$ is convex in π , and $a(\pi)$ is also convex in π . $\square$

Lemma 3.9

$$\mathbb{E}_{x,\pi}[\widehat{a}(S_t) \mid \mathcal{F}_{t-1}^{R,S}] \geq a(\iota(S_{t-1})) \text{ a.s.}$$

Proof of Lemma 3.9. Let $\widehat{a}_f$ be a function from $\mathcal{D}_\pi$ onto $\mathbb{R}_+$ such that

$$\widehat{a}_f(\pi) = \left(\sum_{i=1}^K \mu(i)\pi(i) \right)' \left(\sum_{i=1}^K \Sigma(i)\pi(i) \right)^{-1} \left(\sum_{i=1}^K \mu(i)\pi(i) \right), \quad \pi \in \mathcal{D}_\pi.$$

Then, $\widehat{a}(S_t) = \widehat{a}_f(\iota(S_t))$ and $\widehat{a}_f$ is convex in π since the function $(x, A) \rightarrow x'A^{-1}x$ is convex on $\mathbb{R}^d \times \mathcal{S}_d$ (see Lemma 3.8). Moreover, $\mathbb{E}_{x,\bar{\pi}}[\iota(S_t) \mid \mathcal{F}_{t-1}^{R,S}] = Q'\iota(S_{t-1}) = p(\iota(S_{t-1}))$. Therefore, it follows that

$$\begin{aligned} & \mathbb{E}_{x,\bar{\pi}}[\widehat{a}(\iota(S_t)) \mid \mathcal{F}_{t-1}^{R,S}] \\ & \geq \widehat{a}_f(\mathbb{E}_{x,\bar{\pi}}[\iota(S_t) \mid \mathcal{F}_{t-1}^{R,S}]) \\ & = \left(\sum_{i=1}^K \mu(i)p(i, \iota(S_{t-1})) \right)' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right)^{-1} \left(\sum_{i=1}^K \mu(i)p(i, \iota(S_{t-1})) \right) \\ & = (m^e(\iota(S_{t-1})))' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right)^{-1} (m^e(\iota(S_{t-1}))) \end{aligned}$$

where we have used the Jensen inequality in the first inequality. Fix any $x \in \mathbb{R}^d$ and let $y_i = (\mu^e(i))'x$, $i \in \mathcal{D}$. Then, it holds that

$$\begin{aligned} & x' \left[\sum_{i=1}^K \mu^e(i)(\mu^e(i))'p(i, \iota(S_{t-1})) - \left(\sum_{i=1}^K \mu^e(i)p(i, \iota(S_{t-1})) \right) \left(\sum_{i=1}^K \mu^e(i)p(i, \iota(S_{t-1})) \right)' \right] x \\ & = \sum_{i=1}^K y_i^2 p(i, \iota(S_{t-1})) - \left(\sum_{i=1}^K y_i p(i, \iota(S_{t-1})) \right)^2 \\ & = \sum_{i=1}^K \left(y_i - \sum_{k=1}^K y_k p(k, \iota(S_{t-1})) \right)^2 p(i, \iota(S_{t-1})) \geq 0. \end{aligned}$$

Therefore, for any $x \in \mathbb{R}^d$, we have

$$\begin{aligned} & x'A(\iota(S_{t-1}))x \\ & = x' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right) x \\ & \quad + x' \left[\sum_{i=1}^K \mu^e(i)(\mu^e(i))'p(i, \iota(S_{t-1})) - \left(\sum_{i=1}^K \mu^e(i)p(i, \iota(S_{t-1})) \right) \left(\sum_{i=1}^K \mu^e(i)p(i, \iota(S_{t-1})) \right)' \right] x \\ & \geq x' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right) x. \end{aligned}$$

Hence,

$$x'(A(\iota(S_{t-1})))^{-1}x \leq x' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right)^{-1} x$$

for all $x \in \mathbb{R}^d$. When x is equal to $m^e(\iota(S_{t-1}))$ in the above inequality, we have

$$\begin{aligned} \mathbb{E}_{x,\bar{\pi}}[\widehat{a}(\iota(S_t)) \mid \mathcal{F}_{t-1}^{R,S}] & \geq (m^e(\iota(S_{t-1})))' \left(\sum_{i=1}^K \Sigma(i)p(i, \iota(S_{t-1})) \right)^{-1} m^e(\iota(S_{t-1})) \\ & \geq (m^e(\iota(S_{t-1})))'(A(\iota(S_{t-1})))^{-1} m^e(\iota(S_{t-1})) = a(\iota(S_{t-1})). \end{aligned}$$

□

We can easily see that $E_{x,\bar{\pi}}[\pi_{t-1}] = \bar{\pi}$ and $E_{x,\bar{\pi}}[\iota(S_{t-1})|\mathcal{F}_{t-1}^R] = \pi_{t-1}$. From Lemmas 3.8, 3.9, and the Jensen inequality, we deduce that

$$E_{x,\bar{\pi}}[\hat{a}(S_t)] \geq E_{x,\bar{\pi}}[a(\iota(S_{t-1}))] \geq E_{x,\bar{\pi}}[a(\pi_{t-1})] \geq a(\bar{\pi})$$

Then, the single-period expected return becomes higher as the investor's information becomes more detailed. However, we also deduce that

$$E_{x,\bar{\pi}}[a(\pi_{t-1})] + \text{Var}_{x,\bar{\pi}}(a(\pi_{t-1})) \geq a(\bar{\pi}),$$

$$E_{x,\bar{\pi}}[a(\iota(S_{t-1}))] + \text{Var}_{x,\bar{\pi}}(a(\iota(S_{t-1}))) \geq a(\bar{\pi}),$$

$$E_{x,\bar{\pi}}[\hat{a}(S_t)] + \text{Var}_{x,\bar{\pi}}(\hat{a}(S_t)) \geq a(\bar{\pi}).$$

We see that the single-period variances of the informed investors are also greater than the variance of the myopic investor; therefore, it is possible that the Sharpe ratios of the informed investors are smaller than the Sharpe ratio of the myopic investor.

Proposition 3.10 Suppose that each investor's initial wealth is the same constant x and that the informed investors' trade-off parameters are also the same constant γ . Moreover, we assume that the myopic investor's trade-off parameter is $\gamma^m = \gamma(1 - \delta(1 + r_f))$. Then, we have

$$E_{x,\bar{\pi}}[R_1^{pre}] \geq E_{x,\bar{\pi}}[R_1^{par}] \geq E_{x,\bar{\pi}}[R_1^{per}] \geq E_{x,\bar{\pi}}[R_1^{myo}],$$

and

$$\text{Var}_{x,\bar{\pi}}(R_1^{par}) \geq \text{Var}_{x,\bar{\pi}}(R_1^{myo}), \text{Var}_{x,\bar{\pi}}(R_1^{per}) \geq \text{Var}_{x,\bar{\pi}}(R_1^{myo}), \text{Var}_{x,\bar{\pi}}(R_1^{pre}) \geq \text{Var}_{x,\bar{\pi}}(R_1^{myo}).$$

Guidolin and Ria (2011) report that the tangency portfolio based on the regime switching settings has greater sample mean and variance than the myopic tangency portfolio. Even though our analysis does not test the optimality of the tangency portfolios directly, our theoretical result is consistent with their empirical results.

However, the possibility of mean-variance inefficiency of the informed portfolios may be not a crucial problem in the dynamic portfolio rebalancing. We can compute the

expectation of t -step-ahead wealth under each information level as

$$\begin{aligned} E_{x,\bar{\pi}}[X_t^{par}] &= (1 + r_f)^t x + \lambda \sum_{s=0}^{t-1} (1 + r_f)^{t-1-s} E_{x,\bar{\pi}}[a(\pi_s)], \\ E_{x,\bar{\pi}}[X_t^{per}] &= (1 + r_f)^t x + \lambda \sum_{s=0}^{t-1} (1 + r_f)^{t-1-s} E_{x,\bar{\pi}}[a(\iota(S_s))], \\ E_{x,\bar{\pi}}[X_t^{pre}] &= (1 + r_f)^t x + \lambda \sum_{s=0}^{t-1} (1 + r_f)^{t-1-s} E_{x,\bar{\pi}}[\widehat{a}(S_{s+1})], \\ E_{x,\bar{\pi}}[X_t^{myo}] &= (1 + r_f)^t x + \lambda \sum_{s=0}^{t-1} (1 + r_f)^{t-1-s} a(\bar{\pi}), \end{aligned}$$

for all $t \geq 1$, where $(X_t^{myo})_{t=0}^{\infty}$ is the wealth process when investing in the myopic MV optimal portfolio at each time. By Lemmas 3.8 and 3.9, we have

$$E_{x,\bar{\pi}}[X_t^{pre}] \geq E_{x,\bar{\pi}}[X_t^{per}] \geq E_{x,\bar{\pi}}[X_t^{par}] \geq E_{x,\bar{\pi}}[X_t^{myo}],$$

for all $t \geq 0$. This means that the wealth of the more informed investor grows more rapidly than that of the less informed investor, on average.

Since the optimal Markov policies of the three different information levels and the myopic investor are independent of their wealth at each point in time, the contribution of the risky assets' portfolio to wealth decreases as the wealth increases, for all investors. Therefore, sample variances of the returns of all investors also decrease. Then, we can see that the points of the investors' returns in the risk-return graph are close to the point of the risk-free asset if sufficient time has passed from the initial point. Hence the mean-variance efficiencies of all investors tend to become similar to the risk-free asset. However, since the wealth of the more informed investor grows more rapidly, the mean-variance efficiency of the more informed investor also tends to become similar to the risk-free asset more rapidly. This means that although it is possible that the information benefit measured by the mean-variance efficiency vanishes if the trade-off parameters are constant, the benefit from the amount of wealth of the informed investor still exists, on average.

3.4 Numerical Analysis

Next, we conduct numerical simulations. We assume that a two-state Markov chain drives the market. The transition probability matrix is

$$Q = \begin{pmatrix} 0.97 & 0.03 \\ 0.07 & 0.93 \end{pmatrix}.$$

Then, the steady state probability is $(0.7, 0.3)$. We assume that one risk-free asset and two risky assets are in the market. The means and variances of returns of the risky assets are

$$\begin{aligned} \mu(1) &= \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \quad \mu(2) = \begin{pmatrix} -0.1 \\ -0.2 \end{pmatrix}, \\ \sigma(1)(\sigma(1))' &= \begin{pmatrix} 0.01 & 0.0075 \\ 0.0075 & 0.0225 \end{pmatrix}, \quad \sigma(2)(\sigma(2))' = \begin{pmatrix} 0.04 & 0.048 \\ 0.048 & 0.09 \end{pmatrix}. \end{aligned}$$

The risk-free rate r_f is 0.05. The discount rate δ is 0.95, and the trade-off parameter is constant and equals 5. These settings satisfy the optimality condition that $\delta(1 + r_f) = 0.9975 < 1$. Figure 3.1 shows the conditional mean-variance frontier of returns given the partial information when the current probability is the steady probability.

The optimal portfolio always lies on the tangent line of the conditional mean-variance frontier. When the investor has small amount of wealth, $x = 70$, then she uses financial leverage. On the other hand, the investor buys the risk-free asset when the amount of wealth $x = 200$. This is consistent with the analytical results.

Now, we consider dynamic simulations of the mean-variance portfolio rebalancing under regime switching. We conduct two different simulations, called simulation A and B. Both of the two simulations are driven by the two-state Markov chain defined as above. We denote by μ_A and μ_B the mean vectors of simulation A and B, respectively. μ_A and μ_B are as follows.

$$\begin{aligned} \mu_A(1) &= \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \quad \mu_A(2) = \begin{pmatrix} -0.1 \\ -0.2 \end{pmatrix}, \\ \mu_B(1) &= \begin{pmatrix} 0.25 \\ 0.3 \end{pmatrix}, \quad \mu_B(2) = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}. \end{aligned}$$

Therefore, the difference between the simulations is the conditional mean vector. The trade-off parameter of each information level investor is equal to a constant $\gamma = 5$. The

other parameters, σ , r_f and δ are the same as in the static analysis. The distribution of ϵ_t is the d -dimensional standard normal distribution for all $t \geq 1$. Then, we can compute the filtered probabilities $(\pi_t)_{t=0}^\infty$ for the partial information case using the method of Hamilton (1989).

In each simulation, we consider three different information levels – that is, the partial information, the perfect information, and the prediction. In addition, we also compute the myopic investor's portfolio in each simulation. At each time, the partially and perfectly informed investors and the prediction investor invest in their optimal portfolios derived in Section 3.2. The myopic investor invests in the myopic portfolio of Section 3.3 at each point in time.

We simulate 10000 time-series of the return vectors during 300 investment intervals under regime switching. Of course, these series are independent of each other, but the returns in the same series are correlated. The initial regime in each series is generated using the steady probability $\bar{\pi}$. The initial wealths of all investors are equal to 1.

First, we compute the single-period moments of returns. We compute means, variances, and the Sharpe ratios of the returns in the first investment interval over 10000 scenarios. According to the theoretical analysis in Section 3.3, it has been predicted that the mean and variance of the more informed investor are larger than those of the less informed investor.

Table 3.1 reports the single-period moments. In each simulation, the mean and variance of the more informed investor are larger than those of the less informed investor. This is consistent with our theoretical analysis. Since the initial regime probability is the steady probability, the partially informed investor and the myopic investor invest in the same portfolio at the initial time, so their performance are also the same.

The Sharpe ratios reveal similar tendency in the two simulations; the more informed investor has the greater Sharpe ratio than the less informed investor. This implies that the increase in the variance caused by the information are smaller than that of the mean. However, the differences of the Sharpe ratios in simulation B are smaller than those in simulation A. The difference between simulation A and B is the expected excess returns of the risky assets' returns, where simulation A has the state where the expected

excess returns are negative. Therefore, it is possible that the regime information is more valuable when the recession regime exists.

Next, we consider performances in the whole investment horizon. In each scenario, we compute sample means, variances, and the Sharpe ratios of all investors' returns. Then, we obtain 10000 sample moments mentioned above. Table 3.2 shows the means of these sample moments of these scenarios.

In Table 3.2, the means of the sample means and variances of the more informed investor are larger than those of the less informed investor. The dynamic sample moments of the partially informed investor are different from the single-period moments since the portfolio varies over times in response to movement of the probability π_t . The results for the other investors are close to the single-period case. However, the Sharpe ratios of the partially informed investor in simulation B are smaller than those of the myopic investor. Furthermore, the difference between the partially informed investor and the myopic investor is statistically significant at 1% significance level. The partially informed investor is the most realistic, therefore, this result reveals one of the limitations of portfolio investment when using information about regimes.

However, the mean-variance efficiency is not the only criterion in investment. We define the ratio of the information risk exposure of the information level I at time t by

$$RIRE_t^I = \frac{E_{x,\bar{\pi}}[\max\{X_t^{pre} - X_t^I, 0\}]}{E_{x,\bar{\pi}}[X_t^{pre}]}.$$

$RIRE_t^I$ represents the ratio of the expected loss at time t using the information I compared to the prediction investor. A large $RIRE_t^I$ implies a large loss of wealth when using the information I . Of course, the ratio of the information risk exposure of the prediction investor $RIRE_t^{pre}$ is always 0.

Table 3.3 reports the ratios of the information risk exposure. In Table 3.3, $RIRE_t^I$ decreases as the information increases. In simulation B, where the partially informed investor is less mean-variance efficient than the myopic investor, the ratio of the information risk exposure of the partial information is smaller than that of the myopic at each time. Thus, the partially informed investor has an advantage over the myopic investor in terms of the ratio of the information risk exposure. Furthermore, the ratios

of the information risk exposure in simulation A are larger than those in simulation B. This is an additional evidence that information concerning regimes is more valuable when the recession exists.

Table 3.4 and Figure 3.2 show the optimal portfolio weights for the investors. In Table 3.4, absolute weights of the portfolio of the more informed investor are larger than those of the less informed investor. When $S_{t+1} = 1$ in simulation A, the prediction investor invests $17.037 \times 80 = 1362.96$ dollars in the risky assets. It is about 13 times the amount that the myopic investor invests in the risky assets. The large position in the risky assets may increase the expected return and variance of the more informed investor.

In simulation A and B, we assumed that the trade-off parameters for all investors are the same constant. Next, we relax this assumption. We consider the strategies of buying the tangency portfolio. Assume that

$$\gamma^R(\pi) = \gamma c(\pi), \quad \gamma^{R,S}(i) = \gamma c(\iota(i)), \quad \gamma^{R,S,P}(i) = \gamma \hat{c}(i), \quad \gamma^{myo} = \gamma c(\bar{\pi}).$$

Then, the investors buy the tangency portfolios based on their information. If we assume that γ is positive, we can conduct only simulation B since the sums of the risky assets' portfolio weights are always positive in simulation B. We assume that the other parameters are the same as before. The strategies of buying the tangency portfolio are similar to the investment strategies in the back test of Guidolin and Ria (2011). The strategies of buying the tangency portfolio take the dynamic portfolio rebalancing into consideration, whereas the investment strategies of Guidolin and Ria (2011) are obtained from the single-period problem.

Table 3.5 shows the results of the single-period moments and the dynamic sample moments. As in the case of the constant trade-off parameters, the single-period mean and variance of the more informed investor are larger than those of the less informed investor. However, the differences of the means and variances decrease and the Sharpe ratios get smaller as the investor's information increases. Furthermore, the relationship among the means and variances mentioned above is not evident anymore.

Table 3.6 reports the ratios of the information risk exposure. In all information

levels, the ratios of the information risk exposure decrease to 0 in the long run. This indicates that the advantage of the information measured by the wealth level also vanishes when the investors buy their tangency portfolios.

Figure 3.3 and Table 3.7 report the tangency portfolio weights. Figure 3.3 and Table 3.7 show that the weights of the risky assets are lower than in the case of the constant trade-off parameters.

According to above results, the benefits of the regime information is very limited when investing in the tangency portfolio. Moreover, it is possible that the information causes the worse performance. However, this is consistent with the empirical results of Guidolin and Ria (2011). To avoid these phenomena, the investors should have a constant trade-off parameter. Then, our numerical analysis suggests that there is some benefit from the information about regimes.

3.5 Conclusion

In this chapter, we studied the discrete-time, infinite horizon regime switching MV analysis. We derived the time-consistent, optimal Markov policies for the three different information levels and assessed the values of the regime information. Our results indicate that the expected return increases as the investor becomes informed, but, the variance also increases. However, these surprising results are consistent with the empirical study in Guidolin and Ria (2011). The implication of our results is that in order to obtain higher expected return, the informed investor takes risks more than the uninformed investor and this is not always efficient from the view of unconditional mean-variance efficiency. However, it may not be a crucial problem since the informed investor's wealth will grow more rapidly than the uninformed investor's wealth.

We assume that the investors can have the risk-free asset and the trade-off parameter is independent of her wealth. Under these assumptions, the optimal policies are independent of the investors' wealth, and the mathematical complexity is reduced. However, the leverage effects have become more important recently. For example, Björk, Murgoci, and Zhou (2014) study a continuous time MV analysis with the wealth dependent risk aversion using the game theoretic approach. Therefore, it is worth relaxing the

assumptions under the regime switching settings by using different approach. Another interesting topic is the relationship between the discrete time and the continuous time. If our discrete time results can be extended to a continuous time framework reasonably, then it will help with a mathematical analysis in the continuous time, similar to an option pricing.

Appendix 3.A Tables and Figures

Table 3.1. Single Period Moments. In the column of Mean $\times x/\lambda$, the means of the single-period excess returns of 10000 scenarios times x/λ are displayed, where x is the initial wealth and $\lambda = (\gamma(1 - \delta(1 + r_f)))^{-1}$. Similarly, in the column of Standard Deviation $\times x/\lambda$, the standard deviations of the single-period excess returns of 10000 scenarios times x/λ are displayed. In the column of Share ratio, the Share ratio of the single-period excess returns of 10000 scenarios are displayed.

Investor	Simulation A			Simulation B		
	Mean $\times x/\lambda$	Standard Deviation $\times x/\lambda$	Sharpe Ratio	Mean $\times x/\lambda$	Standard Deviation $\times x/\lambda$	Sharpe Ratio
Partial Information	0.113	0.330	0.342	1.214	1.102	1.102
Perfect Information	1.519	1.433	1.060	2.731	2.260	1.208
Prediction	2.592	2.033	1.275	3.266	2.643	1.236
Myopic	0.113	0.330	0.342	1.214	1.102	1.102

Table 3.2. Dynamic Simulations. Numbers in parenthesis are standard errors.

Investor	Simulation A			Simulation B		
	Mean	Standard Deviation	Sharpe Ratio	Mean	Standard Deviation	Sharpe Ratio
Partial Information	1.269 (0.002)	1.309 (0.001)	0.972 (0.002)	2.220 (0.004)	1.962 (0.001)	1.134 (0.002)
Perfect Information	1.526 (0.002)	1.401 (0.001)	1.095 (0.002)	2.750 (0.004)	2.260 (0.001)	1.223 (0.002)
Prediction	2.577 (0.003)	1.990 (0.001)	1.296 (0.002)	3.306 (0.005)	2.631 (0.001)	1.263 (0.002)
Myopic	0.110 (0.001)	0.321 (0.000)	0.371 (0.003)	1.232 (0.001)	1.100 (0.001)	1.146 (0.002)

Table 3.3. The Ratios of the Information Risk Exposure $RIRE_t^I$.

t	Simulation A				
	10	50	100	200	300
Partial Information	0.582	0.536	0.533	0.533	0.533
Perfect Information	0.406	0.405	0.405	0.405	0.405
Myopic	0.943	0.943	0.943	0.943	0.943

t	Simulation B				
	10	50	100	200	300
Partial Information	0.406	0.353	0.350	0.349	0.349
Perfect Information	0.169	0.165	0.165	0.165	0.165
Myopic	0.642	0.621	0.619	0.619	0.618

Table 3.4. Portfolio Weights of Dynamic Simulations. The investors buy $(\gamma(1 - \delta(1 + r_f)))^{-1} = 80$ times the following risky assets' weights.

		Simulation A			Simulation B		
		Asset 1	Asset 2	Sum	Asset 1	Asset 2	Sum
Perfect Information	$S_t = 1$	5.448	5.149	10.597	12.765	5.246	18.011
	$S_t = 2$	-0.904	-1.628	-2.532	-1.632	2.704	1.071
Prediction	$S_{t+1} = 1$	8.889	8.148	17.037	15.556	5.926	21.481
	$S_{t+1} = 2$	-1.157	-2.160	-3.318	-2.083	2.778	0.694
Myopic		0.604	0.726	1.329	3.557	3.097	6.654

Table 3.5. The Results Concerning Buying of Tangency Portfolios. Numbers in parenthesis represent standard errors.

Investor	Single-Period Moments			Dynamic Sample Moments		
	Mean	Standard Deviation	Sharpe Ratio	Mean	Standard Deviation	Sharpe Ratio
Partial Information	0.232	0.166	1.102	0.055	0.061	0.206
				(0.001)	(0.019)	(0.001)
Perfect Information	0.285	0.307	0.766	0.057	0.058	0.200
				(0.000)	(0.004)	(0.001)
Prediction	0.331	0.461	0.610	0.056	0.121	0.192
				(0.001)	(0.021)	(0.001)
Myopic	0.232	0.166	1.102	0.059	0.083	0.209
				(0.003)	(0.053)	(0.000)

Table 3.6. The Ratios of the Information Risk Exposure $RIRE_t^I$ in Buying the Tangency Portfolios.

t	Simulation B				
	10	50	100	200	300
Partial information	0.244	0.025	0.002	0.00001733	0.00000013
Perfect information	0.196	0.015	0.001	0.00001075	0.00000008
Myopic	0.320	0.038	0.003	0.00002617	0.00000020

Table 3.7. Portfolio Weights for the Tangency Portfolios. The investors buy $(\gamma(1 - \delta(1 + r_f)))^{-1} = 80$ times the following risky assets' weights.

		Asset 1	Asset 2
Perfect Information	$S_t = 1$	0.709	0.291
	$S_t = 2$	-1.523	2.523
Prediction	$S_{t+1} = 1$	0.724	0.276
	$S_{t+1} = 2$	-3.000	4.000
Myopic		0.535	0.465

Figure 3.1. Minimum Variance Frontier Given the Partial Information.

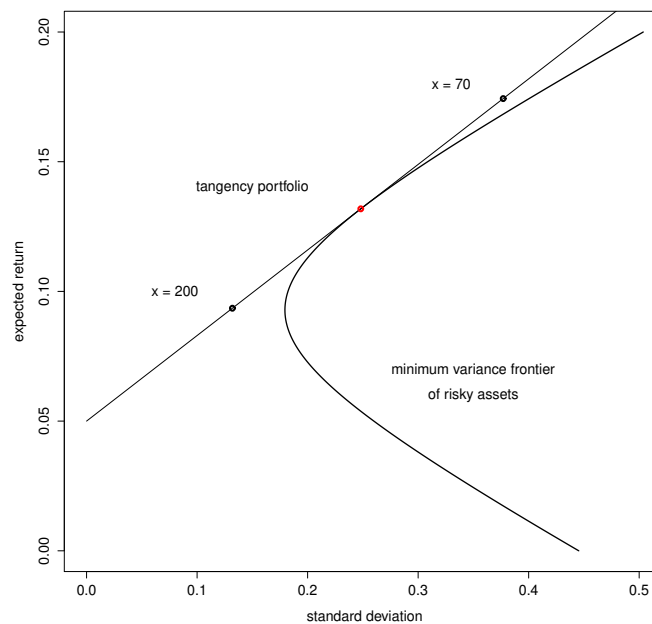


Figure 3.2. Portfolio Weights of the Partial Information Case. The horizontal axis denotes the filtered probability of staying in regime 1. The investors buy $(\gamma(1 - \delta(1 + r_f)))^{-1} = 80$ times the following risky assets' weights.

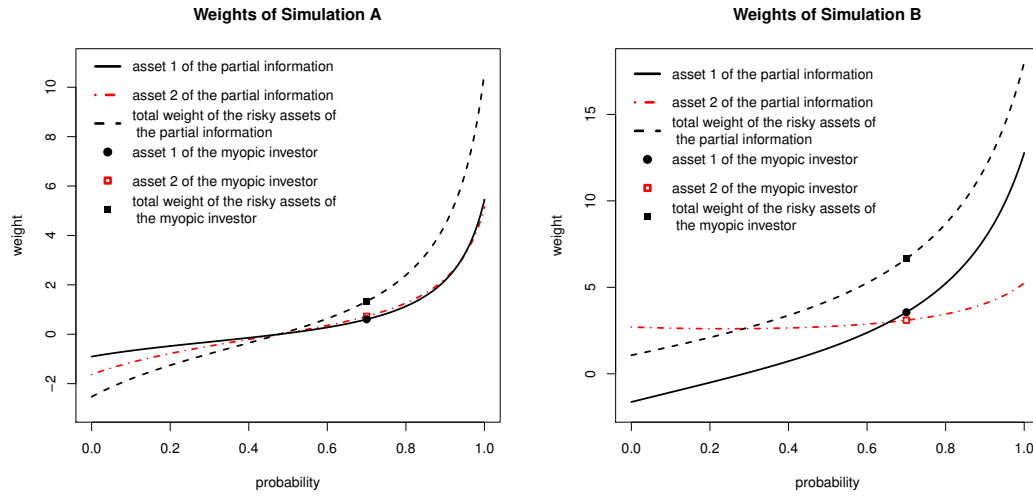
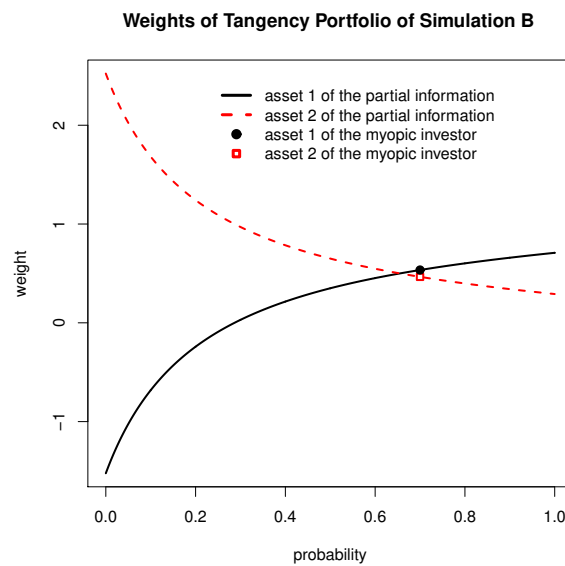


Figure 3.3. The Tangency Portfolio Weights of the Partial Information Case. The horizontal axis denotes the filtered probability of staying in regime 1. The investors buy $(\gamma(1 - \delta(1 + r_f)))^{-1} = 80$ times the following risky assets' weights.



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Chapter 4

Optimality of Naive Investment Strategies in Dynamic Mean-Variance Optimization Problems with Multiple Priors

4.1 Introduction

The mean-variance analysis proposed by Markowitz (1952) and developed by Tobin (1958) and Merton (1972) tells us that the tangency portfolio combined with the risk-free asset is mean-variance efficient, which means that it delivers the highest Sharpe ratio among all portfolios. However, it is not always efficient in practice. For example, Jagannathan and Ma (2003) mention that the global minimum-variance portfolio delivers as large an out-of-sample Sharpe ratio as the other efficient portfolios when the estimated expected returns are used. DeMiguel, Garlappi, and Uppal (2009b) report that in their back tests, the equally weighted portfolio ($1/N$ portfolio) often has a higher Sharpe ratio than other optimal portfolios including the tangency portfolio.

Probable explanations of these practical pitfalls are difficulties in estimating expected returns and sensitivity of the tangency portfolio to the expected returns. Merton (1980) reports that the estimation of expected returns is more difficult than that of a covariance matrix of returns. Best and Grauer (1991) show that the mean-variance efficient portfolios are very sensitive to the values of the expected returns. Therefore, the estimation errors of expected returns have a large impact on optimal portfolio selections. Consequently, in practice, the global minimum-variance portfolio, which is not

mean-variance efficient in theory, and the equally weighted portfolio, which seems to be naive, can become more efficient than the mean-variance efficient portfolios.

In this chapter, we construct plausible mean-variance optimization problems with multiple priors and show that in limiting cases, the solutions to these problems include the equally weighted portfolio and the global minimum-variance portfolio. The mean-variance problems with multiple priors are robust to estimation errors of moments of asset returns. We consider two-dimensional multiple priors: the priors for expected returns and the priors for a covariance matrix of returns. In our framework, the naive and inefficient strategies such as the equally weighted portfolio and the global minimum-variance portfolio are characterized by an investor's suspicion of the estimated parameters. If an investor strongly doubts the estimated expected returns, her optimal portfolio becomes similar to the global minimum-variance portfolio. In contrast to that, it becomes similar to the equally weighted portfolio if an investor strongly doubts the estimated covariances.

Moreover, our framework is based on dynamic optimization problems and allows dependency among the asset returns over time. More specifically, the market model in our framework is the Markovian market model where a Markov process affects the distribution of returns. This includes various return models such as the factor pricing models (e.g., capital asset pricing model (CAPM) by Sharpe (1964) and the Fama-French three factor model by Fama and French (1993)), the Markov regime switching models (e.g., Hamilton (1989) and Ang and Bekaert (2002)) and so on. Furthermore, the dynamic approach justifies the investment strategies which seem myopic. For example, the strategy investing in the global minimum-variance portfolio during economic booming and investing in the equally weighted portfolio during economic recession can be justified by our framework.

In order to investigate the effects of the estimation errors and study the efficiency of the optimal portfolio in our framework, we conduct back tests using various data sets. Similar to previous research, our back tests demonstrate that the global minimum-variance portfolio is sometimes more efficient than the tangency portfolio. On the other hand, the investment strategy that invests in the equally weighted portfolio during the

economic booming and the global minimum-variance portfolio during the economic recession is sometimes the most efficient in all the portfolios. This strategy seems to be naive. However, since it is characterized by the limiting cases in our framework, the reason why the investor chooses it is because of the strong suspicion of the estimated parameters.

In the investment theory, the various methods have been proposed to deal with the estimation errors. They are classified into two major approaches: the Bayesian approach and the non-Bayesian approach. In this chapter, we adopt the non-Bayesian approach, specifically, the max-min approach; maximizing an objective function with respect to portfolios *after* minimizing the objective function with respect to priors of return's distribution. The max-min problem is popular when dealing with the multiple prior optimization. Goldfarb and Iyengar (2003) solve the mean-variance portfolio selections with multiple priors under the factor pricing model. Garlappi, Uppal, and Wang (2007) characterize multiple priors for the expected returns by the confidence interval around the estimated expected returns and find that their multiple-priors optimal portfolios have high Sharpe ratios in their back tests. The max-min approach is economically axiomatized by Gilboa and Schmeidler (1989) in order to describe the ambiguity aversion of investors illustrated by Ellsberg (1961).

In addition, we use one concept from the Bayesian approach. The Bayesian approach is based on the Bayesian statistics, e.g., shrinkage estimators (Jobson and Korkie (1980) and Jorion (1986)), the Black-Litterman model (Black and Litterman (1990)) and the other methods such as Pástor and Stambaugh (2000). Ledoit and Wolf (2003) propose the shrinkage estimator to estimate a covariance matrix of asset returns when facing the small sample problem that is when the number of assets is larger than the number of observations. To compute the shrinkage estimator, they minimize the Frobenius norm of estimated errors of the covariance matrix. Using the idea of Ledoit and Wolf (2003), we introduce the Frobenius norm of estimation errors of a covariance matrix in order to deal with multiple priors.

The results obtained by the mean-variance optimization constraining portfolio norms are similar to the results by our framework. The mean-variance optimization constrain-

ing portfolio norms is proposed by DeMiguel, Garlappi, Nogales, and Uppal (2009a). Especially, their 2-norm-constrained portfolio is basically the same as the one of our limiting cases. The differences between our model and the one in DeMiguel et al. (2009a) are investment horizon and problem formulation. DeMiguel et al. (2009a) consider static portfolio selections such as

$$\min_u u' \hat{\Sigma} u \quad \text{subject to } u'u \leq k,$$

where u is a portfolio vector and u' is its transpose. $\hat{\Sigma}$ is an estimated covariance matrix. k represents the upper bound of the constraint. DeMiguel et al. (2009a) solve the above static minimum-variance problem since it is free from the problem of time-inconsistency¹. The optimal 2-norm-constrained portfolio is based on the covariance estimator of Ledoit and Wolf (2004). If the upper boundary of the constraint is the reciprocal of the number of assets, then the optimal portfolio becomes the equally weighted portfolio.

In contrast to the above problem, we consider the mean-variance-style objective function such that

$$\mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \left(\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V} (X_{t+1} | \mathcal{F}_t^{R,F}) \right) \right],$$

where X_t is a wealth of the investor at time t and $\mathcal{F}_t^{R,F}$ is the information of the investor at time t . $\hat{\gamma}_t/X_t$ represents the trade-off between the expected return and associated risks. The superscripts θ and V express multiple priors and δ is a constant discount rate. Hence, this objective function is an expected value of discounted sum of objectives in conditional mean-variance problems. The above objective function is proposed by Chen, Li, and Zhao (2014) to avoid the problem of time-inconsistency. Chen et al. (2014) consider this objective function with a unique prior, but we consider the objective function with multiple priors.

Although there are several differences, the results of Pflug, Pichler, and Wozabal (2012) are essentially the same as our results. Pflug et al. (2012) characterize a degree of ambiguity as the Kantorovich metric, and show that for various objective functions

¹The standard dynamic programming procedure can not be applied to the simple mean-variance optimization. See Li and Ng (2000) for more details.

including a mean-standard-deviation objective function and for arbitrary distributions of asset returns, the investor chooses the equally weighted portfolio under high model ambiguity. In contrast to Pflug et al. (2012), we focus on mean-variance objective functions and Markovian return models. This allows us to use more simple and clear degrees of ambiguity than the Kantorovich metric, that is, we use a confidence interval around the estimated expected return vector and a relative error of the estimated covariance matrix. Furthermore, our optimization problems are dynamic optimization problems, whereas optimization problems in Pflug et al. (2012) are static optimization problems. Therefore, our models have different implications from the model in Pflug et al. (2012).

The rest of this chapter is organized as follows. Section 4.2 formulates and solves the dynamic mean-variance optimization problem with multiple priors. Section 4.3 investigates limiting behaviors of the solutions. Section 4.4 conducts back tests and reports their results. Section 4.5 is the concluding section. The most proofs of theorems, lemmas and propositions in this chapter are in Appendix.

4.2 Mean-Variance Portfolio Selections with Multiple Priors in a Markovian Market

First, we introduce the following notations.

- $\mathbb{R}^d$, the d -dimensional Euclidean space. Specifically, we write $\mathbb{R} := \mathbb{R}^1$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$.
- $\mathbb{R}^{m,n}$, a set of $m \times n$ matrices.
- x' , the transpose of a vector or matrix x .
- $\mathbf{0}_d$, the d -dimensional real-valued vector whose all elements are 0.
- $\mathbf{1}_d$, the d -dimensional real-valued vector whose all elements are 1.
- I_d , the d -dimensional identity matrix.
- $\mathbb{1}_A(\omega)$, an indicator function. If $\omega \in A$, then $\mathbb{1}_A(\omega) = 1$. If $\omega \notin A$, then $\mathbb{1}_A(\omega) = 0$.
- $(x)^+ := \max\{x, 0\}$ for all $x \in \mathbb{R}$.

- $\|x\|$, a general expression of norms. Specifically, $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\|A\|$ is the Frobenius norm of $A \in \mathbb{R}^{m \times n}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider optimal portfolio selections of an investor during T periods. There are one risk-free asset and d risky assets in the financial market. Moreover, we assume that a sequence of $\mathbb{R}^K$ -valued random vectors, denoted by $(F_t)_{t=0}^T$, influences distributions of the return vectors of the risky assets: The return vector of the risky assets at time t , denoted by R_t , satisfies

$$R_t = \mu(F_t) + \sigma(F_t)\epsilon_t, \quad t \geq 0,$$

where $(\epsilon_t)_{t=1}^T$ is a sequence of mutually independent and d -dimensional standard normal random vectors, and $\mu : \mathbb{R}^K \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^K \rightarrow \mathbb{R}^{d \times d}$ are measurable functions. Furthermore, we assume that F_t and ϵ_t are mutually independent for each $t = 1, \dots, T$. Then, the conditional expectation and variance of R_t given by F_t are

$$\mathbb{E}[R_t | F_t] = \mu(F_t),$$

$$\text{Var}(R_t | F_t) = \sigma(F_t)(\sigma(F_t))',$$

for all $t \geq 1$. Hence, the conditional mean and variance of the risky assets' returns are driven by the movement of $(F_t)_{t=0}^T$. For convenience, we write $R_0 = \mathbf{0}_d$. The risk-free rate is a constant over time and it is denoted by r_f , and $(R_t^e)_{t=1}^T$ is the excess return process, $R_t^e = R_t - r_f \mathbf{1}_d$.

Denote by $\mathbb{F}^{R,F} = (\mathcal{F}_t^{R,F})_{t=0}^T$ and $\mathbb{F}^F = (\mathcal{F}_t^F)_{t=0}^T$ the filtrations generated by $(R_t, F_t)_{t=0}^T$ and $(F_t)_{t=0}^T$, respectively. We assume that an investor can observe the value of F_t and R_t at each $t \geq 0$, so her information at time t is represented by $\mathcal{F}_t^{R,F}$. In addition, we assume the following:

Assumption 4.1 The random process $(F_t)_{t=0}^T$ is a time-homogeneous Markov process with respect to $\mathbb{F}^{R,F}$.

Assumption 4.1 is important for our optimization. Many return models satisfy this assumption.

Example 4.2

1. *Factor pricing models.* Suppose that $(F_t)_{t=0}^T$ satisfies Assumption 4.1. Let $\mu(F) = r_f \mathbf{1}_d + BF$ and $\sigma(F) = \bar{\sigma}$, where $B \in \mathbb{R}^{d,K}$ and $\bar{\sigma} \in \mathbb{R}^{d,d}$ are constant matrices. Then, the return process can be expressed as

$$R_t = r_f \mathbf{1}_d + BF_t + \bar{\sigma} \epsilon_t, \quad t \geq 1.$$

The above return process is the one of factor pricing models such as the capital asset pricing model (CAPM) and the Fama-French three factor model.

2. *Stochastic volatility models.* Suppose that $K = d$ and that $(F_t)_{t=0}^T$ is specified by

$$\log(F_t) = m + B(\log(F_{t-1}) - m) + \eta_t, \quad t \geq 1,$$

where $m \in \mathbb{R}^d$ is a constant vector and $B \in \mathbb{R}^{d,d}$ is a constant diagonal matrix. $(\eta_t)_{t=1}^T$ is a sequence of d -dimensional i.i.d. random vectors that are independent of $(\epsilon_t)_{t=1}^T$. The return process is defined as

$$R_t = \bar{\mu} + \text{diag}(\sqrt{F_t})C\epsilon_t, \quad t \geq 1,$$

where CC' is a constant correlation matrix. Then, $(F_t)_{t=0}^T$ satisfies Assumption 4.1.

3. *Markov regime-switching models.* Suppose that $(F_t)_{t=0}^T$ is a K -states, time-homogeneous Markov chain and that μ and σ are specified by

$$\mu(F_t) = \mu_i, \quad \sigma(F_t) = \sigma_i, \quad \text{if } F_t \text{ is } i\text{th state.}$$

In addition, $(F_t)_{t=0}^T$ and $(\epsilon_t)_{t=1}^T$ are mutually independent. Then, $(F_t)_{t=0}^T$ satisfies Assumption 4.1.

In usual portfolio optimization, researchers assume that an investor knows the conditional mean $\mu(F_t)$ and variance $\sigma(F_t)(\sigma(F_t))'$, but, the investor needs to estimate the moments in practice. In the robust optimization literature, researchers take into account the estimation error of these moments. Similar to the literature, we introduce two statistical error components. Let $(\theta_t)_{t=0}^{T-1}$ be a $\mathbb{R}^d$ -valued random process of

errors of mean and let $(V_t)_{t=0}^{T-1}$ be a $\mathbb{R}^{d,d}$ -valued process of errors of variance. The one-period-ahead conditional mean and variance with respect to the probability measure representing the investor's belief, denoted by $\mathbb{P}^{\theta,V}$, can be expressed as

$$\mathbb{E}^{\theta,V}[R_{t+1}|\mathcal{F}_t^{R,F}] = \mathbb{E}[R_{t+1}|\mathcal{F}_t^{R,F}] + \theta_t, \quad \text{Var}^{\theta,V}(R_{t+1}|\mathcal{F}_t^{R,F}) = \text{Var}(R_{t+1}|\mathcal{F}_t^{R,F}) + V_t, \quad t \geq 0.$$

Hence, θ_t and V_t can be regarded as statistical errors of conditional mean and variance given $\mathcal{F}_t^{R,F}$ at each t . The assumptions regarding $(\theta_t)_{t=0}^{T-1}$ and $(V_t)_{t=0}^{T-1}$ are

Assumption 4.3

1. For each t , θ_t and V_t are $\mathcal{F}_t^{R,F}$ -measurable.
2. For each t , $\text{Var}(R_{t+1}|\mathcal{F}_t^{R,F}) + V_t$ and $\sigma(F_{t+1})(\sigma(F_{t+1}))' + V_t$ are positive-definite matrices.

From the definitions of $(\theta_t)_{t=0}^{T-1}$ and $(V_t)_{t=0}^{T-1}$, Assumption 4.3 is natural.

The important technical issue is whether the probability measure $\mathbb{P}^{\theta,V}$ exists or not. To construct the probability measure $\mathbb{P}^{\theta,V}$, we introduce the random variable,

$$Z_t = \prod_{s=0}^t \zeta_s, \quad t \geq 0,$$

where

$$\begin{aligned} \zeta_{t+1} = & \sqrt{\frac{\det(\sigma(F_{t+1})(\sigma(F_{t+1}))')}{\det(\sigma(F_{t+1})(\sigma(F_{t+1}))' + V_t)}} \\ & \times \exp \left\{ -\frac{1}{2} \theta_t' \left(\sigma(F_{t+1})(\sigma(F_{t+1}))' + V_t \right)^{-1} \theta_t + \theta_t' \left(\sigma(F_{t+1})(\sigma(F_{t+1}))' + V_t \right)^{-1} \sigma(F_{t+1}) \epsilon_{t+1} \right. \\ & \left. + \frac{1}{2} \epsilon_{t+1}' \left(I_d - (\sigma(F_{t+1}))' \left(\sigma(F_{t+1})(\sigma(F_{t+1}))' + V_t \right)^{-1} \sigma(F_{t+1}) \right) \epsilon_{t+1} \right\}, \quad t \geq 1 \end{aligned}$$

and $\zeta_0 = 1$. Then, the following lemma holds.

Lemma 4.4 Under Assumptions 4.1 and 4.3, there exists a probability measure $\mathbb{P}^{\theta,V}$, such that

$$\mathbb{P}^{\theta,V}(A) = \mathbb{E}[\mathbb{1}_A Z_T], \quad A \in \mathcal{F}_T^{R,F}.$$

Under $\mathbb{P}^{\theta,V}$,

$$\begin{aligned} \mathbb{E}^{\theta,V}[R_{t+1}|\mathcal{F}_t^{R,F}] &= \mathbb{E}[R_{t+1}|\mathcal{F}_t^{R,F}] + \theta_t, \\ \text{Var}^{\theta,V}(R_{t+1}|\mathcal{F}_t^{R,F}) &= \text{Var}(R_{t+1}|\mathcal{F}_t^{R,F}) + V_t, \end{aligned}$$

for all $t \geq 0$. Furthermore, the conditional distribution of F_{t+1} given $\mathcal{F}_t^{R,F}$ under $\mathbb{P}^{\theta,V}$ is the same as the conditional distribution of F_{t+1} given $\mathcal{F}_t^{R,F}$ under $\mathbb{P}$.

The proof of Lemma 4.4 is in Section 4.A. Since $(F_t)_{t=0}^T$ is the time-homogeneous Markov process with respect to $\mathbb{F}^{R,F}$, we can express the conditional mean and variance of R_{t+1} as functions depending on F_t , that is,

$$\begin{aligned} m(F) &:= \mathbb{E}[R_{t+1}|F_t = F] = \mathbb{E}[R_{t+1}|\mathcal{F}_t^{R,F}], \\ A(F) &:= \text{Var}(R_{t+1}|F_t = F) = \text{Var}(R_{t+1}|\mathcal{F}_t^{R,F}), \end{aligned}$$

for all $t \geq 0$. We denote the conditional expected excess return by $m^e(F) = m(F) - r_f \mathbf{1}_d$.

Now, let us formulate the optimization problem of the investor. Let $(X_t)_{t=0}^T$ be a wealth process of the investor. We assume that X_t satisfies a self-financing constraint, that is,

$$X_{t+1} = \left((1 + r_f) + (R_{t+1}^e)' u_t \right) X_t, \quad t \geq 0,$$

where $(u_t)_{t=0}^{T-1}$ is a sequence of d -dimensional portfolio vectors for risky assets. For each u_t , the i th element of u_t represents the weight of the wealth invested in the i th risky asset. The investor buys or sells $X_t(1 - (\mathbf{1}_d)' u_t)$ dollar's risk-free asset at each $t \geq 0$. The investor's objective function is

$$\mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \left(\mathbb{E}^{\theta,V}[X_{t+1}|\mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V}(X_{t+1}|\mathcal{F}_t^{R,F}) \right) \right], \quad (4.2.1)$$

where $\hat{\gamma} := (\hat{\gamma}_t)_{t=0}^{T-1}$ is a positive and $\mathbb{F}^F$ -adapted process which represents the investor's trade-off between the expected return and associated risks. We emphasize that $\hat{\gamma}$ is not only $\mathbb{F}^{R,F}$ -adapted, but also $\mathbb{F}^F$ -adapted. The $\mathbb{F}^F$ -adaptedness of $\hat{\gamma}$ is important in the derivation of the explicit form of the value function.

The trade-off parameter is divided by X_t whereas in the other standard problem, it is not divided. This parameterization is used by Björk, Murgoci, and Zhou (2014).

They provide two natural interpretations of the parameterization. One is an adjustment of units. The unit of the conditional expected wealth is (*dollar*), whereas the unit of the conditional variance is (*dollar*)². So the variance needs to be divided by wealth in order to measure the objective function in dollars. Another interpretation is to measure the objective function by the rate of return. We replace the wealth in the objective by the gross return of the wealth, namely, we consider

$$\begin{aligned} & \mathbb{E}^{\theta,V} \left[\frac{X_{t+1}}{X_t} \middle| \mathcal{F}_t^{R,F} \right] - \frac{\widehat{\gamma}_t}{2} \text{Var}^{\theta,V} \left(\frac{X_{t+1}}{X_t} \middle| \mathcal{F}_t^{R,F} \right) \\ &= \frac{1}{X_t} \left\{ \mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\widehat{\gamma}_t}{2X_t} \text{Var}^{\theta,V} (X_{t+1} | \mathcal{F}_t^{R,F}) \right\}. \end{aligned}$$

Then, Björk et al. (2014) argue that the equilibrium in the above objective is the same as that in the following objective,

$$\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\widehat{\gamma}_t}{2X_t} \text{Var}^{\theta,V} (X_{t+1} | \mathcal{F}_t^{R,F}).$$

However, in our settings, the optimal portfolios are different between these two objectives. Our settings, therefore, should be interpreted as the adjustment of units.

We can also interpret our objective function as an approximation of a certainty equivalent of a CRRA utility. Let

$$u_t(x) := \begin{cases} \frac{x^{1-\widehat{\gamma}_t} - 1}{1 - \widehat{\gamma}_t}, & \text{if } \widehat{\gamma}_t \neq 1, \\ \log x, & \text{if } \widehat{\gamma}_t = 1, \end{cases}$$

where $\widehat{\gamma}_t > 0$ is the constant relative risk aversion coefficient at time t . Let us denote by $\bar{c}_t(X)$ the certainty equivalent of a random variable X under the probability measure $\mathbb{P}^{\theta,V}$, that is

$$\bar{c}_t(X) := (u_t)^{-1} \left(\mathbb{E}^{\theta,V} [u_t(X) | \mathcal{F}_t^{R,F}] \right),$$

where $(u_t)^{-1}$ is the functional inverse of u_t . Then, it is well known that $\bar{c}_t(X)$ can be approximated as follows.

$$\bar{c}_t(X_{t+1}) \approx \mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}] + \frac{1}{2} \frac{u_t''(\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}])}{u_t'(\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}])} \text{Var}^{\theta,V} (X_{t+1} | \mathcal{F}_t^{R,F}),$$

where u_t' and u_t'' are the first and the second derivatives of u_t , respectively. Now, we assume $X_t \approx \mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}]$. Then, since u_t is a CRRA utility, we have

$$\frac{u_t''(\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}])}{u_t'(\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}])} = -\frac{\widehat{\gamma}_t}{\mathbb{E}^{\theta,V} [X_{t+1} | \mathcal{F}_t^{R,F}]} \approx -\frac{\widehat{\gamma}_t}{X_t}.$$

Therefore, we have

$$\bar{c}_t(X_{t+1}) \approx E^{\theta,V}[X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V}(X_{t+1} | \mathcal{F}_t^{R,F}).$$

Hence, the expectation of the discounted sum of the certainty equivalents can be approximated as follows:

$$E^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \bar{c}_t(X_{t+1}) \right] \approx E^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \left(E^{\theta,V}[X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V}(X_{t+1} | \mathcal{F}_t^{R,F}) \right) \right].$$

The above approximation coincides with our objective function (4.2.1). Furthermore, the time varying $\hat{\gamma}$ implies that the relative risk aversion coefficient varies over time.

Note that, we assume that X_t is almost surely strictly positive for all $t \geq 0$. Strictly speaking, it does not hold when permitting short selling, whereas it holds if short selling is not allowed. Nevertheless, we assume the strict positivity of X_t in all cases for mathematical convenience. Furthermore, the division of the trade-off parameter by X_t is a crucial assumption for the derivation of the explicit solution to our problem. Without it, we can not derive the explicit solution. We again refer to this assumption in the problem without the risk-free asset.

By the Markov property of $(F_t)_{t=0}^T$, the objective under $\mathbb{P}^{\theta,V}$ can be expressed as

$$\begin{aligned} & E^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \left(E^{\theta,V}[X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V}(X_{t+1} | \mathcal{F}_t^{R,F}) \right) \right] \\ &= E^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) \right], \end{aligned}$$

where $c_t : \mathbb{R}^K \times \mathbb{R}^d \times \mathbb{R}^{d,d} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function such that

$$c_t(F, \theta, V, u) = 1 + r_f + \left(m^e(F) + \theta \right)' u - \frac{\hat{\gamma}_t}{2} u'(A(F) + V)u.$$

Now, we construct the optimization problem with multiple priors. The investor selects the portfolio maximizing the expected utility of the worst case. This means that the investor faces on the following maximization problem:

$$\begin{aligned} V_T(x, f) &= \max_{(u_t)_{t=0}^{T-1} \in \mathcal{A}_T^u} \min_{(\theta_t, V_t)_{t=0}^{T-1} \in \mathcal{A}_T^{\theta,V}} E^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) \right] \\ &\quad \text{subject to } X_{t+1} = \left(1 + r_f + (R_{t+1}^e)' u_t \right) X_t, \quad t \geq 0, \\ &\quad X_0 = x, \quad F_0 = f, \end{aligned} \tag{4.2.2}$$

where $\mathcal{A}_T^u$ is a set of admissible portfolios and $\mathcal{A}_T^{\theta,V}$ is a set of admissible errors, $(\theta_t)_{t=0}^{T-1}$ and $(V_t)_{t=0}^{T-1}$. The set of admissible portfolios can be expressed as

$$\mathcal{A}_T^u = \{u = (u_t)_{t=0}^{T-1} \mid u \text{ is a } \mathbb{R}^d\text{-valued and } \mathbb{F}^{R,F}\text{-adapted process.}\}.$$

So, the investor is permitted to short sell. We later analyze the case when the investor is prohibited from short selling.

The set of admissible errors is characterized by two inequalities. For all $(\theta_t, V_t)_{t=0}^{T-1} \in \mathcal{A}_T^{\theta,V}$, $(\theta_t)_{t=0}^{T-1}$ is a $\mathbb{R}^d$ -valued process and $(V_t)_{t=0}^{T-1}$ is an $\mathbb{R}^{d \times d}$ -valued process. Furthermore, they satisfy Assumption 4.3 and

$$(\eta_t^\theta)^2 \geq \theta_t' (A(F_t) + V_t)^{-1} \theta_t, \quad (4.2.3)$$

$$(\eta_t^V)^2 \text{tr}((A(F_t))^2) \geq \text{tr}(V_t^2), \quad (4.2.4)$$

where $(\eta_t^\theta)_{t=0}^{T-1}$ and $(\eta_t^V)_{t=0}^{T-1}$ are positive $\mathbb{F}^F$ -adapted processes and $\text{tr}(A)$ is the trace of matrix A .

The inequality (4.2.3) represents a confidence interval in statistics. This idea is introduced by Garlappi et al. (2007). We can express $\theta_t'(A(F_t) + V_t)^{-1} \theta_t$ as

$$\begin{aligned} & \theta_t' \left(\text{Var}^{\theta,V}(R_{t+1} | \mathcal{F}_t^{R,F}) \right)^{-1} \theta_t = \\ & \left(\mathbb{E}[R_{t+1} | \mathcal{F}_t^{R,F}] - \mathbb{E}^{\theta,V}[R_{t+1} | \mathcal{F}_t^{R,F}] \right)' \left(\text{Var}^{\theta,V}(R_{t+1} | \mathcal{F}_t^{R,F}) \right)^{-1} \left(\mathbb{E}[R_{t+1} | \mathcal{F}_t^{R,F}] - \mathbb{E}^{\theta,V}[R_{t+1} | \mathcal{F}_t^{R,F}] \right), \end{aligned}$$

so $\theta_t'(A(F_t) + V_t)^{-1} \theta_t$ is the F statistic of the estimated means in the null hypothesis of $\mathbb{E}[R_{t+1} | \mathcal{F}_t^{R,F}] = \mathbb{E}^{\theta,V}[R_{t+1} | \mathcal{F}_t^{R,F}]$.

Garlappi et al. (2007) characterize this constraint of $(\theta_t)_{t=0}^{T-1}$ as the confidence interval of the estimated mean vector. Their discussion is justified if the return vectors of the risky assets are i.i.d.. However, the return vectors in our model are *not* i.i.d. in general. Therefore, our constraint (4.2.3) does not represent the confidence interval. Since the financial models are usually estimated by using the maximum likelihood methods, one should use the inverse of the information matrix as the covariance matrix in our constraint (4.2.3) in order to characterize this constraint as the confidence interval. However, in this chapter, we use this constraint, similar to Garlappi et al. (2007), to reduce the mathematical complexity.

The inequality (4.2.4) can be expressed as

$$\eta_t^V \geq \frac{\text{tr}(V_t^2)}{\text{tr}((A(F_t))^2)} = \frac{\|\text{Var}^{\theta,V}(R_{t+1}|\mathcal{F}_t^{R,F}) - \text{Var}(R_{t+1}|\mathcal{F}_t^{R,F})\|^2}{\|\text{Var}(R_{t+1}|\mathcal{F}_t^{R,F})\|^2},$$

for all $t \geq 0$. Therefore, $(\eta_t^V)_{t=0}^{T-1}$ represents the upper boundary of the ratio of the least square error to the conditional variance. Ledoit and Wolf (2003) consider the shrinkage estimator minimizing the following objective:

$$\min_{\alpha} \text{tr} \left((\widehat{\Sigma}(\alpha) - \Sigma)^2 \right) = \min_{\alpha} \|\widehat{\Sigma}(\alpha) - \Sigma\|^2,$$

where Σ is an actual covariance matrix of returns and $\widehat{\Sigma}(\alpha)$ is a shrunk estimated covariance matrix depending on the parameter α . We use the above objective in the inequality (4.2.4). Therefore, the inequality (4.2.4) represents the upper boundary of the objective in Ledoit and Wolf (2003).

As in the case of a unique prior, we can apply the dynamic programming procedure to the optimization problem (4.2.2).

Theorem 4.5 Let $(Y_t^T)_{t=0}^{T-1}$ be a sequence of random variables, such that

$$\begin{aligned} Y_{T-1}^T &= \max_{u_{T-1} \in \mathbb{R}^d} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta,V} |_{T-1}^{T-1}} \{c_{T-1}(F_{T-1}, \theta_{T-1}, V_{T-1}, u_{T-1})\}, \\ Y_t^T &= \max_{u_t \in \mathbb{R}^d} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta,V} |_t^t} \left\{ c_t(F_t, \theta_t, V_t, u_t) + \delta \mathbb{E}^{\theta,V} \left[Y_{t+1}^T (1 + r_f + (R_{t+1}^e)' u_t) \mid \mathcal{F}_t^{R,S} \right] \right\}, \end{aligned}$$

for all $0 \leq t \leq T-2$, where $\mathcal{A}_T^{\theta,V} |_t^t$ is a set of t -th elements in $\mathcal{A}_T^{\theta,V}$. Then, the above random sequence exists, and

$$V_T(x, f) = \mathbb{E} [Y_0^T \mid F_0 = f] x, \quad x \in \mathbb{R}, f \in \mathbb{R}^K.$$

Furthermore, the optimal portfolio process $(u_t^*)_{t=0}^{T-1}$ derived by the above dynamic programming is time-consistent.

The explicit solution is provided by the following proposition.

Proposition 4.6 Let

$$\begin{aligned} Y_T^T &= 0, \\ m_t^e &= \mathbb{E} \left[(1 + \delta Y_{t+1}^T) (\mu(F_{t+1}) - r_f \mathbf{1}_d) \mid \mathcal{F}_t^{R,F} \right], \quad g_t^e = 1 + \delta \mathbb{E} \left[Y_{t+1}^T \mid \mathcal{F}_t^{R,F} \right], \\ a_t^e &= (m_t^e)' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^e, \end{aligned}$$

for all $0 \leq t \leq T - 1$. Then, $(Y_t^T)_{t=0}^{T-1}$ in Theorem 4.5 can be expressed as

$$Y_t^T = (1 + r_f)g_t^e + \frac{((\sqrt{a_t^e} - |g_t^e|\eta_t^\theta)^+)^2}{2\hat{\gamma}_t},$$

and the optimal portfolio is

$$u_t^* = \frac{1}{\hat{\gamma}_t} \left(1 - \frac{|g_t^e|\eta_t^\theta}{\sqrt{a_t^e}} \right)^+ \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^e. \quad (4.2.5)$$

The proofs of Theorem 4.5 and Proposition 4.6 are in Section 4.B. By the construction, $(Y_t^T)_{t=0}^{T-1}$ is $\mathbb{F}^F$ -adapted. Furthermore, if $\hat{\gamma}_t, \eta_t^\theta$ and η_t^V depend only on the value of F_t for all $0 \leq t \leq T - 1$, Y_t^T also depends only on the value of F_t . Therefore, we can easily compute it numerically using the Markov property of $(F_t)_{t=0}^T$.

The optimal portfolio (4.2.5) is similar to the ambiguity averse minimum-variance portfolio by Pinar (2014). However, the multiple priors for mean and variance affects the portfolio (4.2.5) whereas the ambiguity averse minimum-variance portfolio is only affected by the multiple priors for mean. Therefore, the ambiguity averse minimum-variance portfolio is a special case of our portfolios.

Note that, we obtain similar results to Proposition 4.6 even if r_f varies over time. Let $r_{f,t+1}$ be a time-varying risk-free rate at time $t + 1$ such that

$$r_{f,t+1} = h(F_t), \quad t \geq 0,$$

where h is a measurable function from $\mathbb{R}^K$ onto $\mathbb{R}$. The reason why the risk-free rate at time $t + 1$ depends on the value of F_t is that the risk-free rate needs to be determined before investment decision. Then, the following portfolio u_t^{TV*} is optimal.

$$u_t^{TV*} = \frac{1}{\hat{\gamma}_t} \left(1 - \frac{|g_t^{TVe}|\eta_t^\theta}{\sqrt{a_t^{TVe}}} \right)^+ \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^{TVe},$$

where

$$\begin{aligned} Y_T^{TVT} &= 0, \quad m_t^{TVe} = \mathbb{E} \left[(1 + \delta Y_{t+1}^{TVT}) (\mu(F_{t+1}) - h(F_t) \mathbf{1}_d) | \mathcal{F}_t^{R,F} \right], \\ g_t^{TVe} &= 1 + \delta \mathbb{E} \left[Y_{t+1}^{TVT} | \mathcal{F}_t^{R,F} \right], \quad a_t^{TVe} = (m_t^{TVe})' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^{TVe}, \\ Y_t^{TVT} &= (1 + h(F_t)) g_t^{TVe} + \frac{((\sqrt{a_t^{TVe}} - |g_t^{TVe}|\eta_t^\theta)^+)^2}{2\hat{\gamma}_t}. \end{aligned}$$

It is clear that Y_t^{TVT} is $\mathcal{F}_t^{R,F}$ -measurable for all $t \geq 0$. In addition, u^{TV*} is admissible. The proof is similar to that of Proposition 4.6, so we omit it. The above result indicates that if the movement of $r_{f,t}$ is independent of ϵ_t , then the optimal portfolio with the constant risk-free rate can be naturally extended into the optimal portfolio with the time-varying risk-free rate.

We next consider an optimization problem with multiple priors without the risk-free asset. In this case, the investor's wealth updating formula is

$$X_{t+1} = \left(1 + (R_{t+1})'u_t\right)X_t, \quad 0 \leq t \leq T-1,$$

where u_t satisfies $\mathbf{1}_d' u_t = 1$. The objective can be expressed as

$$\begin{aligned} & \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t \left(\mathbb{E}^{\theta,V}[X_{t+1} | \mathcal{F}_t^{R,F}] - \frac{\hat{\gamma}_t}{2X_t} \text{Var}^{\theta,V}(X_{t+1} | \mathcal{F}_t^{R,F}) \right) \right] \\ &= \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t \hat{c}_t(F_t, \theta_t, V_t, u_t) \right], \end{aligned}$$

where $\hat{c}_t : \mathbb{R}^K \times \mathbb{R}^d \times \mathbb{R}^{d,d} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function such that

$$\hat{c}_t(F, \theta, V, u) = 1 + \left(m(F) + \theta\right)' u - \frac{\hat{\gamma}_t}{2} u'(A(F) + V)u.$$

Hence, the objective without the risk-free asset has the same form as the objective with the risk-free asset. This implies that Theorem 4.5 can be applied to the problem without the risk-free asset.

The optimization problem without the risk-free asset is formulated as

$$\begin{aligned} V_T^{worf}(x, f) &= \max_{(u_t)_{t=0}^{T-1} \in \tilde{\mathcal{A}}_T^u} \min_{(\theta_t, V_t)_{t=0}^{T-1} \in \mathcal{A}_T^{\theta,V}} \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t \hat{c}_t(F_t, \theta_t, V_t, u_t) \right] \\ &\quad \text{subject to } X_{t+1} = \left(1 + (R_{t+1})'u_t\right)X_t, \quad t \geq 0, \\ &\quad X_0 = x, \quad F_0 = f, \end{aligned} \tag{4.2.6}$$

where the set of admissible portfolios $\tilde{\mathcal{A}}_T^u$ is

$$\tilde{\mathcal{A}}_T^u = \{u = (u_t)_{t=0}^{T-1} \mid u \text{ is a } \mathbb{R}^d\text{-valued and } \mathbb{F}^{R,F}\text{-adapted process and } \mathbf{1}_d' u_t = 1\}.$$

The following proposition provides the optimal portfolios and the value functions.

Proposition 4.7 For all $0 \leq t \leq T - 1$, let

$$\begin{aligned}\widehat{Y}_T^T &= 0, \\ g_t &= 1 + \delta \mathbb{E}[\widehat{Y}_{t+1}^T \mid \mathcal{F}_t^{R,F}], \quad m_t = \mathbb{E} \left[(1 + \delta \widehat{Y}_{t+1}^T) \mu(F_{t+1}) \mid \mathcal{F}_t^{R,F} \right], \\ a_t^{worf} &= m'_t \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t, \quad c_t^{worf} = m'_t \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d, \\ b_t^{worf} &= \mathbf{1}_d' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d, \quad d_t^{worf} = a_t^{worf} b_t^{worf} - (c_t^{worf})^2, \\ \widehat{Y}_t^T &= g_t + \frac{c_t^{worf} - (\widehat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)}{b_t^{worf}} + \frac{\widehat{\gamma}_t}{2} (\psi_t^*)^2,\end{aligned}$$

where ψ_t^* is a unique positive solution of the polynomial equation.

$$b_t^{worf} (\psi_t^*)^2 = \frac{d_t^{worf}}{(\widehat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)^2} + 1. \quad (4.2.7)$$

Then, the value function is

$$V_T^{worf}(x, f) = \mathbb{E}[\widehat{Y}_0^T \mid F_0 = f]x, \quad (x, f) \in \mathbb{R}_+ \times \mathbb{R}^K,$$

and the time-consistent optimal portfolio is

$$u_t^* = \frac{1}{\widehat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \left(m_t - \frac{c_t^{worf} - (\widehat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)}{b_t^{worf}} \mathbf{1}_d \right),$$

for all $0 \leq t \leq T - 1$.

The proof of Proposition 4.7 is in Section 4.C. The equation (4.2.7) is essentially the same as the equation (A11) in Garlappi et al. (2007), so it must have a unique positive solution by Garlappi et al. (2007). We give the details of the proofs of the existence of the unique positive solution to the equation (4.2.7) in Section 4.C.

The main difference between Garlappi et al. (2007) and our model is the covariance matrix. The optimal solution in Garlappi et al. (2007) can be regarded as the case when η_t^V is zero in our solution. In our case, the multiple priors for covariance affects the optimal portfolio.

The division of $\widehat{\gamma}_t$ by X_t has an important role in the problem without the risk-free asset. Without this division, we can not derive the explicit solution since the solution of the polynomial (4.2.7) depends on the value of X_t . This implies that the t -step value function $\widehat{Y}_t^T$, also depends on the value of X_t . Then, it is difficult to compute

the conditional expected value of $\widehat{Y}_t^T$ given $\mathcal{F}_{t-1}^{R,F}$ under the probability measure $\mathbb{P}^{\theta,V}$. Hence, the trade-off parameter's division by the wealth is crucial.

Now we introduce a short-selling constraint. Let $\widetilde{\mathcal{A}}_{+T}^u$ be a set of admissible portfolios with a short-selling constraint such that

$$\widetilde{\mathcal{A}}_{+T}^u = \left\{ u = (u_t)_{t=0}^{T-1} \mid u \text{ is a } \mathbb{R}^d\text{-valued and } \mathbb{F}^{R,F}\text{-adapted process, } \mathbf{1}_d' u_t = 1 \text{ and } u_t \in \mathbb{R}_+^d \right\}.$$

Then, the dynamic optimization problem with a short-selling constraint can be expressed as

$$\begin{aligned} V_{+T}^{worf}(x, f) = & \max_{(u_t)_{t=0}^{T-1} \in \widetilde{\mathcal{A}}_{+T}^u} \min_{(\theta_t, V_t)_{t=0}^{T-1} \in \mathcal{A}_T^{\theta,V}} \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t \widehat{c}_t(F_t, \theta_t, V_t, u_t) \right] \\ & \text{subject to } X_{t+1} = \left(1 + (R_{t+1})' u_t \right) X_t, \quad t \geq 0, \\ & X_0 = x, \quad F_0 = f. \end{aligned} \quad (4.2.8)$$

Then, the following proposition holds.

Proposition 4.8 For all $0 \leq t \leq T-1$, let

$$\begin{aligned} \widehat{Y}_T^{+T} &= 0, \\ g_t^+ &= 1 + \delta \mathbb{E}[\widehat{Y}_{t+1}^{+T} \mid \mathcal{F}_t^{R,F}], \quad m_t^+ = \mathbb{E} \left[(1 + \delta \widehat{Y}_{t+1}^{+T}) \mu(F_{t+1}) \mid \mathcal{F}_t^{R,F} \right], \\ \widehat{Y}_t^{+T} &= \max_{u_t \in \overline{\mathcal{C}}^+} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta,V} |^t_t} \left\{ g_t^+ + \left(m_t^+ + g_t^+ \theta_t \right)' u_t - \frac{\widehat{\gamma}_t}{2} u_t' \left(A(F_t) + V_t \right) u_t \right\}, \end{aligned}$$

where $\overline{\mathcal{C}}^+$ is a set of portfolios such that

$$\overline{\mathcal{C}}^+ = \left\{ \phi \in \mathbb{R}_+^d \mid 1 = \mathbf{1}_d' \phi \right\}.$$

Then, the value function is

$$V_{+T}^{worf}(x, f) = \mathbb{E}[\widehat{Y}_0^{+T} \mid F_0 = f]x,$$

and the optimal portfolio at time t is

$$u_t^* = \arg \max_{u_t \in \overline{\mathcal{C}}^+} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta,V} |^t_t} \left\{ g_t^+ + \left(m_t^+ + g_t^+ \theta_t \right)' u_t - \frac{\widehat{\gamma}_t}{2} u_t' \left(A(F_t) + V_t \right) u_t \right\}.$$

The proof of Proposition 4.8 is in Section 4.C. The t stage problem,

$$\max_{u_t \in \mathcal{C}^+} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V} |_t^t} \left\{ g_t^+ + (m_t^+ + g_t^+ \theta_t)' u_t - \frac{\hat{\gamma}_t}{2} u_t' (A(F_t) + V_t) u_t \right\},$$

can be rewritten as follows,

$$\begin{aligned} & \max_{u_t \in \mathcal{C}^+} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V} |_t^t} \left\{ g_t^+ + (m_t^+ + g_t^+ \theta_t)' u_t - \frac{\hat{\gamma}_t}{2} u_t' (A(F_t) + V_t) u_t \right\} = \\ & \max_{u_t \in \mathcal{C}^+} \left\{ g_t^+ + (m_t^+)' u_t - \frac{\hat{\gamma}_t}{2} u_t' (A(F_t) + \eta_t \|A(F_t)\| I_d) u_t - |g_t^+| \eta_t^\theta \sqrt{u_t' (A(F_t) + \eta_t \|A(F_t)\| I_d) u_t} \right\}. \end{aligned}$$

Therefore, the solution to the above problem exists since the above objective is continuous in u_t .

4.3 Limiting Behaviors

In this section, we study limiting behaviors of optimal portfolios. We have postulated that the parameters $(\hat{\gamma}_t)_{t=0}^{T-1}$, $(\eta_t^\theta)_{t=0}^{T-1}$, and $(\eta_t^V)_{t=0}^{T-1}$ are non-negative. We therefore consider the behaviors of the optimal portfolios when these parameters go to infinity.

We first consider the optimal portfolio with the risk-free asset. By Proposition 4.6, the optimal portfolio with the risk-free asset is

$$u_t^* = \frac{1}{\hat{\gamma}_t} \left(1 - \frac{|g_t^e| \eta_t^\theta}{\sqrt{a_t^e}} \right)^+ \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^e,$$

for all $t \geq 0$. Then, m_t^e and g_t^e are determined independently of the values of $\hat{\gamma}_t$, η_t^θ and η_t^V . a_t^e can be expressed as

$$\begin{aligned} a_t^e &= (m_t^e)' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^e \\ &= \frac{1}{\eta_t^V} (m_t^e)' \left(\frac{1}{\eta_t^V} A(F_t) + \|A(F_t)\| I_d \right)^{-1} m_t^e, \end{aligned}$$

therefore, $a_t^e \rightarrow 0$ as $\eta_t^V \rightarrow \infty$. This implies that for any fixed $\hat{\gamma}_t$ and η_t^θ ,

$$\lim_{\eta_t^V \rightarrow \infty} u_t^* = \mathbf{0}_d.$$

So, if the investor has absolutely no confidence in the estimated conditional variance, she quits investing in the risky assets. Similarly, if $\eta_t^\theta \rightarrow \infty$, then

$$\lim_{\eta_t^\theta \rightarrow \infty} \left(1 - \frac{|g_t^e| \eta_t^\theta}{\sqrt{a_t^e}} \right)^+ = 0$$

for any fixed η_t^V . Therefore, u_t^* converges to $\mathbf{0}_d$ when η_t^θ tends to infinity. These behaviors are natural since the investor with strong doubts of the correctnesses of the risky assets' parameters prefers investing in the risk-free asset which yields a deterministic return r_f . Thus, the effects of multiple priors are similar to the risk-averse behavior if the risk-free asset is present. However, with the absence of risk-free asset, limiting portfolios are two famous portfolios, namely, the global minimum-variance portfolio and the equally weighted portfolio.

Proposition 4.9

1. *With the risk-free asset.* If $\hat{\gamma}_t, \eta_t^\theta$ or η_t^V tends to infinity, then the investor invests all wealth in the risk-free asset.
2. *Without the risk-free asset.*

- (a) If $\hat{\gamma}_t$ or η_t^θ tends to infinity and if η_t^V is fixed, then the investor invests in the global minimum-variance portfolio under $\mathbb{P}^{\theta, V}$, that is

$$\lim_{\eta_t^\theta \text{ or } \hat{\gamma}_t \rightarrow \infty} u_t^* = \frac{1}{\mathbf{1}_d' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d,$$

when permitting short selling. When short selling is not allowed, then the optimal portfolio again converges to the global minimum-variance portfolio.

$$\lim_{\eta_t^\theta \rightarrow \infty} u_t^* = u_t^{+GMV} := \arg \min_{u \in \mathcal{C}^+} \left\{ u' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u \right\}.$$

- (b) If η_t^V tends to infinity, then the investor invests in the equally weighted portfolio whether short selling is allowed or not, that is,

$$\lim_{\eta_t^V \rightarrow \infty} u_t^* = \frac{1}{d} \mathbf{1}_d.$$

Proof By Proposition 4.7, the optimal portfolio without the risk-free asset is

$$u_t^* = \frac{1}{\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \left(m_t - \frac{c_t^{worf} - (\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)}{b_t^{worf}} \mathbf{1}_d \right),$$

where

$$\begin{aligned} a_t^{worf} &= m_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t, \quad c_t^{worf} = m_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d, \\ b_t^{worf} &= \mathbf{1}_d' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d, \quad d_t^{worf} = a_t^{worf} b_t^{worf} - (c_t^{worf})^2, \end{aligned}$$

and ψ_t^* is the unique solution of the following quartic equation,

$$b_t^{worf}(\psi_t^*)^2 = \frac{d_t^{worf}}{(\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)^2} + 1.$$

Similar to the case with the risk-free asset, m_t and g_t are determined independently of the values of $\hat{\gamma}_t, \eta_t^\theta$ and η_t^V , so we can regard them as constants. To study the limiting behavior of u_t^* , it is important to use the quartic equation for ψ_t^* .

The quartic equation can be transformed as follows:

$$\left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \frac{d_t^{worf}}{b_t^{worf}(\psi_t^*)^2 - 1}. \quad (4.3.1)$$

The benefit of the above expression is that it separates the terms depending on η_t^θ and η_t^V . In addition,

$$\left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \frac{d_t^{worf}}{b_t^{worf}(\psi_t^*)^2 - 1} \geq \hat{\gamma}_t^2 > 0, \quad (4.3.2)$$

for all $\hat{\gamma}_t, \eta_t^\theta$, and η_t^V . For any fixed $\hat{\gamma}_t$ and η_t^V , two limits of the equation (4.3.1) as $\eta_t^\theta \rightarrow \infty$ can be expected. One is some positive constant $C > 0$. Then, we have

$$\lim_{\eta_t^\theta \rightarrow \infty} \left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \lim_{\eta_t^\theta \rightarrow \infty} \frac{d_t^{worf}}{b_t^{worf}(\psi_t^*)^2 - 1} = C.$$

The right equality follows that

$$\lim_{\eta_t^\theta \rightarrow \infty} \psi_t^* = C_1$$

for some constant $C_1 > 0$. However, this implies

$$\lim_{\eta_t^\theta \rightarrow \infty} \left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \infty.$$

Hence it is a contradiction. Therefore, the equation (4.3.1) diverges when η_t^θ tends to infinity. Indeed,

$$\lim_{\eta_t^\theta \rightarrow \infty} \frac{d_t^{worf}}{b_t^{worf}(\psi_t^*)^2 - 1} = \infty$$

implies that $(\psi_t^*)^2 \rightarrow 1/b_t^{worf}$. Then,

$$\lim_{\eta_t^\theta \rightarrow \infty} \left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \infty.$$

This is consistent. Therefore, the limiting portfolio with respect to η_t^θ is

$$\lim_{\eta_t^\theta \rightarrow \infty} u_t^* = \frac{1}{b_t^{worf}} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d.$$

This limiting portfolio is the global minimum-variance portfolio under $\mathbb{P}^{\theta, V}$.

We consider the limiting behavior with respect to η_t^V . d_t^{worf} and b_t^{worf} can be expressed as

$$\begin{aligned} d_t^{worf} &= \frac{1}{(\eta_t^V)^2} d_t^\eta, & \lim_{\eta_t^V \rightarrow \infty} d_t^\eta &= d_0 > 0, \\ b_t^{worf} &= \frac{1}{\eta_t^V} b_t^\eta, & \lim_{\eta_t^V \rightarrow \infty} b_t^\eta &= b_0 > 0. \end{aligned}$$

Then, the RHS in the equation (4.3.1) becomes

$$\frac{d_t^{worf}}{b_t^{worf}(\psi_t^*)^2 - 1} = \frac{d_t^\eta}{b_t^\eta \eta_t^V \left((\psi_t^*)^2 - \eta_t^V / b_t^\eta \right)}.$$

We first assume that the LHS in the equation (4.3.1) diverges, that is,

$$\lim_{\eta_t^V \rightarrow \infty} \left(\hat{\gamma}_t + \frac{|g_t| \eta_t^\theta}{\psi_t^*} \right)^2 = \infty.$$

This implies that ψ_t^* converges to 0 when $\eta_t^V \rightarrow \infty$. However, for the RHS in the equation (4.3.1),

$$\lim_{\eta_t^V \rightarrow \infty} \frac{d_t^\eta}{b_t^\eta \eta_t^V \left((\psi_t^*)^2 - \eta_t^V / b_t^\eta \right)} = 0.$$

This is a contradiction. It follows that there exists some positive constant $C_2 > 0$ such that

$$\lim_{\eta_t^V \rightarrow \infty} \frac{d_t^\eta}{b_t^\eta \eta_t^V \left((\psi_t^*)^2 - \eta_t^V / b_t^\eta \right)} = C_2.$$

Then, $\eta_t^V \left((\psi_t^*)^2 - \eta_t^V / b_t^\eta \right) \rightarrow C_3$ for some positive constant $C_3 > 0$ as $\eta_t^V \rightarrow \infty$. This implies that ψ_t^* diverges. Then the LHS in the equation (4.3.1) also converges to $\hat{\gamma}_t^2$, so it is consistent with our expectation.

By the definition, it holds that

$$\lim_{\eta_t^V \rightarrow \infty} \frac{c_t^{worf}}{b_t^{worf}} = C_4,$$

for some constant C_4 . Furthermore, we have

$$\lim_{\eta_t^V \rightarrow \infty} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} x = \mathbf{0}_d,$$

for any $x \in \mathbb{R}^d$. Moreover, $\hat{\gamma}_t + |g_t|\eta_t^\theta/\psi_t^*$ converges to $\hat{\gamma}_t$ as $\eta_t^V \rightarrow \infty$. Hence,

$$\lim_{\eta_t^V \rightarrow \infty} \frac{1}{\hat{\gamma}_t + |g_t|\eta_t^\theta/\psi_t^*} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \left(m_t - \frac{c_t^{worf}}{b_t^{worf}} \mathbf{1}_d \right) = \mathbf{0}_d.$$

On the other hand,

$$\lim_{\eta_t^V \rightarrow \infty} \frac{\left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d}{b_t^{worf}} = \lim_{\eta_t^V \rightarrow \infty} \frac{\left(A(F_t)/\eta_t^V + \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d}{\mathbf{1}_d' \left(A(F_t)/\eta_t^V + \|A(F_t)\| I_d \right)^{-1} \mathbf{1}_d} = \frac{1}{d} \mathbf{1}_d.$$

Finally, it holds that

$$\lim_{\eta_t^V \rightarrow \infty} u_t^* = \frac{1}{d} \mathbf{1}_d.$$

Hence, the limiting portfolio as $\eta_t^V \rightarrow \infty$ is the equally weighted portfolio.

In the case when short selling is not allowed, it holds that $u_t^* \rightarrow \mathbf{1}_d/d$ as $\eta_t^V \rightarrow \infty$ since $\mathbf{1}_d/d$ is an interior point of $\overline{\mathcal{C}^+}$. When $\eta_t^\theta \rightarrow \infty$,

$$\lim_{\eta_t^\theta \rightarrow \infty} u_t^* = u_t^{+GMV} = \arg \min_{u \in \mathcal{C}^+} \left\{ u' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u \right\},$$

since the objective function is proportional to

$$\frac{1}{\eta_t^\theta} (m_t^+)' u_t - \frac{\hat{\gamma}_t}{2\eta_t^\theta} u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t - |g_t^+| \sqrt{u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t}$$

and the effects of the first two terms become small when η_t^θ becomes large. $\square$

Interestingly, the optimal portfolio u_t^* also converges to the equally weighted portfolio when $\eta_t^V \rightarrow \infty$ and $\eta_t^\theta = 0$. This means that the investor chooses the equally weighted portfolio under the strong uncertainty for variance even if she believes that the errors of the estimated conditional expected returns do not exist. Therefore, we conclude that the uncertainty about variances has a stronger impact on the investor's portfolio selection than the uncertainty about means.

However, the above result slightly differs from natural intuition. The readers may think that the investor tends to choose the asset having the highest expected return if the degree of suspicion for variances η_t^V , is sufficiently large. This question can be resolved through the inequality constraint (4.2.3). If η_t^V is sufficiently large, then the error of variance V_t , is also large. This implies that

$$\theta_t' \left(A(F_t) + V_t \right)^{-1} \theta_t$$

is very close to 0. Then, the mean errors θ_t in the investor's belief can take an arbitrary value if $\eta^V \rightarrow \infty$. Therefore, the investor's worst scenario is that the means take large negative values which are the same across all risky assets. Hence, the investor chooses the equally weighted portfolio if $\eta^V \rightarrow \infty$.

Without the risk-free asset, the limiting case $\eta_t^\theta \rightarrow \infty$ is the same as the special case of the 2-norm-constrained optimal portfolio in DeMiguel et al. (2009a). However, our framework admits dynamic changes of covariance, so the conditional covariance matrix appears in the limiting portfolio, whereas the unconditional covariance matrix appears in DeMiguel et al. (2009a) since they consider the static optimization.

Unfortunately, without the risk-free asset, the limiting strategies are not applicable straightforwardly since the value functions diverge. These limiting portfolios are only admissible in the initial period of the investment horizon. However, if η_t^θ and η_t^V are sufficiently large, then the optimal portfolios are very similar to the limiting portfolios, so we can use the optimal portfolios with large η_t^θ and η_t^V as proxies of the limiting portfolios. In Section 4.4, we study the cases of large η_t^θ and η_t^V .

Remark 4.10 As mentioned below, in a limiting case, Pflug et al. (2012) reach essentially the same conclusion as us, although there are several differences. Pflug et al. (2012) show that as a model uncertainty measured by the Kantorovich metric increases, an optimal portfolio in a mean-standard-deviation optimization problem converges to the equally weighted portfolio. Specifically, Pflug et al. (2012) consider the following problem:²

$$\begin{aligned} \max_u \min_{\mathbb{Q}: d_2(\mathbb{P}, \mathbb{Q}) \leq \kappa} & \left\{ \mathbb{E}^\mathbb{Q}[X] - \frac{\gamma}{2} \sqrt{\text{Var}^\mathbb{Q}(X)} \right\}, \\ \text{subject to } X &= R'u, \\ 1 &= \mathbf{1}_d'u, \end{aligned} \tag{4.3.3}$$

where d_2 is the Kantorovich metric with order 2, and R is a risky assets' return vector. γ is a non-negative trade-off parameter between returns and risks. $\mathbb{E}^\mathbb{Q}$ and $\text{Var}^\mathbb{Q}$ are expectation and variance operators under a probability measure $\mathbb{Q}$, respectively. κ is a

²We modify the original problem in Pflug et al. (2012) for this to take the same form as in this chapter. It can be easily seen that the problem (4.3.3) is equivalent to the original problem in Pflug et al. (2012).

non-negative constant that represents a degree of uncertainty. Pflug et al. (2012) show that the solution to the problem (4.3.3) converges to the equally weighted portfolio as $\kappa \rightarrow \infty$.

One of the differences between our model and the model in Pflug et al. (2012) is the measure of uncertainty. Pflug et al. (2012) adopts the Kantorovich metric, while it is a mathematically sophisticated concept of a distance among probability measures, it is usually hard to compute, and it is not usually used in practice. In contrast, our measures of uncertainty are a confidence interval around means and a relative error of covariances, which are widely used in practice and easy to compute.

Another difference between our model and the model in Pflug et al. (2012) is the objective function. Let us assume $T = 1$ in our model, so we consider a static problem. In order to apply the approach of Pflug et al. (2012) to our objective function, we need to replace the term of the standard deviation in (4.3.3) to the variance, that is

$$\begin{aligned} \max_u \quad & \min_{\mathbb{Q}: d_2(\mathbb{P}, \mathbb{Q}) \leq \kappa} \left\{ \mathbb{E}^{\mathbb{Q}}[X] - \frac{\gamma}{2} \text{Var}^{\mathbb{Q}}(X) \right\}, \\ \text{subject to } & X = R'u, \\ & 1 = \mathbf{1}_d'u. \end{aligned} \tag{4.3.4}$$

However, since the objective function in the problem (4.3.4) does not satisfy the assumptions in Proposition 1 in Pflug et al. (2012), we can not use the results of Pflug et al. (2012). Therefore, the solution to the problem (4.3.4) may not converge to the equally weighted portfolio. For details about this discussion, we also refer to Wozabal (2014).

Considering a unique prior case (i.e., $\kappa = 0$), we can easily show that the solution to the problem (4.3.3) is proportional to the solution to the problem (4.3.4). However, the trade-off parameter γ has different implications in these problems. As seen in Section 4.2, in our model, we can regard γ as a coefficient of (absolute) risk aversion. On the other hand, economic implication of γ in the problem (4.3.3) is not clear.

Finally, we adopt dynamic mean-variance optimization, whereas the model in Pflug et al. (2012) is static. Therefore, our model allows an investment strategy that changes portfolios in reaction to the state variable F_t . In Section 4.4, we will see that this

investment strategy often performs well in practice.

4.4 Optimality of Naive Investment Strategies in Back Tests

In this section, we conduct back tests of dynamic optimization with multiple priors. We use the following three data sets.

1. **International Equity Indexes.** The four MSCI indexes: US, Japan, UK, and Germany. Each index is month-end US-dollar valued. The data source is the Thomson Reuters Datastream.
2. **Industry Indexes.** The five monthly returns of industry indexes in the US stock market: Consumer, Manufacture, HiTechnology, Health, and Others. The data source is the Kenneth French's web site.
3. **Size- and Value-Sorted Portfolios.** The six monthly returns of the 2×3 size- and book-to-market-sorted portfolios in the US stock market by Fama and French (1993). The data source is the Kenneth French's web site.

We also use monthly returns of 90-day US treasury bill from the Thomson Reuters Datastream as risk-free rates. Each dataset consists of monthly returns from January 1975 to December 2014. In order to focus on the diversification within the risky assets, we consider the asset allocations without the risk-free asset and compare out-of-sample Sharpe ratios.

4.4.1 Methodology of the Back Tests

In each dataset, we assume that the return vector is modeled by a two-state, recursive and time-homogeneous Markov regime-switching model, such that

$$R_{t+1} = \mu(F_{t+1}) + \sigma(F_{t+1})\epsilon_{t+1}, \quad t \geq 0,$$

where $(F_t)_{t=0}^T$ is a two-state, recursive and time-homogeneous Markov chain. $(F_t)_{t=0}^T$ and $(\epsilon_t)_{t=1}^T$ are mutually independent. The transition probability matrix of $(F_t)_{t=0}^T$ is

constant over time by the time-homogeneity of $(F_t)_{t=0}^T$. This model satisfies Assumption 4.1.

We assume that one of the states of F_t is a “good state” and another state is a “bad state”. Whether the market condition is good or bad is determined by the values of conditional means and variances of the return vectors. Consider the state in which more than half of means and reciprocals of variances are larger than those in another state. We call this state the “good state” and call the other state the “bad state”. If these values are the same, as the good state we choose the state in which the sum of the marginal conditional means is larger than that in the other state.

We consider two types of investors. The first type assumes that the returns of the indexes are *not* regime switching. This type of investor always considers the mean and variance of the returns are constants. We call this type “IID”. The second type assumes that the returns are driven by the Markov chain $(F_t)_{t=0}^T$. Hence, this type of investor assumes that the conditional mean and variance of the returns vary over the time. We call this type “RS”. The IID and RS investors compute their portfolios using the dynamic optimization without the risk-free asset proposed in Propositions 4.7 and 4.8.

As for the preference parameters, $\hat{\gamma}_t$ and δ are constant over time and states. We set $\hat{\gamma}_t = 1$ and $\delta = 0.99$. To study the effect of the multiple priors, we consider six different values of $(\eta_t^\theta = 0, 1, 2, 3, 4, 5)$ and five different values of $(\eta_t^V = 0.0, 0.5, 1.0, 3.0, 5.0)$. These parameters are fixed over time and states. Furthermore, we also consider two different cases: the one in which short selling is permitted and the one in which it is not permitted. Therefore, we compute $6 \times 5 \times 2$ portfolios for each investor type. The large η_t^θ and η_t^V represent strong degrees of suspicion of investors, so we can compare the investors with different levels of suspicion. As benchmarks, we also consider the equally weighted portfolio (EW), the portfolio maximizing the single-period empirical Sharpe ratio (max SR), and the single-period global minimum-variance portfolio (GMV). In addition, we use the value-weighted portfolio (VW) as one of the benchmarks for the industry indexes and the size- and value-sorted portfolios data set³. Note that the IID

³Unfortunately, we can not obtain the data of the market values of the US dollar-based MSCI indexes. So, we compute the value-weighted portfolio only for the abovementioned two data sets

investor's max SR at time t maximizes the following objective function:

$$f_{\text{ID}}(u) := \frac{\bar{\mu}'u - r_{f,t}}{\sqrt{u'\bar{\Sigma}u}},$$

where $\bar{\mu}$ and $\bar{\Sigma}$ are sample mean and variance of returns and $r_{f,t}$ is the risk-free rate at time t . On the other hand, the RS investor's max SR at time t with the state $F_t = i$ maximizes the following objective function,

$$f_{\text{RS}}(u) := \frac{\bar{\mu}_i'u - r_{f,t}}{\sqrt{u'(\bar{\sigma}_i\bar{\sigma}_i')u}},$$

where $\bar{\mu}_i$ and $\bar{\sigma}_i\bar{\sigma}_i'$ are estimated mean and variance of returns at state i . By the above definition, max SR delivers the largest Sharpe ratio without estimation errors.

Furthermore, we consider the following six extreme strategies. In these six strategies, η_t^θ and η_t^V take different values in the different states of F_t . The first strategy is “GMV in the good state and EW in the bad state” (GMV-EW). Under this strategy, the investor invests in the global minimum-variance portfolio in the state of good market condition and in the equally weighted portfolio in the state of bad market condition. This strategy can be interpreted as the optimal portfolio when $\eta_t^V = 0$ and η_t^θ tends to infinity in the good state and when η_t^V tends to infinity in the bad state. The second strategy is “EW in the good state and No Error in the bad state” (EW-NE). Under this strategy, the investor invests in the equally weighted portfolio in the good state and in the optimal portfolio of Proposition 4.7 and 4.8 with $\eta_t^\theta = \eta_t^V = 0$ in the bad state. This strategy can be interpreted as the optimal portfolio when $\eta_t^V = 0$ and η_t^θ tends to infinity in the good state and when η_t^θ and η_t^V are 0 in the bad state. The third strategy is “GMV in the good state and No Error in the bad state” (GMV-NE). Under this strategy, the investor invests in the global minimum-variance portfolio in the good state and in the optimal portfolio with $\eta_t^\theta = \eta_t^V = 0$ in the bad state. This strategy can be interpreted as the optimal portfolio when $\eta_t^V = 0$ and η_t^θ tends to infinity in the good state and when η_t^θ and η_t^V are 0 in the bad state. The rest of the strategies are the reverse strategies of the above three strategies “EW in the good state and GMV in the bad state” (EW-GMV), “No Error in the good state and EW in the bad state” (NE-EW), and “No Error in the good state and GMV in the bad state” (NE-GMV). The abbreviations and the six extreme strategies are summarized in Table 4.1.

In the back tests, each investor needs to estimate the distribution parameters from the data. The rolling window of estimation is fixed to 240 months. At each time, the IID investor computes the sample mean and variance of the data over the past 240 months from current time. Similarly, the RS investor estimates the distribution parameters using the data over the past 240 months from current time. The RS investor estimates the parameters by the EM algorithm proposed by Hamilton (1990). At the time when t months have passed after the start of investment, the investors compute the optimal plans of the portfolios for $240 - t$ periods and invest in their optimal portfolios in the initial period. For example, suppose that the investors would like to decide the portfolios in January 2000. They first compute the optimal portfolio plans for 180 months, from January 2000 to December 2014. Then, they invest in the portfolios in the initial period in January 2000.

However, the RS investors can not determine their portfolios since the state variable F in actual data is not observable. To determine their portfolios, they regard the state having the highest conditional probability as the current state, that is, the current state at time t is the following i_t ,

$$i_t = \arg \max_{i \in \{1,2\}} \mathbb{P}(F_t \text{ is in the } i \text{ th state.} \mid R_1, R_2, \dots, R_t),$$

where R_s is the return vector of the indexes at time $s = 1, \dots, t$.

The optimality of the portfolios derived in Proposition 4.7 and 4.8 are not guaranteed in the above rolling-window approach. However, in general, investors usually choose their portfolios based on the latest information. Moreover, the existing literature (e.g., Garlappi et al. (2007) and DeMiguel et al. (2009b)) adopts the rolling-window approach. For these reasons, we also use this approach.

4.4.2 Results of the Back Tests

Tables 4.2, 4.3, and 4.4 display the out-of-sample Sharpe ratios obtained from the back tests. In all data sets, the extreme strategies tend to deliver larger Sharpe ratios.

In the international diversification (Table 4.2), max SR has larger Sharpe ratios than the other typical portfolios, GMV and EW, in all cases (whether the investor type is IID or RS and with or without permitting short selling). When permitting short

selling, max SR of the IID investor has the largest Sharpe ratio (0.1014) except for the extreme strategies. However, the IID investor's optimal portfolio with $\eta_t^\theta = 0.0$ and $\eta_t^V = 5.0$ when not permitting short selling delivers the largest Sharpe ratio (0.1081) among all portfolios. Also, the Sharpe ratios when short selling is not permitted tend to be larger than those when it is permitted. This is consistent with the well-known results of Jagannathan and Ma (2003), which states that a short-selling constraint can improve the investment performance.

According to Table 4.2, the two extreme strategies, GMV-EW and GMV-NE, have good performances whether short selling is permitted or not. GMV-EW with short selling delivers the largest Sharpe ratio (0.1032) among the portfolios with short selling. On the other hand, GMV-NE without short selling has the second largest Sharpe ratio (0.1078) among all portfolios. Furthermore, the Sharpe ratios of GMV-EW without short selling and GMV-NE with short selling, 0.0922 and 0.0891 respectively, are not small. Among the typical portfolios (max SR, GMV, and EW) and the typical extreme strategies (GMV-EW and EW-GMV), GMV-EW when permitting short selling has the largest Sharpe ratio. Taking into account the simplicity of computation, GMV-EW when permitting short selling works effectively.

Table 4.3 reports the results of the industry indexes data set. According to Table 4.3, GMV has larger Sharpe ratio than max SR, EW and VW in all cases. Precisely, the IID investor's GMV with short selling has the largest Sharpe ratio among all portfolios. Among portfolios without short selling, the RS investor's GMV has the largest Sharpe ratio. However, the two extreme strategies, GMV-EW and EW-GMV, are not so bad either. Both of GMV-EW and EW-GMV defeat max SR, EW, and VW regardless of whether short selling is permitted or not. Comparing GMV-EW and EW-GMV, EW-GMV performs better than GMV-EW. When short selling is permitted, the Sharpe ratio of GMV-EW is 0.2234, whereas the Sharpe ratio of EW-GMV is 0.2564. Without short selling, the Sharpe ratio of GMV-EW is 0.2281, whereas the Sharpe ratio of EW-GMV is 0.2473.

Table 4.4 reports the results of the size- and value-sorted portfolios data set. The IID investor's optimal portfolio with $\eta_t^\theta = 1.0$ and $\eta_t^V = 0.0$ when permitting short selling

has the largest Sharpe ratio (0.3406) among all portfolios. On the other hand, similar to the industry indexes data set, GMV has a larger Sharpe ratio than max SR, EW, and VW in all cases. Furthermore, regardless of whether permitting short selling or not, GMV-EW defeats max SR, EW, VW, and GMV, except for the IID investor's GMV. Precisely, GMV-EW has the largest Sharpe ratio (0.2336) among portfolios without short selling.

In summary, GMV-EW and GMV tend to perform well in all data sets. DeMiguel et al. (2009b) report relatively good performances of GMV, so our results are consistent with their results. Unlike DeMiguel et al. (2009b), EW is less efficient in our data sets. However, GMV-EW delivers larger Sharpe ratio in the international indexes data set when short selling is permitted and in the size- and value-sorted portfolios data set when short selling is not permitted. Furthermore, GMV-EW defeats max SR in the industry indexes data set regardless of whether permitting short selling or not. Therefore, EW is efficient under particular situations and GMV-EW effectively uses the efficiency of EW.

Note that the IID investor's max SRs deliver larger Sharpe ratios than the RS investor's max SRs in all cases. One explanation of this result is the difference between the objective functions. The RS investor's max SR maximizes the conditional Sharpe ratio whereas the IID investor's max SR maximizes the unconditional Sharpe ratio. Since our back tests compute out-of-sample Sharpe ratios, which are sample analogs of the unconditional Sharpe ratio, it is not surprising that the IID investor's max SRs perform better than the RS investor's max SRs.

As seen in Section 4.3, the extreme strategies can not be justified by our framework since the value functions also diverge. However, optimal portfolios with sufficiently large η_t^θ and η_t^V will work as extreme strategies. We show behaviors of the optimal portfolios with sufficiently large η_t^θ and η_t^V .

We consider the following metrics from the optimal portfolios of EW and GMV:

$$\|u_t^*\|_{GMV} := \|u_t^* - u_t^{GMV}\|, \quad \|u_t^*\|_{EW} := \|u_t^* - \mathbf{1}_d/d\|,$$

where u_t^* is the optimal portfolio at time t . These metrics represent the root square

errors from EW and GMV, so we call these the portfolio errors from EW and GMV. Figure 4.1 shows the portfolio errors of the international indexes data set from EW and GMV in the case when short selling is permitted. From the upper figures in Figure 4.1, we see that the high η_t^θ reduces the portfolio errors from GMV for both IID and RS investors. Similarly, the lower figures in Figure 4.1 show that the high η_t^V reduces the portfolio errors from EW for the both investors⁴. This observation is consistent with our theoretical results in Section 4.3.

Figure 4.1 suggests that the optimal portfolios with sufficiently large η_t^θ and η_t^V can be good proxies for the extreme strategies. We try to approximate GMV-EW and EW-GMV by these optimal portfolios. We denote these approximated portfolios by mGMV-EW and mEW-GMV. The first letter m means *mimicking* extreme strategy.

Table 4.5 reports the Sharpe ratios of the optimal portfolios with sufficiently large η_t^θ and η_t^V . The second, third, fourth, and fifth columns are the specific values of η_t^θ and η_t^V of the mimicking strategies. In Table 4.5, the averages of the root square errors of the mimicking strategies from the (original) extreme strategies are sufficiently small in all the data sets. This implies that the mimicking strategies are good proxies for the extreme strategies. Consequently, the mimicking strategies' Sharpe ratios are close to the original extreme strategies' Sharpe ratios. Hence, we conclude that the investors can asymptotically justify the extreme strategies and exploit the portfolio selections that are extremely robust to the estimation errors.

GMV-EW and GMV-EW are very simple strategies, which means investing in the global minimum-variance portfolio or the equally weighted portfolio with respect to the market condition. These strategies seem to be naive and ad hoc, but they are one of the results of the plausible portfolio optimization; For GMV-EW, the investor assumes that the estimates of the conditional mean in economic booming and the conditional variance in recession are not credible at all, whereas for EW-GMV, she assumes that the estimated variances in economic booming and the estimated means in recession are not credible. Therefore, the investor chooses GMV-EW or EW-GMV as the extremely robust portfolio to estimation errors.

⁴In the lower figures in Figure 4.1, we fix $\eta_t^\theta = 5$. Since the portfolio errors from EW with $\eta_t^\theta = 1$ is larger than those with $\eta_t^\theta = 5$, we do not report the results of $\eta_t^\theta = 1$

4.5 Conclusion

In this chapter, we derive the optimal portfolios in the dynamic mean-variance problems with multiple priors. Furthermore, we show that the optimal portfolios include the equally weighted portfolio and the global minimum-variance portfolio in the limiting cases.

In the back tests, we find that the extreme strategies, especially GMV-EW tends to be relatively mean-variance efficient in the various data sets. In addition, the portfolios with sufficiently large η_t^θ and η_t^V can be good proxies for the extreme strategies. Therefore, we can mathematically justify the extreme strategies although the extreme strategies seem to be naive; the reason why the investors choose the extreme strategies is that the investors have strong suspicion of the estimated expected return and covariance.

Our analysis has an important implication about asset pricing. As seen above, the standard mean-variance analysis does not work in practice, neither does the capital asset pricing model. However, according to our framework, two mean-variance inefficient in theory, and naive portfolios, the global minimum portfolio and the equally weighted portfolio, can be justified as rational selections of the investors. Thus, it is possible that the two portfolios develop asset pricing models. Then, our framework can be used to analyze the rational reasoning behind the investors' choices of these portfolios.

Appendix 4.A Proof of Lemma 4.4

Proof of Lemma 4.4. We first prove that

$$\mathbb{E} \left[\exp\{u' R_{t+1}\} \zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] = \exp \left\{ u' \left(\mu(F_{t+1}) + \theta_t \right) + \frac{1}{2} u' (\sigma(F_{t+1}) (\sigma(F_{t+1}))' + V_t) u \right\} \quad (4.A.1)$$

for all $t \geq 0$ and $u \in \mathbb{R}^d$. To simplify the notations, we write $\mu_{t+1} = \mu(F_{t+1})$ and $\sigma_{t+1} = \sigma(F_{t+1})$. For any fixed constant vector $u \in \mathbb{R}^d$, we have

$$\begin{aligned}
& \mathbb{E} \left[\exp\{u' R_{t+1}\} \zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\det(\sigma_{t+1} \sigma'_{t+1} + V_t)}} \exp \left\{ -\frac{1}{2} \theta'_t (\sigma_{t+1} \sigma'_{t+1} + V_t)^{-1} \theta_t + u' \mu_{t+1} \right. \\
&\quad \left. + \left((\sigma_{t+1} \sigma'_{t+1} + V_t)^{-1} \theta_t + u \right)' \sigma_{t+1} \epsilon_{t+1} + \frac{1}{2} \epsilon'_{t+1} \left(I_d - \sigma'_{t+1} (\sigma_{t+1} \sigma'_{t+1} + V_t)^{-1} \sigma_{t+1} \right) \epsilon_{t+1} \right. \\
&\quad \left. - \frac{1}{2} \epsilon'_{t+1} \epsilon_{t+1} \right\} d\epsilon_{t+1} \\
&= \exp \left\{ u' (\mu_{t+1} + \theta_t) + \frac{1}{2} u' (\sigma_{t+1} \sigma'_{t+1} + V_t) u \right\} \\
&\quad \times \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\det(\sigma_{t+1} \sigma'_{t+1} + V_t)}} \\
&\quad \times \exp \left\{ -\frac{1}{2} (\sigma_{t+1} \epsilon_{t+1} - \text{mean}_{t+1})' (\sigma_{t+1} \sigma'_{t+1} + V_t)^{-1} (\sigma_{t+1} \epsilon_{t+1} - \text{mean}_{t+1}) \right\} d\epsilon_{t+1},
\end{aligned}$$

where $\text{mean}_{t+1} = \theta_t + (\sigma_{t+1} \sigma'_{t+1} + V_t) u$. Then, the above integral is equal to 1 and the equation (4.A.1) holds.

If $u = \mathbf{0}_d$, we have

$$\mathbb{E} \left[\zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] = \mathbb{E} \left[\exp\{\mathbf{0}'_d R_{t+1}\} \zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] = 1.$$

Then, the process $(Z_t)_{t=0}^T$ is a martingale with respect to $\mathbb{F}^{R,F}$ since

$$\mathbb{E} \left[Z_{t+1} | \mathcal{F}_t^{R,F} \right] = \mathbb{E} \left[Z_t \mathbb{E} \left[\zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] | \mathcal{F}_t^{R,F} \right] = Z_t,$$

for all $t \geq 0$. Moreover, $\mathbb{E}[Z_t] = \mathbb{E}[Z_0] = 1$. These equalities imply that we can define the probability measure $\mathbb{P}^{\theta,V}$ such that

$$\mathbb{P}^{\theta,V}(A) = \mathbb{E}[\mathbb{1}_A Z_T], \quad A \in \mathcal{F}_T^{R,F}.$$

Let $\sqrt{-1}$ be an imaginary unit. For any fixed $u \in \mathbb{R}^K$, by the Bayes rule, it holds that

$$\begin{aligned}
\mathbb{E}^{\theta,V} \left[\exp\{\sqrt{-1} u' F_{t+1}\} | \mathcal{F}_t^{R,F} \right] &= \mathbb{E} \left[\exp\{\sqrt{-1} u' F_{t+1}\} \zeta_{t+1} | \mathcal{F}_t^{R,F} \right] \\
&= \mathbb{E} \left[\exp\{\sqrt{-1} u' F_{t+1}\} \mathbb{E} \left[\zeta_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1} \right] | \mathcal{F}_t^{R,F} \right] \\
&= \mathbb{E} \left[\exp\{\sqrt{-1} u' F_{t+1}\} | \mathcal{F}_t^{R,F} \right].
\end{aligned}$$

This implies that the conditional distribution of F_{t+1} given $\mathcal{F}_t^{R,F}$ under $\mathbb{P}^{\theta,V}$ is the same as the conditional distribution of F_{t+1} given $\mathcal{F}_t^{R,F}$ under $\mathbb{P}$. It follows that

$$\begin{aligned} \mathbb{E}^{\theta,V} \left[R_{t+1} \middle| \mathcal{F}_t^{R,F} \right] &= \mathbb{E}^{\theta,V} \left[\mathbb{E}^{\theta,V} \left[R_{t+1} \middle| \mathcal{F}_t^{R,F}, F_{t+1} \right] \middle| \mathcal{F}_t^{R,F} \right] \\ &= \mathbb{E}^{\theta,V} \left[\mu(F_{t+1}) \middle| \mathcal{F}_t^{R,F} \right] + \theta_t = \mathbb{E} \left[\mu(F_{t+1}) \middle| \mathcal{F}_t^{R,F} \right] + \theta_t \\ &= \mathbb{E} \left[R_{t+1} \middle| \mathcal{F}_t^{R,F} \right] + \theta_t, \end{aligned}$$

for all $t \geq 0$. Similarly, it holds that

$$\begin{aligned} \text{Var}^{\theta,V}(R_{t+1} | \mathcal{F}_t^{R,F}) &= \text{Var}^{\theta,V}(\mathbb{E}^{\theta,V}[R_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1}] | \mathcal{F}_t^{R,F}) \\ &\quad + \mathbb{E}^{\theta,V}[\text{Var}^{\theta,V}(R_{t+1} | \mathcal{F}_t^{R,F}, F_{t+1}) | \mathcal{F}_t^{R,F}] \\ &= \text{Var}^{\theta,V}(\mu(F_{t+1}) | \mathcal{F}_t^{R,F}) + \mathbb{E}^{\theta,V}[\sigma(F_{t+1})(\sigma(F_{t+1}))' | \mathcal{F}_t^{R,F}] + V_t \\ &= \text{Var}(\mu(F_{t+1}) | \mathcal{F}_t^{R,F}) + \mathbb{E}[\sigma(F_{t+1})(\sigma(F_{t+1}))' | \mathcal{F}_t^{R,F}] + V_t \\ &= \text{Var}(R_{t+1} | \mathcal{F}_t^{R,F}) + V_t, \end{aligned}$$

for all $t \geq 0$. □

Appendix 4.B Proofs of Theorem 4.5 and Proposition 4.6

To prove Theorem 4.5, we need to show two lemmas. The objective function in the problem (4.2.2) can be written as

$$\begin{aligned} \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) \right] &= \mathbb{E} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) Z_t \right] \\ &= \mathbb{E} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) \prod_{s=0}^t \zeta_s \right]. \end{aligned}$$

Let ρ_t be a mapping such that

$$\rho_t(W) = \delta \mathbb{E} \left[W \zeta_{t+1} \middle| \mathcal{F}_t^{R,S} \right], \quad 0 \leq t \leq T-1.$$

Then, we can express the objective as

$$\begin{aligned} J_T(x, f, \theta, V, u) &:= \mathbb{E}^{\theta,V} \left[\sum_{t=0}^{T-1} \delta^t X_t c_t(F_t, \theta_t, V_t, u_t) \right] \\ &= x c_0(f, \theta_0, V_0, u_0) + \rho_0 \left(X_1 c_1(F_1, \theta_1, V_1, u_1) + \rho_1 \left(\cdots \right. \right. \\ &\quad \left. \left. + \rho_{T-2} \left(X_{T-1} c_{T-1}(F_{T-1}, \theta_{T-1}, V_{T-1}, u_{T-1}) \right) \cdots \right) \right), \end{aligned}$$

for all $u \in \mathcal{A}_T^u$ and $(\theta, V) = (\theta_t, V_t)_{t=0}^{T-1} \in \mathcal{A}_T^{\theta, V}$, where $X_0 = x$ and $F_0 = f$. The above recursive structure is important for the optimization. To simplify the notations, we write

$$c_t^{u, \theta, V} = X_t c_t(F_t, \theta_t, V_t, u_t), \quad 0 \leq t \leq T-1.$$

We define $\rho_{t, r-1}$ as

$$\rho_{t, r-1}(W_t, \dots, W_r) = W_t + \rho_t \left(W_{t+1} + \rho_{t+1} \left(\dots + \rho_{r-1}(W_r) \right) \right).$$

Then, for all $u \in \mathcal{A}_T^u$ and $(\theta, V) \in \mathcal{A}_T^{\theta, V}$, we have

$$J_T(x, f, \theta, V, u) = \rho_{0, t-1} \left(c_0^{u, \theta, V}, \dots, c_{t-1}^{u, \theta, V}, \rho_{t, T-2} \left(c_t^{u, \theta, V}, \dots, c_{T-2}^{u, \theta, V}, c_{T-1}^{u, \theta, V} \right) \right),$$

for all $0 \leq t \leq T-1$. We first prove that the order of the minimization with respect to (θ, V) can be exchangeable. The proof of the following lemma is essentially the same as the proof of Lemma 1 in Chen et al. (2014).

Lemma 4.11 Fix any $u \in \mathcal{A}_T^u$. Then, it holds that

$$\begin{aligned} & \min_{(\theta_r, V_r)_{r=t}^{T-1} \in \mathcal{A}_T^{\theta, V}|_t^{T-1}} \rho_{t, T-1}(c_t^{u, \theta, V}, \dots, c_{T-1}^{u, \theta, V}) \\ &= \min_{(\theta_r, V_r)_{r=t}^{s-1} \in \mathcal{A}_T^{\theta, V}|_t^{s-1}} \rho_{t, s-1} \left(c_0^{u, \theta, V}, \dots, c_{s-1}^{u, \theta, V}, \min_{(\theta_r, V_r)_{r=s}^{T-1} \in \mathcal{A}_T^{\theta, V}|_s^{T-1}} \rho_{s, T-2} \left(c_s^{u, \theta, V}, \dots, c_{T-1}^{u, \theta, V} \right) \right), \end{aligned} \quad (4.B.1)$$

for all $0 \leq t \leq s \leq T-1$, where $\mathcal{A}_T^{\theta, V}|_t^r$ is a set of $(\theta_s, V_s)_{s=t}^r$ which are components of $(\theta, V) \in \mathcal{A}_T^{\theta, V}$ from the time t to r .

Proof Fix any $u \in \mathcal{A}_T^u, x \in \mathbb{R}$ and $f \in \mathbb{R}^K$. Fix any $0 \leq t \leq s \leq T-1$. We define $(\tilde{\theta}, \tilde{V})$ as

$$(\tilde{\theta}, \tilde{V}) = (\tilde{\theta}_r, \tilde{V}_r)_{r=s}^{T-1} \in \arg \min_{(\theta_r, V_r)_{r=s}^{T-1} \in \mathcal{A}_T^{\theta, V}|_s^{T-1}} \rho_{s, T-2} \left(c_s^{u, \theta, V}, \dots, c_{T-2}^{u, \theta, V}, c_{T-1}^{u, \theta, V} \right).$$

Then, for every $(\theta, V) \in \mathcal{A}_T^{\theta, V}$, it holds that

$$\rho_{s, T-2} \left(c_s^{u, \theta, V}, \dots, c_{T-2}^{u, \theta, V}, c_{T-1}^{u, \theta, V} \right) \geq \rho_{s, T-2} \left(c_s^{u, \tilde{\theta}, \tilde{V}}, \dots, c_{T-2}^{u, \tilde{\theta}, \tilde{V}}, c_{T-1}^{u, \tilde{\theta}, \tilde{V}} \right).$$

Since $\rho_r(W) \leq \rho_r(W')$ for all r if $W \leq W'$, it holds that

$$\rho_{t, s-1} \left(W_t, \dots, W_{s-1}, W_s \right) \leq \rho_{t, s-1} \left(W_t, \dots, W_{s-1}, W' \right),$$

if $W_s \leq W'$. Therefore, we have

$$\rho_{t,T-1}(c_t^{u,\theta,V}, \dots, c_{T-1}^{u,\theta,V}) \geq \rho_{t,s-1}\left(c_t^{u,\theta,V}, \dots, c_{s-1}^{u,\theta,V}, \rho_{s,T-2}\left(c_s^{u,\tilde{\theta},\tilde{V}}, \dots, c_{T-2}^{u,\tilde{\theta},\tilde{V}}, c_{T-1}^{u,\tilde{\theta},\tilde{V}}\right)\right).$$

Minimizing the above inequality, we obtain

$$\begin{aligned} & \min_{(\theta_r, V_r)_{r=t}^{T-1} \in \mathcal{A}_T^{\theta,V} |_t^{T-1}} \rho_{t,T-1}(c_t^{u,\theta,V}, \dots, c_{T-1}^{u,\theta,V}) \\ & \geq \min_{(\theta_r, V_r)_{r=t}^{s-1} \in \mathcal{A}_T^{\theta,V} |_t^{s-1}} \rho_{t,s-1}\left(c_t^{u,\theta,V}, \dots, c_{s-1}^{u,\theta,V}, \rho_{s,T-2}\left(c_s^{u,\tilde{\theta},\tilde{V}}, \dots, c_{T-2}^{u,\tilde{\theta},\tilde{V}}, c_{T-1}^{u,\tilde{\theta},\tilde{V}}\right)\right) \\ & = \min_{(\theta_r, V_r)_{r=t}^{s-1} \in \mathcal{A}_T^{\theta,V} |_t^{s-1}} \rho_{t,s-1}\left(c_t^{u,\theta,V}, \dots, c_{s-1}^{u,\theta,V}, \min_{(\theta_r, V_r)_{r=s}^{T-1} \in \mathcal{A}_T^{\theta,V} |_s^{T-1}} \rho_{s,T-2}\left(c_s^{u,\theta,V}, \dots, c_{T-1}^{u,\theta,V}\right)\right). \end{aligned}$$

We denote by $(\hat{\theta}, \hat{V})$ the following minimizer:

$$(\hat{\theta}, \hat{V}) := (\hat{\theta}_r, \hat{V}_r)_{r=t}^{s-1} \in \arg \min_{(\theta_r, V_r)_{r=t}^{s-1} \in \mathcal{A}_T^{\theta,V} |_t^{s-1}} \rho_{t,s-1}\left(c_t^{u,\theta,V}, \dots, c_{s-1}^{u,\theta,V}, \rho_{s,T-2}\left(c_s^{u,\tilde{\theta},\tilde{V}}, \dots, c_{T-1}^{u,\tilde{\theta},\tilde{V}}\right)\right).$$

Then, $(\bar{\theta}, \bar{V}) := ((\hat{\theta}_t \hat{V}_t), \dots, (\hat{\theta}_{s-1}, \hat{V}_{s-1}), (\tilde{\theta}_s, \tilde{V}_s), \dots, (\tilde{\theta}_{T-1}, \tilde{V}_{T-1}))$ is in $\mathcal{A}_T^{\theta,V} |_t^{T-1}$. It follows that

$$\begin{aligned} & \min_{(\theta_r, V_r)_{r=t}^{T-1} \in \mathcal{A}_T^{\theta,V} |_t^{T-1}} \rho_{t,T-1}(c_t^{u,\theta,V}, \dots, c_{T-1}^{u,\theta,V}) \\ & \leq \rho_{t,T-1}(c_t^{u,\bar{\theta},\bar{V}}, \dots, c_{T-1}^{u,\bar{\theta},\bar{V}}) = \rho_{t,s-1}\left(c_t^{u,\hat{\theta},\hat{V}}, \dots, c_{s-1}^{u,\hat{\theta},\hat{V}}, \rho_{s,T-2}\left(c_s^{u,\tilde{\theta},\tilde{V}}, \dots, c_{T-2}^{u,\tilde{\theta},\tilde{V}}, c_{T-1}^{u,\tilde{\theta},\tilde{V}}\right)\right) \\ & = \min_{(\theta_r, V_r)_{r=t}^{s-1} \in \mathcal{A}_T^{\theta,V} |_t^{s-1}} \rho_{t,s-1}\left(c_t^{u,\theta,V}, \dots, c_{s-1}^{u,\theta,V}, \min_{(\theta_r, V_r)_{r=s}^{T-1} \in \mathcal{A}_T^{\theta,V} |_s^{T-1}} \rho_{s,T-2}\left(c_s^{u,\theta,V}, \dots, c_{T-1}^{u,\theta,V}\right)\right). \end{aligned}$$

Therefore, we can conclude that the equality (4.B.1) holds. $\square$

By Lemma 4.11, we can minimize the objective function with respect to (θ, V) , iteratively. Let

$$\begin{aligned} \bar{\rho}_{T-1}(u_{T-1}) &= \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta,V} |_{T-1}^{T-1}} c_{T-1}^{u,\theta,V}, \\ \bar{\rho}_t(W, u_t) &= \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta,V} |_t^t} \left\{ c_t^{u,\theta,V} + \delta \mathbb{E} \left[W \zeta_{t+1} | \mathcal{F}_t^{R,F} \right] \right\}, \quad 0 \leq t \leq T-2, \\ \bar{\rho}_{t,s-1}(W, (u_r)_{r=t}^{s-1}) &= \bar{\rho}_t(\bar{\rho}_{t+1}(\dots, \bar{\rho}_{s-1}(W, u_{s-1}) \dots, u_{t+1}), u_t) \end{aligned}$$

Then, by Lemma 4.11, we derive that

$$J_T^*(x, f, u) := \min_{(\theta, V) \in \mathcal{A}_T^{\theta,V}} J_T(x, f, \theta, V, u) = \bar{\rho}_{0,T-1}(\bar{\rho}_{T-1}(u_{T-1}), (u_t)_{t=0}^{T-2}),$$

for all $u = (u_t)_{t=0}^{T-1} \in \mathcal{A}_T^u$.

The investor needs to maximize $J_T^*(x, f, u)$ over $u \in \mathcal{A}_T^u$. In the proof of Lemma 4.11, the monotonicity of ρ_t plays a key role in the exchangeability of the minimization with respect to (θ, V) . Similarly, the monotonicity of $\bar{\rho}_t$ is important to the the order of the maximization with respect to u . $\bar{\rho}_t$ satisfies the monotonicity, that is, for all $y \in \mathbb{R}^d$ and $0 \leq t \leq T-1$, $\bar{\rho}_t(W, y) \leq \bar{\rho}_t(W', y)$ holds if $W \leq W'$. This implies that for any $u \in \mathcal{A}_T^u$,

$$\bar{\rho}_{t,r-1}(W_r, (u_s)_{s=t}^{r-1}) \leq \bar{\rho}_{t,r-1}(W', (u_s)_{s=t}^{r-1}),$$

if $W_r \leq W'$. Therefore, using the same argument as in Lemma 4.11, we can prove the following lemma.

Lemma 4.12 For all $0 \leq t \leq s \leq T-1$, it holds that

$$\begin{aligned} & \max_{(u_r)_{r=t}^{T-1} \in \mathcal{A}_T^u|_t^{T-1}} \bar{\rho}_{t,T-1}(\bar{\rho}_{T-1}(u_{T-1}), (u_r)_{r=t}^{T-2}) \\ &= \max_{(u_r)_{r=t}^{s-1} \in \mathcal{A}_T^u|_t^{s-1}} \bar{\rho}_{t,s-1} \left(\max_{(u_r)_{r=s}^{T-1} \in \mathcal{A}_T^u|_s^{T-1}} \bar{\rho}_{s,T-2} \left(\rho_{T-1}(u_{T-1}), (u_r)_{r=s}^{T-2} \right), (u_r)_{r=t}^{s-1} \right), \end{aligned}$$

where $\mathcal{A}_T^u|_t^r$ is a set of $(u_s)_{s=t}^r$ which are components of $u \in \mathcal{A}_T^u$ from the time t to r .

Using Lemma 4.12, the dynamic programming of the multiple priors optimization can be justified. Now, we prove Theorem 4.5 and Proposition 4.6.

Proof of Theorem 4.5 and Proposition 4.6. Consider the time $T-1$ problem

$$\begin{aligned} & \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} \bar{\rho}_{T-1}(u_{T-1}) \\ &= \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} c_{T-1}^{u, \theta, V} \\ &= \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} X_{T-1} c_{T-1}(F_{T-1}, \theta_{T-1}, V_{T-1}, u_{T-1}) \\ &= \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} X_{T-1} \left\{ 1 + r_f + (m^e(F_{T-1}) + \theta_{T-1})' u_{T-1} \right. \\ & \quad \left. - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + V_{T-1} \right) u_{T-1} \right\}. \end{aligned} \tag{4.B.2}$$

Then, the minimization problem in the problem (4.B.2) is reduced to the following problem,

$$\min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} \left\{ \theta_{T-1}' u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' V_{T-1} u_{T-1} \right\}. \tag{4.B.3}$$

The Lagrange function of the problem (4.B.3) is

$$\begin{aligned}\mathcal{L}^{\theta,V} = & \theta'_{T-1}u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2}u'_{T-1}V_{T-1}u_{T-1} - \lambda_{\theta}\left((\eta_{T-1}^{\theta})^2 - \theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1}\right) \\ & - \lambda_V\left((\eta_{T-1}^V)^2\|A(F_{T-1})\|^2 - \|V_{T-1}\|^2\right),\end{aligned}$$

where λ_{θ} and λ_V are Lagrange multipliers. We first consider the case when $u_{T-1} \neq \mathbf{0}_d$.

The first order condition for θ_{T-1} is

$$u_{T-1} + 2\lambda_{\theta}(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1} = \mathbf{0}_d.$$

Hence, we have

$$\theta_{T-1} = -\frac{1}{2\lambda_{\theta}}(A(F_{T-1}) + V_{T-1})u_{T-1}.$$

Since the constraint for θ_{T-1} is binded, it holds that

$$(\eta_{T-1}^{\theta})^2 = \theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1} = \frac{1}{4\lambda_{\theta}^2}u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}.$$

Hence,

$$\lambda_{\theta} = \frac{1}{2\eta_{T-1}^{\theta}}\sqrt{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}},$$

and

$$\theta_{T-1} = -\frac{\eta_{T-1}^{\theta}}{\sqrt{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}}}(A(F_{T-1}) + V_{T-1})u_{T-1}.$$

We denote the $i \times j$ th element of V_{T-1} by $v_{T-1}^{i,j}$. Then, the first order condition for $v_{T-1}^{i,j}$ is

$$\begin{aligned}& -\widehat{\gamma}_{T-1}u_{T-1}^iu_{T-1}^j + 4\lambda_Vv_{T-1}^{i,j} \\ & = \text{tr}\left\{(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1}\theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1}N^{i,j}\right\}, \text{ if } i \neq j, \\ & -\frac{\widehat{\gamma}_{T-1}}{2}(u_{T-1}^i)^2 + 2\lambda_Vv_{T-1}^{i,i} \\ & = \text{tr}\left\{(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1}\theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1}N^{i,i}\right\}, \text{ if } i = j,\end{aligned}$$

where $N^{i,j}$ is a d -dimensional matrix whose $i \times j$ th element and $j \times i$ th element are 1 and the other elements are zero. Substituting θ_{T-1} into $(A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1}\theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1}$, we have

$$\begin{aligned}& (A(F_{T-1}) + V_{T-1})^{-1}\theta_{T-1}\theta'_{T-1}(A(F_{T-1}) + V_{T-1})^{-1} \\ & = \frac{(\eta_{T-1}^{\theta})^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}}u_{T-1}u'_{T-1}.\end{aligned}$$

Hence, it holds that

$$\begin{aligned} & \text{tr} \left\{ (A(F_{T-1}) + V_{T-1})^{-1} \theta_{T-1} \theta'_{T-1} (A(F_{T-1}) + V_{T-1})^{-1} N^{i,j} \right\} \\ &= \begin{cases} \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} u_{T-1}^i u_{T-1}^j, & \text{if } i \neq j \\ \frac{(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} (u_{T-1}^i)^2, & \text{if } i = j \end{cases} \end{aligned}$$

It follows that

$$-\widehat{\gamma}_{T-1} u_{T-1}^i u_{T-1}^j + 4\lambda_V v_{T-1}^{i,j} = \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} u_{T-1}^i u_{T-1}^j,$$

for all i and j . Using the matrix notation, we have

$$-\widehat{\gamma}_{T-1} u_{T-1} u'_{T-1} + 4\lambda_V V_{T-1} = \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} u_{T-1} u'_{T-1}.$$

Hence,

$$V_{T-1} = \frac{1}{4\lambda_V} \left(\widehat{\gamma}_{T-1} + \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} \right) u_{T-1} u'_{T-1}.$$

Since the constraint for V_{T-1} is also binded, it holds that

$$\begin{aligned} & (\eta_{T-1}^V)^2 \|A(F_{T-1})\|^2 \\ &= \|V_{T-1}\|^2 = \frac{1}{16\lambda_V^2} \left(\widehat{\gamma}_{T-1} + \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} \right)^2 \|u_{T-1}\|^4. \end{aligned}$$

Therefore, the Lagrange multiplier λ_V is

$$\lambda_V = \frac{1}{4\eta_{T-1}^V \|A(F_{T-1})\|} \left(\widehat{\gamma}_{T-1} + \frac{2(\eta_{T-1}^\theta)^2}{u'_{T-1}(A(F_{T-1}) + V_{T-1})u_{T-1}} \right) \|u_{T-1}\|^2.$$

Hence, we have

$$V_{T-1} = \frac{\eta_{T-1}^V \|A(F_{T-1})\|}{\|u_{T-1}\|^2} u_{T-1} u'_{T-1}.$$

Then,

$$(A(F_{T-1}) + V_{T-1})u_{T-1} = (A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d)u_{T-1}$$

and

$$\theta_{T-1} = -\frac{\eta_{T-1}^\theta (A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d)u_{T-1}}{\sqrt{u'_{T-1}(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d)u_{T-1}}}.$$

If $u_{T-1} = \mathbf{0}_d$, then every $(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta,V}|_{T-1}^{T-1}$ is a solution. In this case, we choose

$\theta_{T-1} = \mathbf{0}_d$ and $V_{T-1} = \mathbf{0}_d$ as the solution. In all cases, (θ_{T-1}, V_{T-1}) is in $\mathcal{A}_T^{\theta,V}|_{T-1}^{T-1}$.

Substituting θ_{T-1} and V_{T-1} into the original time $T-1$ optimization problem (4.B.2), we obtain

$$\begin{aligned} \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} X_{T-1} & \left\{ 1 + r_f + (m^e(F_{T-1}))' u_{T-1} \right. \\ & - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \\ & \left. - \eta_{T-1}^\theta \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}} \right\}. \end{aligned} \quad (4.B.4)$$

We first consider the following case.

$$a_{T-1}^e := (m^e(F_{T-1}))' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} m^e(F_{T-1}) > (\eta_{T-1}^\theta)^2. \quad (4.B.5)$$

The first order condition of this problem (4.B.4) is

$$m^e(F_{T-1}) - \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}} \right) \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} = \mathbf{0}_d,$$

where

$$\psi_{T-1} = \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}}.$$

Hence, we have

$$u_{T-1} = \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}} \right)^{-1} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} m^e(F_{T-1}).$$

Furthermore, we also have

$$\begin{aligned} \psi_{T-1}^2 &= u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \\ &= \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}} \right)^{-2} a_{T-1}^e. \end{aligned}$$

Thus, we obtain the following quadratic equation for ψ_{T-1} .

$$\hat{\gamma}_{T-1}^2 \psi_{T-1}^2 + 2\hat{\gamma}_{T-1} \eta_{T-1}^\theta \psi_{T-1} + (\eta_{T-1}^\theta)^2 - a_{T-1}^e = 0.$$

By the inequality (4.B.5), the above equation has a unique positive solution such that

$$\psi_{T-1} = \frac{\sqrt{a_{T-1}^e} - \eta_{T-1}^\theta}{\hat{\gamma}_{T-1}}.$$

Hence, the optimal portfolio at time $T-1$ is

$$u_{T-1} = \frac{1}{\hat{\gamma}_{T-1}} \left(1 - \frac{\eta_{T-1}^\theta}{\sqrt{a_{T-1}^e}} \right) \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} m^e(F_{T-1}).$$

Let $K_{T-1} = A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d$. If the inequality (4.B.5) does not hold, then the Cauchy-Schwartz inequality leads to

$$\begin{aligned} \eta_{T-1}^\theta \sqrt{u'_{T-1} K_{T-1} u_{T-1}} &\geq \sqrt{a_{T-1}^e} \sqrt{u'_{T-1} K_{T-1} u_{T-1}} \\ &= \sqrt{(m^e(F_{T-1}))' K_{T-1}^{-1} m^e(F_{T-1})} \sqrt{u'_{T-1} K_{T-1} (K_{T-1})^{-1} K_{T-1} u_{T-1}} \\ &\geq \left| (m^e(F_{T-1}))' (K_{T-1})^{-1} K_{T-1} u_{T-1} \right| = \left| (m^e(F_{T-1}))' u_{T-1} \right|. \end{aligned}$$

Hence,

$$\begin{aligned} &\max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} X_{T-1} \left\{ 1 + r_f + (m^e(F_{T-1}))' u_{T-1} \right. \\ &\quad \left. - \frac{\widehat{\gamma}_{T-1}}{2} u'_{T-1} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \right. \\ &\quad \left. - \eta_{T-1}^\theta \sqrt{u'_{T-1} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}} \right\} \\ &\leq \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}} X_{T-1} \left\{ 1 + r_f + (m^e(F_{T-1}))' u_{T-1} - \left| (m^e(F_{T-1}))' u_{T-1} \right| \right\} \end{aligned}$$

The solution of the later problem is an arbitrarily vector $u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}$, such that $(m^e(F_{T-1}))' u_{T-1} \geq 0$. We choose $u_{T-1} = \mathbf{0}_d$, and then the optimal value of the later problem becomes $X_{T-1}(1 + r_f)$. Moreover, we have

$$\begin{aligned} &1 + r_f + (m^e(F_{T-1}))' u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2} u'_{T-1} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \\ &\quad - \eta_{T-1}^\theta \sqrt{u'_{T-1} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}} \\ &= 1 + r_f, \end{aligned}$$

if $u_{T-1} = \mathbf{0}_d$. Therefore, $u_{T-1} = \mathbf{0}_d$ is also a solution of the former problem.

In each case, we can express the optimal portfolio at time $T - 1$ as

$$u_{T-1} = \frac{1}{\widehat{\gamma}_{T-1}} \left(1 - \frac{\eta_{T-1}^\theta}{\sqrt{a_{T-1}^e}} \right)^+ \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} m^e(F_{T-1}).$$

It is clear that $u_{T-1} \in \mathcal{A}_T^u|_{T-1}^{T-1}$. In addition, the value function is

$$\begin{aligned} \max_{u_{T-1} \in \mathcal{A}_T^u|_{T-1}^T} \bar{\rho}_{T-1}(u_{T-1}) &= Y_{T-1}^T X_{T-1} \\ Y_{T-1}^T &= \max_{u_{T-1} \in \mathbb{R}^d} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} \{c_{T-1}(F_{T-1}, \theta_{T-1}, V_{T-1}, u_{T-1})\} \\ &= 1 + r_f + \frac{\left((\sqrt{a_{T-1}^e} - \eta_{T-1}^\theta)^+ \right)^2}{2\widehat{\gamma}_{T-1}}. \end{aligned}$$

By the definition, Y_{T-1}^T is $\mathcal{F}_{T-1}^F$ -measurable.

Now, we assume that

$$\max_{(u_r)_{r=t+1}^{T-1} \in \mathcal{A}_T^u|_{t+1}^{T-1}} \min_{(\theta_r, V_r)_{r=t+1}^{T-1} \in \mathcal{A}_T^{\theta, V}|_{t+1}^{T-1}} \rho_{t+1, T-1}(c_{t+1}^{u, \theta, V}, \dots, c_{T-1}^{u, \theta, V}) = Y_{t+1}^T X_{t+1},$$

for some $0 \leq t \leq T-2$. Then, we have

$$\begin{aligned} & \mathbb{E}^{\theta, V}[Y_{t+1}^T X_{t+1} | \mathcal{F}_t^{R, F}] \\ &= \mathbb{E}^{\theta, V}[Y_{t+1}^T (1 + r_f + (R_{t+1}^e)' u_t) X_t | \mathcal{F}_t^{R, F}] \\ &= (1 + r_f) \mathbb{E}^{\theta, V}[Y_{t+1}^T | \mathcal{F}_t^{R, F}] X_t + \left(\mathbb{E}^{\theta, V}[Y_{t+1}^T R_{t+1} | \mathcal{F}_t^{R, F}] \right)' u_t X_t \\ &= (1 + r_f) \mathbb{E}[Y_{t+1}^T | \mathcal{F}_t^{R, F}] X_t + \left(\mathbb{E}[Y_{t+1}^T \mu^e(F_{t+1}) | \mathcal{F}_t^{R, F}] + \mathbb{E}[Y_{t+1}^T | \mathcal{F}_t^{R, F}] \theta_t \right)' u_t X_t, \end{aligned}$$

where we have used Lemma 4.4 and $\mathcal{F}_{t+1}^F$ -measurability of Y_{t+1}^T . Therefore, the objective in the time t problem is

$$c_t^{u, \theta, V} + \delta \mathbb{E}^{\theta, V}[Y_{t+1}^T X_{t+1} | \mathcal{F}_t^{R, F}] = X_t \left\{ (1 + r_f) g_t^e + (m_t^e + g_t^e \theta_t)' u_t - \frac{\hat{\gamma}_t}{2} u_t' (A_t + V_t) u_t \right\}.$$

Hence, the objective function in the time t problem has the same form as the objective function in the problem (4.B.2). Therefore, the error of the conditional mean is

$$\theta_t = - \frac{\text{sgn}(g_t^e) \eta_t^\theta (A(F_t) + \eta_t^V \|A(F_t)\| I_d) u_t}{\sqrt{u_t' (A(F_t) + \eta_t^V \|A(F_t)\| I_d) u_t}},$$

where sgn is a sign function such that

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The error of the conditional variance is

$$V_t = \frac{\eta_t^V \|A(F_t)\|}{\|u_t\|^2} u_t u_t'.$$

The optimal portfolio at time t is

$$u_t = \frac{1}{\hat{\gamma}_t} \left(1 - \frac{|g_t^e| \eta_t^\theta}{\sqrt{a_t^e}} \right)^+ \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} m_t^e.$$

It is clear that $u_t \in \mathcal{A}_T^u|_t^t$ and $(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V}|_t^t$. The value function is

$$\begin{aligned} Y_t^T X_t &= \max_{(u_r)_{r=t}^{T-1} \in \mathcal{A}_T^u|_t^{T-1}} \min_{(\theta_r, V_r)_{r=t}^{T-1} \in \mathcal{A}_T^{\theta, V}|_t^{T-1}} \rho_{t, T-1}(c_t^{u, \theta, V}, \dots, c_{T-1}^{u, \theta, V}), \\ Y_t^T &= (1 + r_f) g_t^e + \frac{\left((\sqrt{a_t^e} - |g_t^e| \eta_t^\theta)^+ \right)^2}{2 \hat{\gamma}_t}. \end{aligned}$$

By construction, Y_t^T is $\mathcal{F}_t^F$ -measurable.

For any $0 \leq t \leq T-1$, by the mathematical induction, the value function at time t can be expressed as

$$Y_t^T X_t = \max_{(u_r)_{r=t}^{T-1} \in \mathcal{A}_T^u|_t^{T-1}} \min_{(\theta_r, V_r)_{r=t}^{T-1} \in \mathcal{A}_T^{\theta, V}|_t^{T-1}} \rho_{t, T-1}(c_t^{u, \theta, V}, \dots, c_{T-1}^{u, \theta, V}),$$

where

$$Y_t^T = (1 + r_f)g_t^e + \frac{\left((\sqrt{a_t^e} - |g_t^e| \eta_t^\theta)^+\right)^2}{2\hat{\gamma}_t}.$$

The optimal portfolio at time t is

$$u_t^* = \frac{1}{\hat{\gamma}_t} \left(1 - \frac{|g_t^e| \eta_t^\theta}{\sqrt{a_t^e}}\right)^+ \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d\right)^{-1} m_t^e.$$

Also, it is obvious that the optimal $(u_t^*)_{t=0}^{T-1}$ is time-consistent. $\square$

Appendix 4.C Proofs of Proposition 4.7 and 4.8

Proof of Proposition 4.7. By Theorem 4.5, we can use the following dynamic programming procedure.

$$\begin{aligned} \hat{Y}_{T-1}^T &= \max_{u_{T-1} \in \tilde{\mathcal{A}}_T^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} \{\hat{c}_{T-1}(F_{T-1}, \theta_{T-1}, V_{T-1}, u_{T-1})\}, \\ \hat{Y}_t^T &= \max_{u_t \in \tilde{\mathcal{A}}_T^u|_t^t} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V}|_t^t} \left\{ \hat{c}_t(F_t, \theta_t, V_t, u_t) + \delta \mathbb{E}^{\theta, V} \left[Y_{t+1}^T (1 + (R_{t+1})' u_t) \mid \mathcal{F}_t^{R, S} \right] \right\}. \end{aligned}$$

Consider the time $T-1$ problem.

$$\begin{aligned} \hat{Y}_{T-1}^T &= \max_{u_{T-1} \in \tilde{\mathcal{A}}_T^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} \left\{ 1 + (m(F_{T-1}) + \theta_{T-1})' u_{T-1} \right. \\ &\quad \left. - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + V_{T-1} \right) u_{T-1} \right\}. \quad (4.C.1) \end{aligned}$$

The inner minimization problem in the problem (4.C.1) can be reduced to the following:

$$\min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V}|_{T-1}^{T-1}} \left\{ \theta_{T-1}' u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' V_{T-1} u_{T-1} \right\}.$$

The above problem is the same as the problem (4.B.3) in the proof of Theorem 4.5.

Therefore, the solutions are

$$\begin{aligned}\theta_{T-1} &= -\frac{\eta_{T-1}^\theta(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d)u_{T-1}}{\sqrt{u'_{T-1}(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d)u_{T-1}}}, \\ V_{T-1} &= \frac{\eta_{T-1}^V\|A(F_{T-1})\|}{\|u_{T-1}\|^2}u_{T-1}u'_{T-1}.\end{aligned}$$

Unlike Theorem 4.5, we do not need to consider the case $u_{T-1} = \mathbf{0}_d$ since $\mathbf{0}_d \notin \tilde{\mathcal{A}}_T^u|_{T-1}^{T-1}$.

Therefore, the time $T - 1$ problem is

$$\begin{aligned}\hat{Y}_{T-1}^T = \max_{u_{T-1} \in \tilde{\mathcal{A}}_T^u|_{T-1}^{T-1}} & \left\{ 1 + (m(F_{T-1}))'u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2}u'_{T-1}\left(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d\right)u_{T-1} \right. \\ & \left. - \eta_{T-1}^\theta\sqrt{u'_{T-1}(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d)u_{T-1}} \right\}.\end{aligned}$$

The Lagrange function of the above problem is

$$\begin{aligned}\mathcal{L}^u &= 1 + (m(F_{T-1}))'u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2}u'_{T-1}\left(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d\right)u_{T-1} \\ &\quad - \eta_{T-1}^\theta\sqrt{u'_{T-1}(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d)u_{T-1}} + \lambda_u(1 - \mathbf{1}'_d u_{T-1}),\end{aligned}$$

where λ_u is the Lagrange multiplier. The first order condition is

$$m(F_{T-1}) - \lambda_u \mathbf{1}_d - \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right) \left(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d \right) u_{T-1} = \mathbf{0}_d,$$

where

$$\psi_{T-1}^* = \sqrt{u'_{T-1}(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d)u_{T-1}}.$$

Hence,

$$u_{T-1} = \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^{-1} \left(A(F_{T-1}) + \eta_{T-1}^V\|A(F_{T-1})\|I_d \right)^{-1} \left(m(F_{T-1}) - \lambda_u \mathbf{1}_d \right).$$

Since the constraint is binded, we have

$$1 = \mathbf{1}'_d u_{T-1} = \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^{-1} \left(c_{T-1}^{worf} - \lambda_u b_{T-1}^{worf} \right).$$

Therefore, we obtain

$$\lambda_u = \frac{1}{b_{T-1}^{worf}} \left(c_{T-1}^{worf} - \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right) \right).$$

The optimal portfolio is

$$u_{T-1} = \frac{1}{\hat{\gamma}_{T-1} + \eta_{T-1}^\theta / \psi_{T-1}^*} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} \\ \times \left(m(F_{T-1}) - \frac{c_{T-1}^{worf} - (\hat{\gamma}_{T-1} + \eta_{T-1}^\theta / \psi_{T-1}^*)}{b_{T-1}^{worf}} \mathbf{1}_d \right).$$

Then,

$$\begin{aligned} & b_{T-1}^{worf} (\psi_{T-1}^*)^2 \\ &= b_{T-1}^{worf} u_{T-1}' (A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d) u_{T-1} \\ &= \frac{b_{T-1}^{worf}}{\left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^2} \left(a_{T-1}^{worf} - 2 \frac{c_{T-1}^{worf} - \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)}{b_{T-1}^{worf}} c_{T-1}^{worf} + \frac{\left(c_{T-1}^{worf} - \left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right) \right)^2}{b_{T-1}^{worf}} \right) \\ &= \frac{b_{T-1}^{worf}}{\left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^2} \left(a_{T-1}^{worf} - \frac{(c_{T-1}^{worf})^2}{b_{T-1}^{worf}} + \frac{\left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^2}{b_{T-1}^{worf}} \right) \\ &= \frac{a_{T-1}^{worf} b_{T-1}^{worf} - (c_{T-1}^{worf})^2}{\left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^2} + 1 = \frac{d_{T-1}^{worf}}{\left(\hat{\gamma}_{T-1} + \frac{\eta_{T-1}^\theta}{\psi_{T-1}^*} \right)^2} + 1. \end{aligned}$$

Therefore, we can consider the following polynomial equation for ψ_{T-1}^* :

$$b_{T-1}^{worf} (\psi_{T-1}^*)^2 = \frac{d_{T-1}^{worf}}{\left(\hat{\gamma}_{T-1} + \eta_{T-1}^\theta / \psi_{T-1}^* \right)^2} + 1. \quad (4.C.2)$$

Expanding it, we obtain

$$\begin{aligned} q(\psi_{T-1}^*) &:= b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 (\psi_{T-1}^*)^4 + 2b_{T-1}^{worf} \hat{\gamma}_{T-1} \eta_{T-1}^\theta (\psi_{T-1}^*)^3 \\ &\quad + \left(b_{T-1}^{worf} (\eta_{T-1}^\theta)^2 - (d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2) \right) (\psi_{T-1}^*)^2 \\ &\quad - 2\hat{\gamma}_{T-1} \eta_{T-1}^\theta \psi_{T-1}^* - (\eta_{T-1}^\theta)^2 = 0. \end{aligned} \quad (4.C.3)$$

The equation $q(\psi_{T-1}^*) = 0$ is essentially the same as the equation (A11) in Garlappi et al. (2007), therefore it has a unique positive solution by the discussion in Garlappi et al. (2007). We show the existence of the unique positive solution to the equation (4.C.3). Since the equation (4.C.3) is a quartic equation, it has four solutions. Since $q(0) = -(\eta_{T-1}^\theta)^2 < 0$ and $b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 > 0$, the equation (4.C.3) has at least two real solutions: the one is a positive solution and the one is a negative solution. We show that the positive solution is unique. Consider the following quadratic equation.

$$\begin{aligned} \frac{d^2 q(\psi_{T-1}^*)}{d(\psi_{T-1}^*)^2} &= 12b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 (\psi_{T-1}^*)^2 + 12b_{T-1}^{worf} \hat{\gamma}_{T-1} \eta_{T-1}^\theta \psi_{T-1}^* \\ &\quad + 2 \left(b_{T-1}^{worf} (\eta_{T-1}^\theta)^2 - (d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2) \right) = 0. \end{aligned} \quad (4.C.4)$$

The discriminant of the quadratic equation (4.C.4) is

$$\begin{aligned} D &= 36(b_{T-1}^{worf})^2 \hat{\gamma}_{T-1}^2 (\eta_{T-1}^\theta)^2 - 24b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 \left(b_{T-1}^{worf} (\eta_{T-1}^\theta)^2 - (d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2) \right) \\ &= 12(b_{T-1}^{worf})^2 \hat{\gamma}_{T-1}^2 (\eta_{T-1}^\theta)^2 + 24b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 (d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2) > 0. \end{aligned}$$

Hence, the equation (4.C.4) has two real solutions. Moreover, we have

$$\frac{d^2 q(\psi_{T-1}^*)}{d(\psi_{T-1}^*)^2} = 12b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 \left(\psi_{T-1}^* + \frac{\eta_{T-1}^\theta}{2\hat{\gamma}_{T-1}} \right)^2 - \left(b_{T-1}^{worf} (\eta_{T-1}^\theta)^2 + d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2 \right).$$

Since $\eta_{T-1}^\theta / (2\hat{\gamma}_{T-1}) > 0$, the equation (4.C.4) has at least one negative solution. Next, consider the first derivative of q such that

$$\begin{aligned} \frac{dq(\psi_{T-1}^*)}{d\psi_{T-1}^*} &= 4b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 (\psi_{T-1}^*)^3 + 6b_{T-1}^{worf} \hat{\gamma}_{T-1} \eta_{T-1} (\psi_{T-1}^*)^2 \\ &\quad + 2 \left(b_{T-1}^{worf} (\eta_{T-1}^\theta)^2 - (d_{T-1}^{worf} + \hat{\gamma}_{T-1}^2) \right) \psi_{T-1}^* - 2\hat{\gamma}_{T-1} \eta_{T-1}^\theta = 0. \end{aligned} \quad (4.C.5)$$

Then, the cubic equation (4.C.5) has at least one positive solution since $4b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 > 0$ and $-2\hat{\gamma}_{T-1} \eta_{T-1}^\theta < 0$. However, since the equation (4.C.4) has at least one negative solution, we conclude that the equation (4.C.5) has a unique positive solution. Now, let us consider the quartic equation (4.C.3) again. Since $q(0) = -(\eta_{T-1}^\theta)^2 < 0$ and $b_{T-1}^{worf} \hat{\gamma}_{T-1}^2 > 0$ hold, and since the equation (4.C.5) has a unique positive solution, the equation (4.C.3) has at most one local minimum on $\psi_{T-1}^* > 0$. Furthermore, we also have

$$\left. \frac{dq(\psi_{T-1}^*)}{d\psi_{T-1}^*} \right|_{\psi_{T-1}^*=0} = -2\hat{\gamma}_{T-1} \eta_{T-1}^\theta < 0.$$

These facts imply that the equation (4.C.3) has only one positive solution. Therefore, the positive solution of the equation (4.C.3) is unique. We also write ψ_{T-1}^* as the positive solution of the equation (4.C.3). Then, the optimal portfolio is

$$\begin{aligned} u_{T-1} &= \frac{1}{\hat{\gamma}_{T-1} + \eta_{T-1}^\theta / \psi_{T-1}^*} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} \\ &\quad \times \left(m(F_{T-1}) - \frac{c_{T-1}^{worf} - (\hat{\gamma}_{T-1} + \eta_{T-1}^\theta / \psi_{T-1}^*)}{b_{T-1}^{worf}} \mathbf{1}_d \right) \\ &= \frac{1}{\hat{\gamma}_{T-1} + |g_{T-1}| \eta_{T-1}^\theta / \psi_{T-1}^*} \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right)^{-1} \\ &\quad \times \left(m_{T-1} - \frac{c_{T-1}^{worf} - (\hat{\gamma}_{T-1} + |g_{T-1}| \eta_{T-1}^\theta / \psi_{T-1}^*)}{b_{T-1}^{worf}} \mathbf{1}_d \right), \end{aligned}$$

where we have used $g_{T-1} = 1$ and $m_{T-1} = m(F_{T-1})$. Since the coefficients of the quadratic equation (4.C.3) are $\mathcal{F}_{T-1}^{R,F}$ -measurable, its solution ψ_{T-1}^* is also $\mathcal{F}_{T-1}^{R,F}$ -measurable. Therefore, u_{T-1} is $\mathcal{F}_{T-1}^{R,F}$ -measurable. The value function at time $T - 1$ is

$$\begin{aligned}\hat{Y}_{T-1}^T &= 1 + (m(F_{T-1}))'u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2 - \eta_{T-1}^\theta \psi_{T-1}^* \\ &= g_{T-1} + \frac{1}{\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*} \left(a_{T-1}^{worf} - \frac{c_{T-1}^{worf} - (\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*)}{b_{T-1}^{worf}} c_{T-1}^{worf} \right) \\ &\quad - \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2 - |g_{T-1}|\eta_{T-1}^\theta \psi_{T-1}^* \\ &= g_{T-1} + \frac{c_{T-1}^{worf}}{b_{T-1}^{worf}} + \frac{d_{T-1}^{worf}}{b_{T-1}^{worf}(\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*)} - \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2 - |g_{T-1}|\eta_{T-1}^\theta \psi_{T-1}^*,\end{aligned}$$

where we have used $g_{T-1} = 1$. Using the equation (4.C.2), we have

$$\begin{aligned}&\frac{d_{T-1}^{worf}}{b_{T-1}^{worf}(\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*)} - \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2 - |g_{T-1}|\eta_{T-1}^\theta \psi_{T-1}^* \\ &= \left((\psi_{T-1}^*)^2 - \frac{1}{b_{T-1}^{worf}} \right) \left(\hat{\gamma}_{T-1} + \frac{|g_{T-1}|\eta_{T-1}^\theta}{\psi_{T-1}^*} \right) - \left(\hat{\gamma}_{T-1} + \frac{|g_{T-1}|\eta_{T-1}^\theta}{\psi_{T-1}^*} \right) (\psi_{T-1}^*)^2 \\ &\quad + \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2 \\ &= -\frac{\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*}{b_{T-1}^{worf}} + \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2.\end{aligned}$$

Hence, the value function is

$$\hat{Y}_{T-1}^T = g_{T-1} + \frac{c_{T-1}^{worf} - (\hat{\gamma}_{T-1} + |g_{T-1}|\eta_{T-1}^\theta/\psi_{T-1}^*)}{b_{T-1}^{worf}} + \frac{\hat{\gamma}_{T-1}}{2}(\psi_{T-1}^*)^2$$

and it is $\mathcal{F}_{T-1}^F$ -measurable.

Now, we assume that $\hat{Y}_{t+1}^T$ is $\mathcal{F}_{t+1}^F$ -measurable for some $0 \leq t \leq T - 2$. Then, we have

$$\begin{aligned}\hat{c}_t(F_t, \theta_t, V_t, u_t) + \delta E^{\theta, V} \left[Y_{t+1}^T (1 + (R_{t+1})'u_t) \mid \mathcal{F}_t^{R, S} \right] \\ = g_t + (m_t + g_t \theta_t)'u_t - \frac{\hat{\gamma}_t}{2} u_t' \left(A(F_t) + V_t \right) u_t.\end{aligned}$$

Then, the minimization problem for (θ_t, V_t) can be reduced to

$$\min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V} \mid \mathcal{I}_t^t} \left\{ g_t \theta_t' u_t - \frac{\hat{\gamma}_t}{2} u_t' V_t u_t \right\}.$$

Similarly to the time $T - 1$ problem, the solution of the above problem is

$$\begin{aligned}\theta_t &= -\frac{\text{sgn}(g_t)\eta_t^\theta(A(F_t) + \eta_t^V\|A(F_t)\|I_d)u_t}{\sqrt{u_t'(A(F_t) + \eta_t^V\|A(F_t)\|I_d)u_t}}, \\ V_t &= \frac{\eta_t^V\|A(F_t)\|}{\|u_t\|^2}u_tu_t'.\end{aligned}$$

Therefore, the maximization problem for u_t is

$$\max_{u_t \in \tilde{\mathcal{A}}_T|_t^t} \left\{ g_t + m_t' u_t - \frac{\hat{\gamma}_t}{2} u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t - |g_t| \eta_t^\theta \sqrt{u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t} \right\}.$$

Since the above problem has the same form as the time $T - 1$ problem, the optimal portfolio is

$$u_t = \frac{1}{\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*} \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right)^{-1} \left(m_t - \frac{c_t^{worf} - (\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)}{b_t^{worf}} \mathbf{1}_d \right),$$

where ψ_t^* is the unique positive solution of the following quartic equation:

$$b_t^{worf}(\psi_t^*)^2 = \frac{d_t^{worf}}{(\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)^2} + 1.$$

The value function is

$$\hat{Y}_t^T = g_t + \frac{c_t^{worf} - (\hat{\gamma}_t + |g_t| \eta_t^\theta / \psi_t^*)}{b_t^{worf}} + \frac{\hat{\gamma}_t}{2} (\psi_t^*)^2,$$

and it is $\mathcal{F}_t^F$ -measurable.

By the mathematical induction, we therefore conclude that the optimal portfolio and that the value function is determined as in Proposition 4.7. $\square$

Proof of Proposition 4.8. In this case, we can also apply the dynamic programming procedure. Consider the time $T - 1$ problem,

$$\begin{aligned}\hat{Y}_{T-1}^{+T} &= \max_{u_{T-1} \in \tilde{\mathcal{A}}_{+T}^u|_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_{T-1}^{\theta, V}|_{T-1}^{T-1}} \left\{ 1 + \left(m(F_{T-1}) + \theta_{T-1} \right)' u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + V_{T-1} \right) u_{T-1} \right\}.\end{aligned}$$

Since $g_{T-1}^+ = 1$ and $m_{T-1}^+ = m(F_{T-1})$ by the definition, the objective function of the time $T - 1$ problem can be expressed as

$$g_{T-1}^+ + \left(m_{T-1}^+ + g_{T-1}^+ \theta_{T-1}\right)' u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + V_{T-1}\right) u_{T-1}.$$

Hence, it suffices to consider the following optimization problem.

$$\max_{u_{T-1} \in \tilde{\mathcal{A}}_{+T}^u |_{T-1}^{T-1}} \min_{(\theta_{T-1}, V_{T-1}) \in \mathcal{A}_T^{\theta, V} |_{T-1}^{T-1}} \left\{ g_{T-1}^+ + \left(m_{T-1}^+ + g_{T-1}^+ \theta_{T-1}\right)' u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + V_{T-1}\right) u_{T-1} \right\}.$$

Solving the inner minimization problem, we can express the above problem as

$$\max_{u_{T-1} \in \tilde{\mathcal{A}}_{+T}^u |_{T-1}^{T-1}} \left\{ g_{T-1}^+ + (m_{T-1}^+)' u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1} - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1}} \right\}. \quad (4.C.6)$$

Now, we consider the auxiliary problem of (4.C.6), whose feasible set of portfolios is $\overline{\mathcal{C}^+}$. If we regard $\overline{\mathcal{C}^+}$ as a correspondence from $\mathbb{R}^K$ to $\mathbb{R}^d$, then its graph is

$$Gr(\overline{\mathcal{C}^+}) := \{(f, \phi) \in \mathbb{R}^K \times \mathbb{R}^d \mid \phi \in \overline{\mathcal{C}^+}\} = \mathbb{R}^K \times \overline{\mathcal{C}^+}.$$

It is clear that $Gr(\overline{\mathcal{C}^+})$ is a Borel subset on $\mathbb{R}^K \times \mathbb{R}^d$. This implies that $\overline{\mathcal{C}^+}$ is Borel measurable. Furthermore, the objective function of the problem (4.C.6) is continuous in u_{T-1} and also $\mathcal{F}_{T-1}^F$ -measurable for any fixed u_{T-1} . Moreover, $\overline{\mathcal{C}^+}$ is a non-empty compact subset of $\mathbb{R}^d$ endowed with the Euclidean topology. Hence, by the measurable selection theorem (see Appendix D in Hernández-Lerma and Lasserre (1996)), there exists an $\mathcal{F}_{T-1}^F$ -measurable random vector $u_{T-1}^* \in \overline{\mathcal{C}^+}$ such that

$$\begin{aligned} & g_{T-1}^+ + (m_{T-1}^+)' u_{T-1}^* - \frac{\widehat{\gamma}_{T-1}}{2} (u_{T-1}^*)' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1}^* \\ & \quad - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{(u_{T-1}^*)' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1}^*} \\ &= \max_{u_{T-1} \in \overline{\mathcal{C}^+}} \left\{ g_{T-1}^+ + (m_{T-1}^+)' u_{T-1} - \frac{\widehat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1} - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d\right) u_{T-1}} \right\}. \end{aligned}$$

Moreover, since $u_{T-1}^* \in \tilde{\mathcal{A}}_{+T}^u|_{T-1}^{T-1}$ and $\tilde{\mathcal{A}}_{+T}^u|_{T-1}^{T-1} \subseteq \bar{\mathcal{C}}^+$, we conclude that

$$\begin{aligned}
& g_{T-1}^+ + (m_{T-1}^+)' u_{T-1}^* - \frac{\hat{\gamma}_{T-1}}{2} (u_{T-1}^*)' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}^* \\
& \quad - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{(u_{T-1}^*)' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}^*} \\
& = \max_{u_{T-1} \in \tilde{\mathcal{A}}_{+T}^u|_{T-1}^{T-1}} \left\{ g_{T-1}^+ + (m_{T-1}^+)' u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \right. \\
& \quad \left. - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}} \right\} \\
& = \max_{u_{T-1} \in \bar{\mathcal{C}}^+} \left\{ g_{T-1}^+ + (m_{T-1}^+)' u_{T-1} - \frac{\hat{\gamma}_{T-1}}{2} u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1} \right. \\
& \quad \left. - |g_{T-1}^+| \eta_{T-1}^\theta \sqrt{u_{T-1}' \left(A(F_{T-1}) + \eta_{T-1}^V \|A(F_{T-1})\| I_d \right) u_{T-1}} \right\} \\
& = \hat{Y}_{T-1}^{+T}.
\end{aligned}$$

This implies that u_{T-1}^* is the maximizer of the problem (4.C.6). Moreover, $\hat{Y}_{T-1}^{+T}$ is $\mathcal{F}_{T-1}^F$ -measurable by the measurable selection theorem.

Now, we hypothesize that at some time $t+1$, $0 \leq t+1 \leq T-1$, $\hat{Y}_{t+1}^{+T}$ is $\mathcal{F}_{t+1}^S$ -measurable. Then, we have

$$\begin{aligned}
\hat{Y}_t^{+T} &:= \max_{u_t \in \tilde{\mathcal{A}}_{+T}^u|_t^t} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V}|_t^t} \left\{ 1 + (m(F_t) + \theta_t)' u_t - \frac{\hat{\gamma}_t}{2} u_t' \left(A(F_t) + V_t \right) u_t \right. \\
& \quad \left. + \delta \mathbb{E}^{\theta, V} \left[\hat{Y}_{t+1}^{+T} \mid \mathcal{F}_t^{R, F} \right] \right\} \\
&= \max_{u_t \in \tilde{\mathcal{A}}_{+T}^u|_t^t} \min_{(\theta_t, V_t) \in \mathcal{A}_T^{\theta, V}|_t^t} \left\{ g_t^+ + \left(m_t^+ + g_t^+ \theta_t \right)' u_t - \frac{\hat{\gamma}_t}{2} u_t' \left(A(F_t) + V_t \right) u_t \right\} \\
&= \max_{u_t \in \tilde{\mathcal{A}}_{+T}^u|_t^t} \left\{ g_t^+ + (m_t^+)' u_t - \frac{\hat{\gamma}_t}{2} u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t \right. \\
& \quad \left. - |g_t^+| \eta_t^\theta \sqrt{u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t} \right\}. \quad (4.C.7)
\end{aligned}$$

The problem (4.C.7) has the same form as the problem (4.C.6), and m_t^+ , g_t^+ , $\hat{\gamma}_t$, $A(F_t)$, η_t^θ and η_t^V are $\mathcal{F}_t^F$ -measurable. Therefore, we can use the measurable selection theorem in the same manner as we did in the case of the problem (4.C.6). By the measurable

selection theorem, there exists an $\mathcal{F}_t^F$ -measurable random vector $u_t^* \in \overline{\mathcal{C}^+}$ such that

$$\begin{aligned}
& g_t^+ + (m_t^+)'u_t^* - \frac{\widehat{\gamma}_t}{2}(u_t^*)' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t^* \\
& \quad - |g_t^+| \eta_t^\theta \sqrt{(u_t^*)' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t^*} \\
& = \max_{u_t \in \widetilde{\mathcal{A}}_{+T}^u |_t^t} \left\{ g_t^+ + (m_t^+)'u_t - \frac{\widehat{\gamma}_t}{2} u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t \right. \\
& \quad \left. - |g_t^+| \eta_t^\theta \sqrt{u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t} \right\} \\
& = \max_{u_t \in \overline{\mathcal{C}^+}} \left\{ g_t^+ + (m_t^+)'u_t - \frac{\widehat{\gamma}_t}{2} u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t \right. \\
& \quad \left. - |g_t^+| \eta_t^\theta \sqrt{u_t' \left(A(F_t) + \eta_t^V \|A(F_t)\| I_d \right) u_t} \right\} = \widehat{Y}_t^{+T}.
\end{aligned}$$

Moreover, $\widehat{Y}_t^{+T}$ is $\mathcal{F}_t^F$ -measurable.

By the mathematical induction, $\widehat{Y}_t^{+T}$ is $\mathcal{F}_t^F$ -measurable for all $0 \leq t \leq T-1$.

Therefore, the value function of the original optimization problem is

$$V_{+T}^{worf}(x, f) = \mathbb{E}[\widehat{Y}_t^{+T} \mid F_0 = f]x,$$

and the optimal portfolio at time t is u_t^* . □

Appendix 4.D Tables and Figures

Table 4.1. The Abbreviations of Typical Portfolios and Six Extreme Strategies. Panel A displays the abbreviations of typical portfolios, and Panel B displays the details concerning the extreme strategies. The first column in Panel B lists the abbreviations of the extreme strategies. The second and third columns represent portfolios in different states. The forth and fifth columns display behaviors of parameters which justify the extreme strategies.

Panel A	
Abbreviation	Portfolio
max SR	The portfolio maximizing the empirical Sharpe ratio
GMV	The single-period global minimum-variance portfolio
EW	The equally weighted portfolio
VW	The value-weighted portfolio
NE (No Error)	The no error portfolio (always $\eta_t^\theta = \eta_t^V = 0$)

Panel B				
Abbreviation	Portfolio		Parameters	
	Good State	Bad State	Good State	Bad State
GMV-EW	GMV	EW	$\eta_t^V = 0$ and $\eta_t^\theta \rightarrow \infty$	$\eta_t^V \rightarrow \infty$
EW-NE	EW	No Error	$\eta_t^V \rightarrow \infty$	$\eta_t^\theta = \eta_t^V = 0$
GMV-NE	GMV	No Error	$\eta_t^V = 0$ and $\eta_t^\theta \rightarrow \infty$	$\eta_t^\theta = \eta_t^V = 0$
EW-GMV	EW	GMV	$\eta_t^V \rightarrow \infty$	$\eta_t^V = 0$ and $\eta_t^\theta \rightarrow \infty$
NE-EW	No Error	EW	$\eta_t^\theta = \eta_t^V = 0$	$\eta_t^V \rightarrow \infty$
NE-GMV	No Error	GMV	$\eta_t^\theta = \eta_t^V = 0$	$\eta_t^V = 0$ and $\eta_t^\theta \rightarrow \infty$

Table 4.2. International Diversification ($d = 4$). This table reports the Sharpe ratios of various strategies obtained from the back test of the international asset allocations. “IID” represents the investor who believes that the number of market states is 1. “RS” represents the investor who believes that the number of market states is 2. The abbreviations of portfolios are listed in Table 4.1. In the case of $\eta_t^\theta = 0$ with short selling, we can not compute the optimal portfolios since they diverge. Therefore, we use the portfolios of $\eta_t^\theta = 1$ and $\eta_t^V = 0$ as the No Error portfolios in the extreme strategies.

Short selling is permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.1014	0.0	-	0.0954	0.0941	0.0936	0.0934	0.0933	0.0927
	0.5	-	0.0878	0.0872	0.0869	0.0868	0.0867	0.0864
	η_t^V	1.0	0.0849	0.0843	0.0842	0.0841	0.0840	0.0838
	3.0	-	0.0818	0.0815	0.0814	0.0814	0.0813	0.0812
	5.0	-	0.0810	0.0808	0.0807	0.0807	0.0807	0.0806
RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.0889	0.0	-	0.0837	0.0855	0.0859	0.0860	0.0861	0.0863
	0.5	-	0.0856	0.0852	0.0850	0.0849	0.0849	0.0847
	η_t^V	1.0	0.0835	0.0831	0.0830	0.0829	0.0829	0.0827
	3.0	-	0.0813	0.0811	0.0810	0.0809	0.0809	0.0808
	5.0	-	0.0808	0.0805	0.0805	0.0804	0.0804	0.0803

Short selling is not permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.1014	0.0	0.1025	0.1028	0.1021	0.1018	0.1017	0.1016	0.1012
	0.5	0.0896	0.0878	0.0872	0.0869	0.0868	0.0867	0.0864
	η_t^V	1.0	0.1078	0.0849	0.0843	0.0842	0.0841	0.0838
	3.0	0.1019	0.0818	0.0815	0.0814	0.0814	0.0813	0.0812
	5.0	0.1081	0.0810	0.0808	0.0807	0.0807	0.0807	0.0806
RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.0889	0.0	0.0597	0.0973	0.0952	0.0942	0.0937	0.0934	0.0920
	0.5	0.0674	0.0856	0.0852	0.0850	0.0849	0.0849	0.0847
	η_t^V	1.0	0.0583	0.0835	0.0831	0.0830	0.0829	0.0827
	3.0	0.0577	0.0813	0.0811	0.0810	0.0809	0.0809	0.0808
	5.0	0.0570	0.0808	0.0805	0.0805	0.0804	0.0804	0.0803
EW		Extreme Strategies						
0.0795		Short selling is permitted.			Short selling is not permitted.			
		GMV-EW	EW-NE	GMV-NE	GMV-EW	EW-NE	GMV-NE	
		0.1032	0.0664	0.0891	0.0922	0.0949	0.1078	
		EW-GMV	NE-EW	NE-GMV	EW-GMV	NE-EW	NE-GMV	
		0.0636	0.0976	0.0809	0.0783	0.0457	0.0433	

Table 4.3. Diversification across Industries ($d = 5$). This table reports the Sharpe ratios of various strategies obtained from the back test of the asset allocations across industries. “IID” represents the investor who believes that the number of market states is 1. “RS” represents the investor who believes that the number of market states is 2. The abbreviations of portfolios are listed in Table 4.1. In the case of $\eta_t^\theta = 0$ with short selling, we can not compute the optimal portfolios since they diverge. Therefore, we use the portfolios of $\eta_t^\theta = 1$ and $\eta_t^V = 0$ as the No Error portfolios in the extreme strategies.

Short selling is permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2210	0.0	-	0.2732	0.2791	0.2808	0.2817	0.2821	0.2839
	0.5	-	0.2213	0.2220	0.2222	0.2224	0.2224	0.2227
η_t^V	1.0	-	0.2165	0.2169	0.2170	0.2171	0.2171	0.2173
	3.0	-	0.2127	0.2129	0.2129	0.2130	0.2130	0.2131
	5.0	-	0.2119	0.2120	0.2121	0.2121	0.2121	0.2121

RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.1968	0.0	-	0.1912	0.2524	0.2601	0.2632	0.2649	0.2707
	0.5	-	0.2238	0.2269	0.2278	0.2283	0.2286	0.2297
η_t^V	1.0	-	0.2181	0.2202	0.2209	0.2212	0.2214	0.2222
	3.0	-	0.2129	0.2141	0.2145	0.2147	0.2148	0.2153
	5.0	-	0.2118	0.2127	0.2130	0.2131	0.2132	0.2136

Short selling is not permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2210	0.0	0.1822	0.2496	0.2547	0.2561	0.2570	0.2576	0.2596
	0.5	0.1643	0.2213	0.2220	0.2222	0.2224	0.2224	0.2227
η_t^V	1.0	0.1724	0.2165	0.2169	0.2170	0.2171	0.2171	0.2173
	3.0	0.1783	0.2127	0.2129	0.2129	0.2130	0.2130	0.2131
	5.0	0.1487	0.2119	0.2120	0.2121	0.2121	0.2121	0.2121

RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.1968	0.0	0.0660	0.2394	0.2502	0.2543	0.2573	0.2592	0.2664
	0.5	0.0796	0.2239	0.2269	0.2278	0.2283	0.2286	0.2297
η_t^V	1.0	0.0769	0.2181	0.2202	0.2209	0.2212	0.2214	0.2222
	3.0	0.0653	0.2129	0.2141	0.2145	0.2147	0.2148	0.2153
	5.0	0.0841	0.2118	0.2127	0.2130	0.2131	0.2132	0.2136

EW		Extreme Strategies					
0.0795		Short selling is permitted.			Short selling is not permitted.		
		GMV-EW	EW-NE	GMV-NE	GMV-EW	EW-NE	GMV-NE
VW		0.2234	0.2457	0.2592	0.2281	0.1199	0.1344
0.1825		EW-GMV	NE-EW	NE-GMV	EW-GMV	NE-EW	NE-GMV
		0.2564	0.1657	0.1964	0.2473	0.1410	0.1678

Table 4.4. Diversification among Sizes and Values ($d = 6$). This table reports the Sharpe ratios of various strategies obtained from the back test of the asset allocations among the 2×3 size- and book-to-market-sorted portfolios by Fama and French (1993). “IID” represents the investor who believes that the number of market states is 1. “RS” represents the investor who believes that the number of market states is 2. The abbreviations of portfolios are listed in Table 4.1. In the case of $\eta_t^\theta = 0$ with short selling, we can not compute the optimal portfolios since they diverge. Therefore, we use the portfolios of $\eta_t^\theta = 1$ and $\eta_t^V = 0$ as the No Error portfolios in the extreme strategies.

Short selling is permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2139	0.0	-	0.3406	0.3293	0.3243	0.3215	0.3197	0.3122
	0.5	-	0.2237	0.2228	0.2224	0.2223	0.2222	0.2218
	η_t^V 1.0	-	0.2176	0.2170	0.2168	0.2167	0.2166	0.2164
	3.0	-	0.2126	0.2123	0.2122	0.2122	0.2121	0.2120
	5.0	-	0.2115	0.2113	0.2112	0.2111	0.2111	0.2110

RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2081	0.0	-	0.2389	0.2473	0.2486	0.2489	0.2491	0.2488
	0.5	-	0.2184	0.2195	0.2198	0.2200	0.2200	0.2203
	η_t^V 1.0	-	0.2147	0.2154	0.2156	0.2157	0.2158	0.2160
	3.0	-	0.2113	0.2117	0.2118	0.2119	0.2119	0.2120
	5.0	-	0.2105	0.2108	0.2109	0.2109	0.2110	0.2110

Short selling is not permitted.

IID								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2139	0.0	0.2202	0.2260	0.2263	0.2255	0.2253	0.2257	0.2256
	0.5	0.2289	0.2237	0.2228	0.2224	0.2223	0.2222	0.2218
	η_t^V 1.0	0.2207	0.2176	0.2170	0.2168	0.2167	0.2166	0.2164
	3.0	0.2271	0.2126	0.2123	0.2122	0.2122	0.2121	0.2120
	5.0	0.2291	0.2115	0.2113	0.2112	0.2111	0.2111	0.2110

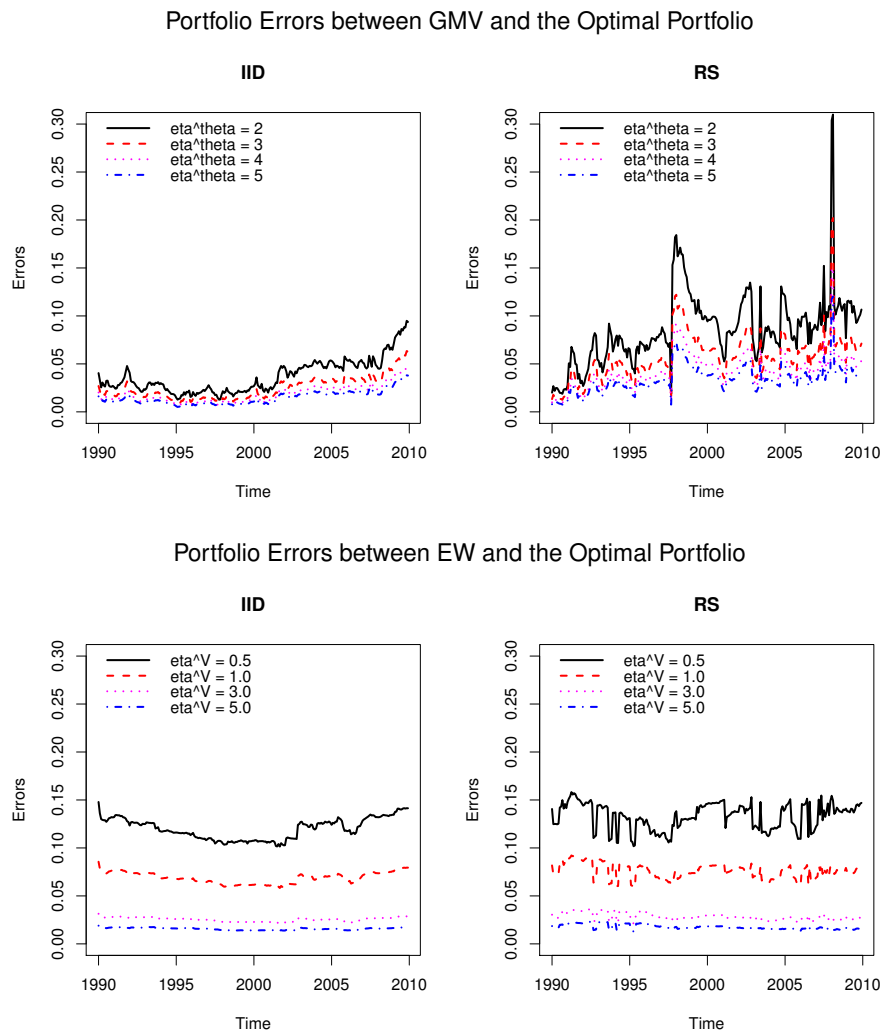
RS								
		η_t^θ						GMV
max SR		0.0	1.0	2.0	3.0	4.0	5.0	∞
0.2081	0.0	0.1791	0.2101	0.2133	0.2146	0.2155	0.2158	0.2175
	0.5	0.1874	0.2184	0.2195	0.2198	0.2199	0.2200	0.2203
	η_t^V 1.0	0.1949	0.2147	0.2154	0.2156	0.2157	0.2158	0.2160
	3.0	0.1856	0.2113	0.2117	0.2118	0.2119	0.2119	0.2120
	5.0	0.1999	0.2105	0.2108	0.2109	0.2109	0.2110	0.2110

EW		Extreme Strategies					
0.0795		Short selling is permitted.			Short selling is not permitted.		
		GMV-EW	EW-NE	GMV-NE	GMV-EW	EW-NE	GMV-NE
	VW	0.2763	0.1697	0.2281	0.2336	0.1790	0.1979
0.2010		EW-GMV	NE-EW	NE-GMV	EW-GMV	NE-EW	NE-GMV
		0.1876	0.2861	0.2592	0.1939	0.2069	0.1920

Table 4.5. The Results of the Mimicking Strategies' Sharpe Ratios Obtained from the Back Tests. The second and third columns display parameters in the good state, and the fourth and fifth columns display parameters in the bad state. The sixth column shows whether short selling is permitted or not. The seventh column reports the Sharpe ratios of the mimicking extreme strategies. The eighth column reports the Sharpe ratios of the original extreme strategies. The ninth column reports the average root square errors between the original extreme strategies and the mimicking strategies. The extreme strategies are listed in Table 4.1.

International Indexes								
	Good State		Bad State		Short Selling	Sharpe Ratio	Original Sharpe Ratio	Ave. RSE
	η_t^θ	η_t^V	η_t^θ	η_t^V				
mGMV-EW	30	0	10	5	Yes	0.1038	0.1032	0.0116
					No	0.0929	0.0922	0.0102
mEW-GMV	10	5	30	0	Yes	0.0640	0.0636	0.0116
					No	0.0788	0.0783	0.0112
Industry Indexes								
	Good State		Bad State		Short Selling	Sharpe Ratio	Original Sharpe Ratio	Ave. RSE
	η_t^θ	η_t^V	η_t^θ	η_t^V				
mGMV-EW	30	0	10	5	Yes	0.2252	0.2234	0.0134
					No	0.2303	0.2281	0.0089
mEW-GMV	10	5	30	0	Yes	0.2567	0.2564	0.0095
					No	0.2470	0.2473	0.0079
Size- and Value-Sorted Portfolios								
	Good State		Bad State		Short Selling	Sharpe Ratio	Original Sharpe Ratio	Ave. RSE
	η_t^θ	η_t^V	η_t^θ	η_t^V				
mGMV-EW	30	0	10	5	Yes	0.2776	0.2763	0.0248
					No	0.2343	0.2336	0.0118
mEW-GMV	10	5	30	0	Yes	0.1882	0.1876	0.0199
					No	0.1945	0.1939	0.0117

Figure 4.1. Portfolio Errors of International Indexes Data Set When Short Selling Is Permitted. The following four figures display portfolio errors $\|u_t^*\|_{GMV}$ and $\|u_t^*\|_{EW}$ at each time t . The upper two figures show the portfolio errors from GMV, $\|u_t^*\|_{GMV}$, with $\eta_t^V = 0$. The lower two figures represent the portfolio errors from EW, $\|u_t^*\|_{EW}$, with $\eta_t^\theta = 5$. The left two figures represent the portfolio errors of the IID investor and the right two figures show those of the RS investor.



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Chapter 5

Optimal Switching under Ambiguity and Its Applications in Finance

5.1 Introduction

Optimal switching problems are widely used to describe many situations in finance and economics. For example, they are applied to natural resource extractions (Brennan and Schwartz (1985) and Brekke and Øksendal (1994)), reversible investments (Ly Vath and Pham (2007)), and entry and exit decisions of firms (Dixit (1989)). In plain words, the optimal switching problems are the problems that a decision maker chooses her actions from a discrete state space to maximize her profit (objective function).

In this chapter, our aims are to construct optimal switching problems under ambiguity and to derive general properties of solutions to these problems. A concept of ambiguity aversion is one of prominent issues in recent finance and economics. The ambiguity aversion (also known as the Knightian uncertainty aversion or the model uncertainty aversion) is the behavior that an economic agent prefers avoiding the event whose occurrence probability is unknown. Ellsberg (1961) first provides illustrative examples of the ambiguity aversion and Gilboa and Schmeidler (1989) and Schmeidler (1989) economically axiomatize the ambiguity aversion. After these works, Chen and Epstein (2002) establish a model of the ambiguity aversion in continuous time and Riedel (2009) and Cheng and Riedel (2013) construct optimal stopping problems under ambiguity.

Using a concept of the ambiguity aversion, one can describe the properties not captured by a usual trade-off between returns and risks. Therefore, we can consider a

more practical optimal switching problem. In existing literature, Hamadène and Zhang (2010) mention that their model can be applied to the optimal switching problems under ambiguity, but in our knowledge, there does not exist a study of general results of optimal switching problems under ambiguity. Therefore, it is worth studying optimal switching problems under ambiguity.

To deal with optimal switching problems under ambiguity, we use frameworks of backward stochastic differential equations (hereafter BSDEs). BSDEs are introduced by Bismut (1978) and Pardoux and Peng (1990) establish a general theory of BSDEs. Many researchers (e.g., El Karoui, Peng, and Quenez (1997b), Rouge and El Karoui (2000), Chen and Epstein (2002) and Cheng and Riedel (2013)) apply the theory of BSDEs to various problems in finance and economics. Recently, a theory of multidimensional reflected BSDEs (hereafter multidimensional RBSDEs) is developed by Hamadène and Zhang (2010), Hu and Tang (2010) and Hamadène and Morlais (2013) to study the optimal switching problems. This approach makes us naturally incorporate ambiguity aversion into the optimal switching problems. Therefore, multidimensional RBSDEs have an important role in this study.

In this chapter, our contributions are as follows.

1. We characterize the optimal switching problems under ambiguity in both of the finite horizon and infinite horizon using multidimensional RBSDEs.
2. We show that value functions of the optimal switching problems under ambiguity are viscosity solutions to some system of partial differential equations.
3. Unlike existing literature, we do not assume non-negativity of switching costs.

We first define the optimal switching problems under ambiguity and characterize them using the theory of multidimensional RBSDEs by Hamadène and Zhang (2010). Hamadène and Zhang (2010) assume non-negativity of the switching costs and this assumption has an important role in their study. However, there are optimal switching problems that definitely need negative switching costs (i.e., positive switching benefits) such as the buy low and sell high problem (Zhang and Zhang (2008)) and the pair-trading problem (Ngo and Pham (2016)). Therefore, we do not assume the non-negativity of the

switching costs, and we need to modify the proof of Hamadène and Zhang (2010) to allow negative switching costs. In order to allow negative switching costs, we add a weak assumption of the switching costs. Since existing literature usually assumes non-negativity of switching costs (for example, Hamadène and Zhang (2010), Hu and Tang (2010) and Hamadène and Morlais (2013)), our results are more general than those of the existing literature in the sense of allowing negative switching costs. Furthermore, using the results of Hamadène and Morlais (2013), we show that value functions of the optimal switching problems under ambiguity are viscosity solutions of some system of partial differential equations.

Moreover, we show that under some conditions, the value function in the finite horizon problem converges to the value function in the infinite horizon. El Asri (2010) studies the infinite horizon problem using multidimensional RBSDEs under a non-negativity assumption of switching costs, but the most of existing studies mainly focus to the finite horizon problem. Therefore, our results may provide new insights in the optimal switching problems using multidimensional RBSDEs.

Finally, we give some examples of optimal switching problems under ambiguity in finance. We show that under certain conditions, the optimal switching problems under ambiguity can be interpreted as the optimal switching problems under a certain probability measure determined a priori. Therefore, the results of existing literature can be used to optimal switching problems under ambiguity. However, the problems not meeting these conditions provide more interesting results. In Section 5.6.3, we consider the buy low and sell high problem under ambiguity, which does not satisfy these conditions. Our results indicate that effects of ambiguity in this problem can not be reproduced by a simple change of the probability measure.

The rest of this chapter is organized as follows. Section 5.2 defines the optimal switching problems under ambiguity in the finite horizon using the concept of multiple priors introduced by Chen and Epstein (2002). Section 5.3 introduces multidimensional RBSDEs and proves the existence of their solutions. Section 5.4 verifies that the value functions of the optimal switching problems under ambiguity are characterized by the solutions to the multidimensional RBSDEs, and derives the system of partial differential

equations, which the value functions satisfy. Section 5.5 considers the infinite horizon problem. Section 5.6 provides some applications of optimal switching problems under ambiguity in finance. Lengthy proofs are in Appendix.

5.2 Preliminaries and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a d -dimensional Brownian motion $W = (W_t)_{t \geq 0}$. Let $T > 0$ be a finite constant time. We first consider an optimal switching problem during $[0, T]$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be an augmentation of the natural filtration generated by W .

We denote by $\alpha = (\alpha_t)_{t \geq 0}$ a control process such that

$$\alpha_t = \sum_{k \geq 0} i_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(t), \quad (5.2.1)$$

where $(i_k)_{k \geq 0}$ is a regime process taking values in a discrete state space $\mathcal{I} = \{1, \dots, I\}$, $I > 0$, and $(\tau_k)_{k \geq 0}$ is a non-decreasing sequence of stopping times. $\mathbb{1}_A(x)$ is an indicator function such that for a given set A ,

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

We suppose that each i_k is $\mathcal{F}_{\tau_k}$ -measurable. Under a control α , a decision maker chooses a regime i_k on $[\tau_k, \tau_{k+1})$ for all $k \geq 0$. For convenience, we also write a control as a sequence of pairs of regimes and stopping times: $\alpha = (\tau_k, i_k)_{k \geq 0}$.

Let $X = (X_t)_{0 \leq t \leq T}$ be a d -dimensional stochastic process satisfying the following stochastic differential equation (hereafter SDE):

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t, \quad (5.2.2)$$

where $\alpha = (\alpha_t)_{0 \leq t \leq T}$ is a control process. b and σ are measurable functions satisfying the following.

Assumption 5.1 $b : [0, T] \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^{d \times d}$ satisfy the following Lipschitz condition and quadratic growth condition:

$$\|b(t, x, i) - b(t, y, i)\| + \|\sigma(t, x, i) - \sigma(t, y, i)\| \leq L\|x - y\|, \quad (5.2.3)$$

$$\|b(t, x, i)\|^2 + \|\sigma(t, x, i)\|^2 \leq L^2(1 + \|x\|^2), \quad (5.2.4)$$

for every $t \in [0, T]$, $i \in \mathcal{I}$, and $x, y \in \mathbb{R}^d$, where L is a positive constant and $\|x\|$ is the Euclid norm of $x \in \mathbb{R}^d$.

Let $L_t^q(\mathbb{R}^d)$ be a set of d -dimensional, q -th integrable (that is, an L^q norm on $(\Omega, \mathcal{F}, \mathbb{P})$ is finite), and $\mathcal{F}_t$ -measurable random vectors. Let $\mathcal{T}_t^T$ be a set of stopping times taking values in $[t, T]$. Let $\widetilde{\mathcal{I}}_t$ be a set of $\mathcal{F}_t$ -measurable random variables taking values in $\mathcal{I}$. We define $\widetilde{\mathcal{K}}_T^q$ and $\overline{\mathcal{K}}_T$ as follows,

$$\widetilde{\mathcal{K}}_T^q := \left\{ (\nu, \eta, \iota) \mid \nu \in \mathcal{T}_0^T, \eta \in L_\nu^q(\mathbb{R}^d), \iota \in \widetilde{\mathcal{I}}_\nu \right\}, \quad (5.2.5)$$

$$\overline{\mathcal{K}}_T := [0, T] \times \mathbb{R}^d \times \mathcal{I}. \quad (5.2.6)$$

By Assumption 5.1, for every $(\nu, \eta, \iota) \in \widetilde{\mathcal{K}}_T^2$ and progressively measurable control α starting from $\alpha_\nu = \iota$, there exists a unique strong solution to the SDE (5.2.2) on $[\nu, T]$ starting from $X_\nu = \eta$ and controlled by α . We denote this controlled process by $X^{\nu, \eta, \iota, \alpha} = (X_s^{\nu, \eta, \iota, \alpha})_{\nu \leq s \leq T}$. Furthermore, it is well known that the moments of X is upper bounded (e.g., Corollary 2.5.12 in Krylov (1980) and Theorem 5.2.9 in Karatzas and Shreve (1991)). We shortly summarize the results of the moment estimates of X .

Proposition 5.2 Under Assumption 5.1, for every $q > 0$, there exist constants $C_{q,X} \geq 1$ and $C_q > 0$ such that

$$\mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q \right] \leq C_{q,X} (1 + \|x\|^q) e^{C_q(T-t)}, \quad (5.2.7)$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$, $i \in \mathcal{I}$ and control α . Note that $C_{q,X}$ and C_q do not depend on t, T, x, i and α . Furthermore, if a constant ρ is sufficiently large such that $\rho > C_q$, then there exists a positive constant $C_{q,X}^\infty$ such that

$$\mathbb{E} \left[\max_{s \geq t} e^{-\rho s} \left(1 + \|X_s^{t,x,i,\alpha}\|^q \right) \right] \leq C_{q,X}^\infty (1 + \|x\|^q) e^{-(\rho - C_q)t}, \quad (5.2.8)$$

for all $0 \leq t$, $x \in \mathbb{R}^d$, $i \in \mathcal{I}$ and control α . Note that $C_{q,X}^\infty$ does not depend on t, x, i and α .

The proof of Proposition 5.2 is in Appendix 5.A. Moreover, we can easily show that the results of Proposition 5.2 hold in the case when the initial time is a stopping time. For every $\nu \in \mathcal{T}_0^T$, $\eta \in L_\nu^{2q}(\mathbb{R}^d)$, $i \in \mathcal{I}$ and control α , we have

$$\mathbb{E} \left[\max_{\nu \leq s \leq T} \|X_s^{\nu, \eta, i, \alpha}\|^q \mid \mathcal{F}_\nu \right] \leq C_{q,X} (1 + \|\eta\|^q) e^{C_q(T-\nu)}.$$

We first consider an optimal switching problem without ambiguity. An objective function of the optimal switching problem without ambiguity is

$$J^{na}(t, x, i, \alpha) := \mathbb{E} \left[\int_t^T D_s^{t,x,i,\alpha} \psi(s, X_s^{t,x,i,\alpha}, \alpha_s) ds + D_T^{t,x,i,\alpha} g(X_T^{t,x,i,\alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \mid \mathcal{F}_t \right], \quad (5.2.9)$$

where ψ, g , and c are measurable functions. ψ represents running rewards for the switching problem without ambiguity. g represents a terminal payoff. c is a switching cost function. $c_{i,j}(t, x)$ represents a switching cost from regime i to j at time t and $X_t = x$. $D^{t,x,i,\alpha}$ is a discount factor such that for any $(t, x, i) \in \overline{\mathcal{K}_T}$ and control α ,

$$D_s^{t,x,i,\alpha} = \exp \left\{ - \int_t^s \rho(t, X_u^{t,x,i,\alpha}, \alpha_u) du \right\}, \quad s \in [t, T], \quad (5.2.10)$$

where $\rho(t, x, i)$ is a bounded measurable function. By the definition (5.2.10), we allow the discount rate to be random and controllable. Therefore, the objective function (5.2.9) represents the expected and discounted total profit on $[t, T]$.

For all $\nu \in \mathcal{T}_0^T$ and $\iota \in \tilde{\mathcal{I}}_\nu$, let $\mathbb{A}_\iota[\nu, T]$ be a set of controls such that

$$\mathbb{A}_\iota[\nu, T] := \left\{ \alpha = (\alpha_s)_{\nu \leq s \leq T} \mid \mathbb{E} \left[\left| \sum_{\nu \leq \tau_k \leq T} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, x, \iota, \alpha}) \right|^2 \right] < \infty, \forall x \in \mathbb{R}^d, \text{ and } \alpha_\nu = \iota. \right\}. \quad (5.2.11)$$

We call a control in $\mathbb{A}_\iota[\nu, T]$ an admissible control. The optimal switching problem without ambiguity is

$$\sup_{\alpha \in \mathbb{A}_i[t, T]} J^{na}(t, x, i, \alpha), \quad (5.2.12)$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$.

The optimal switching problems expressed as (5.2.12) are well studied in many researchers (e.g., Brekke and Øksendal (1994), Ly Vath and Pham (2007), Djehiche, Hamadène, and Popier (2009), and Bayraktar and Egami (2010)). However, one of the weakness of the optimal switching problem (5.2.12) is not to take into account ambiguity. The problem (5.2.12) assumes that the decision maker knows the functional

form of the distribution parameters b and σ a priori, whereas we do not know them in practice. Therefore, it needs to take into account uncertainty about the distribution of X in order to derive more useful switching strategies. Hence, we consider an optimal switching problem under ambiguity hereafter.

We first define a set of degrees of ambiguity. For $t \in [0, T]$, let Θ_t be a set of d -dimensional $\mathcal{F}_t$ -measurable random variables. We assume the form of Θ_t as follows.

Assumption 5.3

1. There exists a non-negative constant C such that

$$\mathbb{P}(\|\theta_t\| \leq C, \forall \theta_t \in \Theta_t, t \in [0, T]) = 1.$$

2. Θ_t is convex and compact valued for all $t \in [0, T]$.
3. Θ_t is a progressively measurable correspondence for all $t \in [0, T]$.
4. $0 \in \Theta_t$ $dt \otimes \mathbb{P}$ -a.e..

Let

$$\Theta[t, T] = \left\{ \theta = (\theta_s)_{t \leq s \leq T} \mid \begin{array}{l} \theta \text{ is right-continuous with left limits and} \\ \theta_s \in \Theta_s \text{ for all } s \in [t, T]. \end{array} \right\}. \quad (5.2.13)$$

For all $\theta \in \Theta[t, T]$, we define a density process $\zeta^{\theta, t} = (\zeta_s^{\theta, t})_{t \leq s \leq T}$ such that

$$\zeta_s^{\theta, t} := \exp \left\{ - \int_t^s \theta'_u dW_u - \frac{1}{2} \int_t^s \|\theta_u\|^2 du \right\}, \quad s \in [t, T], \quad (5.2.14)$$

where x' is a transpose of a vector $x \in \mathbb{R}^d$. By Assumption 5.3, for all $\theta \in \Theta[t, T]$, $\zeta^{\theta, t}$ is a martingale with respect to $\mathbb{F}$. Therefore, for all $\theta \in \Theta[t, T]$, we can define a new probability measure such that

$$\mathbb{P}_T^\theta(A) := \mathbb{E}[\mathbb{1}_A \zeta_T^{\theta, t}], \quad A \in \mathcal{F}_T. \quad (5.2.15)$$

We denote by $\mathbb{E}_T^\theta$ the expectation operator under the probability measure $\mathbb{P}_T^\theta$.

Under the probability measure $\mathbb{P}_T^\theta$, by the Girsanov theorem, the SDE (5.2.2) can be expressed as

$$dX_t = \left(b(t, X_t, \alpha_t) - \sigma(t, X_t, \alpha_t) \theta_t \right) dt + \sigma(t, X_t, \alpha_t) dW_t^\theta, \quad t \in [0, T],$$

where W^θ is a d -dimensional Brownian motion under $\mathbb{P}_T^\theta$. This implies that we can take account of the ambiguity about the drift of X under $\mathbb{P}_T^\theta$.

Θ represents a set of priors of the decision maker. Chen and Epstein (2002) establish a decision making problem under ambiguity in continuous time, which means that the decision maker would like to avoid the event whose occurrence probability is unknown. To incorporate ambiguity into an optimal switching problem, we use the concept of Chen and Epstein (2002). In their model, the decision maker chooses her subjective probability measure before choosing her decision as if her expected utility is minimized. They succeed to pose such a decision making problem under Assumption 5.3. They called Assumption 5.3 the rectangular condition.

The objective function under ambiguity is

$$J(t, x, i, \alpha) := \inf_{\theta \in \Theta[t, T]} E_T^\theta \left[\int_t^T D_s^{t, x, i, \alpha} \left(\psi(s, X_s^{t, x, i, \alpha}, \alpha_s) - \theta'_s \phi(s, X_s^{t, x, i, \alpha}, \alpha_s) \right) ds \right. \\ \left. + D_T^{t, x, i, \alpha} g(X_T^{t, x, i, \alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{t, x, i, \alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{t, x, i, \alpha}) \mid \mathcal{F}_t \right], \quad (5.2.16)$$

where ϕ is a measurable function from $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ onto $\mathbb{R}^d$. ϕ determines a running premium for ambiguity. Our settings allow choices of ambiguity levels to affect the running rewards through the term $\theta'_s \phi(s, X_s^{t, x, i, \alpha})$. The optimal switching problem under ambiguity is

$$\sup_{\alpha \in \mathbb{A}_i[t, T]} J(t, x, i, \alpha), \quad (5.2.17)$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$.

Furthermore, we assume the functions, ρ, ψ, ϕ, g , and c as follows.

Assumption 5.4

1. $\rho(\cdot, \cdot, i)$ is a continuous, non-negative and upper bounded function for all $i \in \mathcal{I}$.
2. *Polynomial growth condition*

$\psi(\cdot, \cdot, i)$, $\phi(\cdot, \cdot, i)$, $g(\cdot, i)$ and $c_{i,j}(\cdot, \cdot)$ are continuous for all $i, j \in \mathcal{I}$, and $c_{i,i}(t, x) = 0$ for all $(t, x, i) \in \overline{\mathcal{K}_T}$. Furthermore, there exist positive constants C_f and q such that

$$|\psi(t, x, i)| + \|\phi(t, x, i)\| + |g(t, x, i)| + |c_{i,j}(t, x)| \leq C_f(1 + \|x\|^q), \quad (5.2.18)$$

for all $(t, x, i, j) \in [0, T] \times \mathbb{R}^d \times (\mathcal{I})^2$. Without loss of generality, we assume $q \geq 1$.

3. Non-free loop conditions

- (a) For all finite loop $(i_0, i_1, \dots, i_m) \in \mathcal{I}^{m+1}$ with $i_0 = i_m$ and $i_0 \neq i_1$ and for all $(t, x) \in [0, T] \times \mathbb{R}^d$, c satisfies

$$c_{i_0, i_1}(t, x) + \dots + c_{i_{m-1}, i_m}(t, x) > 0. \quad (5.2.19)$$

- (b) g satisfies the following inequality,

$$g(x, i) \geq \max_{j \in \mathcal{I} \setminus \{i\}} \{g(x, j) - c_{i,j}(T, x)\}, \quad (5.2.20)$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$.

4. Strong triangular condition

Let

$$\mathcal{N} = \left\{ i \in \mathcal{I} \mid \exists j \in \mathcal{I}, j \neq i, \int_{[0, T] \times \mathbb{R}^d} \mathbb{1}_{\{c_{i,j}(t, x) < 0\}}(t, x) dt dx > 0 \right\},$$

$$C_i = - \min_{j \in \mathcal{I}, x \in \mathbb{R}^d, t \in [0, T]} \frac{c_{i,j}(t, x)}{1 + \|x\|^q}, \quad i \in \mathcal{N},$$

where q is defined in Assumption 5.4.2. Then, for all $i \in \mathcal{N}$,

$$c_{k,j}(t, x) \leq c_{k,i}(t, x) - C_i(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}), \quad (5.2.21)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $(j, k) \in \mathcal{I}$ with $j \neq i$ and $k \neq i$, where $C_{q,X}$ and C_q are defined in Proposition 5.2.

Assumption 5.4.1 implies that the discount rate is upper bounded and non-negative. The non-negativity is usual, and the assumption of upper boundedness guarantees the Lipschitz condition of a generator in the BSDE literature. Assumption 5.4.2 and Proposition 5.2 guarantee the value function of our optimal switching problem to be finite. Therefore, it is needed in order to consider meaningful problems.

The non-free loop conditions (Assumption 5.4.3) say that whenever one first stands in some regime (call regime A), next instantaneously goes to the other regimes, and finally goes back to the regime A at the same time, then she has to pay a positive cost.

Hence, the non-free loop conditions exclude the possibility that one can gain a positive profit by a looping switching strategy at the same time. If the non-free loop conditions are not postulated, then the value function diverges as the decision maker obtains an infinitely large reward by such a looping strategy. Since it is an arbitrage, the non-free loop conditions are natural in the optimal switching problems.

Unlike the previous literature, we do not assume non-negativity of the cost functions. Our specification of ambiguity allows this generalization. However, we need an additional assumption in this case. If some cost function can take a negative value, it needs to satisfy the strong triangular condition (Assumption 5.4.4).

The strong triangular condition means that the switching benefits are not too large to take these benefits. Heuristically speaking, if one first stands in the regime k and if $c_{i,j} < 0$, then the cost that she goes to the regime j via the regime i is at least as large as the cost that she directly goes to the regime j . The strong triangular condition implies the standard triangle inequality. Indeed, by the inequality (5.2.21), we have

$$\begin{aligned} c_{k,i}(t, x) + c_{i,j}(t, x) &\geq c_{k,i}(t, x) - C_i(1 + \|x\|^q) \\ &\geq c_{k,i}(t, x) - C_i(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}) \geq c_{k,j}(t, x), \end{aligned}$$

for all $i \in \mathcal{N}$, $(j, k) \in \mathcal{I}$, $(t, x) \in [0, T] \times \mathbb{R}^d$ with $k \neq i$ and $j \neq i$. Therefore, our triangular condition (5.2.21) is stronger than the standard triangle inequality.

By Proposition 5.2 and Assumption 5.4, we can show that an expected total cost does not diverge for every admissible control.

Proposition 5.5 Under Assumption 5.1 and 5.4,

$$\mathbb{E} \left[- \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \right] \leq C_f(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}), \quad (5.2.22)$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$ and $\alpha = (\tau_k, i_k)_{k \geq 0} \in \mathbb{A}_i[t, T]$.

Proof of Proposition 5.5. Fix an arbitrary $(t, x, i) \in \overline{\mathcal{K}_T}$ and $\alpha = (\tau_k, i_k)_{k \geq 0} \in \mathbb{A}_i[t, T]$.

We first prove

$$\mathbb{E} \left[- \sum_{k=1}^n D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \right] \leq C_f(1 + C_{q,X}(1 + \|x\|^q)e^{C_q(T-t)}),$$

for all $n \geq 1$. If $\mathbb{P}(i_{n-1} \in \mathcal{N} \mid \mathcal{F}_{\tau_{n-1}}) = 0$, then $c_{i_{n-1}, i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \geq 0$. Hence, we have

$$\begin{aligned} & -D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1}, i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \\ & \leq -D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}). \end{aligned} \quad (5.2.23)$$

If $\mathbb{P}(i_{n-1} \in \mathcal{N} \mid \mathcal{F}_{\tau_{n-1}}) > 0$, then, by Proposition 5.2, we have

$$\begin{aligned} & \mathbb{E} \left[-D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1}, i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \mid \mathcal{F}_{\tau_{n-1}} \right] \\ & \leq -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} \left(c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - C_{i_{n-1}} \left(1 + \|X_{\tau_n}^{t,x,i,\alpha}\|^q \right) \right) \mathbb{1}_{\{i_{n-1} \in \mathcal{N}\}} \right. \\ & \quad \left. + D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mathbb{1}_{\{i_{n-1} \notin \mathcal{N}\}} \mid \mathcal{F}_{\tau_{n-1}} \right] \\ & \leq -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} \left(c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \right. \right. \\ & \quad \left. \left. - C_{i_{n-1}} \left(1 + C_{q,X} \left(1 + \|X_{\tau_{n-1}}^{t,x,i,\alpha}\|^q \right) e^{C_q(T-\tau_{n-1})} \right) \right) \mathbb{1}_{\{i_{n-1} \in \mathcal{N}\}} \right. \\ & \quad \left. + D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mathbb{1}_{\{i_{n-1} \notin \mathcal{N}\}} \mid \mathcal{F}_{\tau_{n-1}} \right]. \end{aligned}$$

By Assumption 5.4.4, there exists an $\mathcal{F}_{\tau_{n-1}}$ -measurable random variable $\tilde{i}_{n-1}$ taking values in $\mathcal{I}$ such that

$$\begin{aligned} & -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} \left(c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \right. \right. \\ & \quad \left. \left. - C_{i_{n-1}} \left(1 + C_{q,X} \left(1 + \|X_{\tau_{n-1}}^{t,x,i,\alpha}\|^q \right) e^{C_q(T-\tau_{n-1})} \right) \right) \mathbb{1}_{\{i_{n-1} \in \mathcal{N}\}} \mid \mathcal{F}_{\tau_{n-1}} \right] \\ & \leq -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, \tilde{i}_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mathbb{1}_{\{i_{n-1} \in \mathcal{N}\}} \mid \mathcal{F}_{\tau_{n-1}} \right]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \mathbb{E} \left[-D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) - D_{\tau_n}^{t,x,i,\alpha} c_{i_{n-1}, i_n}(\tau_n, X_{\tau_n}^{t,x,i,\alpha}) \mid \mathcal{F}_{\tau_{n-1}} \right] \\ & \leq -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} \left(c_{i_{n-2}, \tilde{i}_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mathbb{1}_{\{i_{n-1} \in \mathcal{N}\}} \right. \right. \\ & \quad \left. \left. + c_{i_{n-2}, i_{n-1}}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mathbb{1}_{\{i_{n-1} \notin \mathcal{N}\}} \right) \mid \mathcal{F}_{\tau_{n-1}} \right] \\ & \leq -\mathbb{E} \left[D_{\tau_{n-1}}^{t,x,i,\alpha} c_{i_{n-2}, i_{n-1}^*}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha}) \mid \mathcal{F}_{\tau_{n-1}} \right], \end{aligned} \quad (5.2.24)$$

where

$$i_{n-1}^* = \arg \min_{j \in \mathcal{I} \setminus \{i_{n-2}\}} \{c_{i_{n-2}, j}(\tau_{n-1}, X_{\tau_{n-1}}^{t,x,i,\alpha})\},$$

and i_{n-1}^* is obviously $\mathcal{F}_{\tau_{n-1}}$ -measurable. Therefore, the inequalities (5.2.23) and (5.2.24)

lead to

$$\begin{aligned}
\mathbb{E} \left[- \sum_{k=1}^n D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \right] &\leq \mathbb{E} \left[-D_{\tau_1}^{t,x,i,\alpha} c_{i,i_1^*}(\tau_1, X_{\tau_1}^{t,x,i,\alpha}) \right] \\
&\leq C_f (1 + \mathbb{E} [\|X_{\tau_1}^{t,x,i,\alpha}\|^q]) \\
&\leq C_f \left(1 + \mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q \right] \right) \\
&\leq C_f (1 + C_{q,X} (1 + \|x\|^q) e^{C_q(T-t)}).
\end{aligned}$$

Since $\alpha \in \mathbb{A}_i[t, T]$, by the Lebesgue dominated convergence theorem, we obtain the inequality (5.2.22). $\square$

Proposition 5.5 has an important role in our switching problem. The other studies assuming non-negativity of switching costs naturally derive a lower boundary of the total expected costs, that is 0. However, we do not naturally say that the total costs are non-negative since our switching costs can take a negative value. Therefore, we need to estimate a lower boundary of the total expected costs by Proposition 5.5.

Remark 5.6 Even if the cost functions do not satisfy the strong triangular condition, it is possible that Proposition 5.5 holds. In this case, the following discussion in this chapter also holds. Essentially, we need

$$\mathbb{E} \left[- \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{t,x,i,\alpha} c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{t,x,i,\alpha}) \right] \leq C(1 + \|x\|^q),$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$ and $\alpha \in \mathbb{A}_i[t, T]$, where C is a positive constant not depending on (t, x, i) and α .

5.3 Multidimensional Reflected BSDEs

Next, we consider a representation of the objective function by BSDEs.

For all $\nu \in \mathcal{T}_0^T$, we denote by $\mathbb{S}^2[\nu, T]$ the set of real-valued progressively measurable processes Y such that

$$\mathbb{E} \left[\sup_{\nu \leq t \leq T} |Y_t|^2 \right] < \infty,$$

and by $\mathbb{H}_d^2[\nu, T]$ the set of $\mathbb{R}^d$ -valued progressively measurable processes Z such that

$$\mathbb{E} \left[\int_{\nu}^T \|Z_t\|^2 dt \right] < \infty.$$

Especially, we denote by $\mathbb{S}_c^2[\nu, T]$ a set of all continuous processes in $\mathbb{S}^2[\nu, T]$ and by $\mathbb{K}^2[\nu, T]$ a set of all non-decreasing processes in $\mathbb{S}^2[\nu, T]$.

We consider the following BSDE: For given $(\nu, \eta, \iota) \in \widetilde{\mathcal{K}}_T^{2q}$, $\theta \in \Theta[\nu, T]$ and $\alpha \in \mathbb{A}_\iota[\nu, T]$,

$$\begin{aligned} -dY_t^{\nu, \eta, \iota, \theta, \alpha} &= \left(\psi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) - \rho(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) Y_t^{\nu, \eta, \iota, \theta, \alpha} \right. \\ &\quad \left. - \theta'_t \left(\phi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) + Z_t^{\nu, \eta, \iota, \theta, \alpha} \right) \right) dt \\ &\quad - (Z_t^{\nu, \eta, \iota, \theta, \alpha})' dW_t - dA_t^{\nu, \eta, \iota, \alpha}, \quad t \in [\nu, T], \\ Y_T^{\nu, \eta, \iota, \theta, \alpha} &= g(X_T^{\nu, \eta, \iota, \alpha}, \alpha_T), \quad A_t^{\nu, \eta, \iota, \alpha} = \sum_{t \leq \tau_k \leq T} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, \eta, \iota, \alpha}), \quad t \in [\nu, T], \quad (5.3.1) \\ (Y^{\nu, \eta, \iota, \theta, \alpha}, Z^{\nu, \eta, \iota, \theta, \alpha}) &\in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]. \end{aligned}$$

Since $g(X_T^{\nu, \eta, \iota, \alpha}, \alpha_T) \in L_T^2(\mathbb{R})$ and $(\phi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t))_{\nu \leq t \leq T}, (\psi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t))_{\nu \leq t \leq T} \in \mathbb{H}_1^2[\nu, T]$ and since θ and ρ are uniformly bounded by Assumption 5.1, 5.3 and 5.4, the BSDE (5.3.1) has a unique solution in $\mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]$. Furthermore, by Proposition 2.2 in El Karoui et al. (1997b), the solution of the BSDE (5.3.1), also denoted by $(Y_t^{\nu, \eta, \iota, \theta, \alpha}, Z_t^{\nu, \eta, \iota, \theta, \alpha})_{\nu \leq t \leq T}$, can be represented as the following form.

$$\begin{aligned} Y_t^{\nu, \eta, \iota, \theta, \alpha} &= \frac{1}{D_t^{\nu, \eta, \iota, \alpha} \zeta_t^{\theta, \nu}} \mathbb{E} \left[\int_t^T D_s^{\nu, \eta, \iota, \alpha} \zeta_s^{\theta, \nu} \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \theta'_s \phi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) \right) ds \right. \\ &\quad \left. + D_T^{\nu, \eta, \iota, \alpha} \zeta_T^{\theta, \nu} g(X_T^{\nu, \eta, \iota, \alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{\nu, \eta, \iota, \alpha} \zeta_{\tau_k}^{\theta, \nu} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, \eta, \iota, \alpha}) \mid \mathcal{F}_t \right] \\ &= \mathbb{E}_T^\theta \left[\int_t^T \frac{D_s^{\nu, \eta, \iota, \alpha}}{D_t^{\nu, \eta, \iota, \alpha}} \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \theta'_s \phi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) \right) ds \right. \\ &\quad \left. + \frac{D_T^{\nu, \eta, \iota, \alpha}}{D_t^{\nu, \eta, \iota, \alpha}} g(X_T^{\nu, \eta, \iota, \alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} \frac{D_{\tau_k}^{\nu, \eta, \iota, \alpha}}{D_t^{\nu, \eta, \iota, \alpha}} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, \eta, \iota, \alpha}) \mid \mathcal{F}_t \right], \quad (5.3.2) \end{aligned}$$

where we have used the Bayes rule in the second equality.

Now, we also consider another BSDE such that

$$\begin{aligned} -dY_t^{\nu, \eta, \iota, \alpha} &= \left(\psi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) - \rho(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) Y_t^{\nu, \eta, \iota, \alpha} \right. \\ &\quad \left. - \max_{\theta_t \in \Theta_t} \left\{ \theta'_t \left(\phi(t, X_t^{\nu, \eta, \iota, \alpha}, \alpha_t) + Z_t^{\nu, \eta, \iota, \alpha} \right) \right\} \right) dt \\ &\quad - (Z_t^{\nu, \eta, \iota, \alpha})' dW_t - dA_t^{\nu, \eta, \iota, \alpha}, \quad t \in [\nu, T], \quad (5.3.3) \end{aligned}$$

$$\begin{aligned} Y_T^{\nu, \eta, \iota, \alpha} &= g(X_T^{\nu, \eta, \iota, \alpha}, \alpha_T), \quad A_t^{\nu, \eta, \iota, \alpha} = \sum_{t \leq \tau_k \leq T} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, \eta, \iota, \alpha}), \quad t \in [\nu, T], \\ (Y^{\nu, \eta, \iota, \alpha}, Z^{\nu, \eta, \iota, \alpha}) &\in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]. \end{aligned}$$

The BSDE (5.3.5) also has a unique solution in $\mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]$. From the comparison theorem, the solution of the BSDE (5.3.5) is a minimum value of $Y_t^{\nu, \eta, t, \theta, \alpha}$ over $\theta \in \Theta[\nu, T]$, that is, the following inequality holds.

$$Y_t^{\nu, \eta, t, \theta, \alpha} \geq Y_t^{\nu, \eta, t, \alpha}, \quad (5.3.4)$$

$\mathbb{P}$ -almost surely for all $t \in [\nu, T]$ and $\theta \in \Theta[\nu, T]$.

Combining the inequality (5.3.4) with the equality (5.3.2), we deduce that

$$\begin{aligned} Y_t^{t, x, i, \alpha} &= \inf_{\theta \in \Theta[t, T]} E_T^\theta \left[\int_t^T D_s^{t, x, i, \alpha} \left(\psi(s, X_s^{t, x, i, \alpha}, \alpha_s) - \theta'_s \phi(s, X_s^{t, x, i, \alpha}, \alpha_s) \right) ds \right. \\ &\quad \left. + D_T^{t, x, i, \alpha} g(X_T^{t, x, i, \alpha}, \alpha_T) - \sum_{s \leq \tau_k \leq T} D_{\tau_k}^{t, x, i, \alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{t, x, i, \alpha}) \mid \mathcal{F}_t \right] \\ &= J(t, x, i, \alpha), \end{aligned}$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$ and $\alpha \in \mathbb{A}_i[t, T]$. Therefore, $Y_t^{t, x, i, \alpha}$ is the objective function under ambiguity.

For the sake of brevity, we assume as follows.

Assumption 5.7 Suppose that Θ_t is measurable with respect to the σ -algebra generated by X_t and α_t for all $t \in [0, T]$. We denote by $\Theta_t^{x, i}$ a Θ_t with $X_t = x$ and $\alpha_t = i$. For all $(t, x, i) \in \overline{\mathcal{K}_T}$ and $z \in \mathbb{R}^d$, let

$$\varsigma(t, x, i, z) := \max_{\theta_t \in \Theta_t^{x, i}} \left\{ \theta'_t \left(\phi(t, x, i) + z \right) \right\}.$$

Then, suppose that ς is a deterministic and measurable function. Moreover, suppose that $\varsigma(\cdot, \cdot, i, \cdot)$ is continuous for all $i \in \mathcal{I}$.

By Assumption 5.3.1 and 5.4, ς satisfy the polynomial growth condition with respect to x and z and the Lipschitz condition with respect to z : There exists a positive constant C_ς such that

$$|\varsigma(t, x, i, z)| \leq C_\varsigma(1 + \|x\|^q + \|z\|), \quad |\varsigma(t, x, i, z) - \varsigma(t, x, i, \tilde{z})| \leq C_\varsigma \|z - \tilde{z}\|,$$

for all $(t, x, i, z, \tilde{z}) \in \overline{\mathcal{K}_T} \times (\mathbb{R}^d)^2$.

Under Assumption 5.7, the BSDE (5.3.3) can be expressed as

$$\begin{aligned} -dY_t^{\nu,\eta,\iota,\alpha} = & \left(\psi(t, X_t^{\nu,\eta,\iota,\alpha}, \alpha_t) - \rho(t, X_t^{\nu,\eta,\iota,\alpha}, \alpha_t) Y_t^{\nu,\eta,\iota,\alpha} - \varsigma(t, X_t^{\nu,\eta,\iota,\alpha}, \alpha_t, Z_t^{\nu,\eta,\iota,\alpha}) \right) dt \\ & - (Z_t^{\nu,\eta,\iota,\alpha})' dW_t - dA_t^{\nu,\eta,\iota,\alpha}, \quad t \in [\nu, T], \end{aligned} \quad (5.3.5)$$

$$\begin{aligned} Y_T^{\nu,\eta,\iota,\alpha} = & g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T), \quad A_t^{\nu,\eta,\iota,\alpha} = \sum_{t \leq \tau_k \leq T} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha}), \quad t \in [\nu, T], \\ (Y^{\nu,\eta,\iota,\alpha}, Z^{\nu,\eta,\iota,\alpha}) \in & \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]. \end{aligned}$$

Now, let us consider the multidimensional RBSDE. For given $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ and for all $i \in \mathcal{I}$,

$$\begin{aligned} -dY_t^{\nu,\eta,i} = & \left(\psi(t, X_t^{\nu,\eta,i}, i) - \rho(t, X_t^{\nu,\eta,i}, i) Y_t^{\nu,\eta,i} - \varsigma(t, X_t^{\nu,\eta,i}, i, Z_t^{\nu,\eta,i}) \right) dt \\ & - (Z_t^{\nu,\eta,i})' dW_t + dK_t^{\nu,\eta,i}, \quad t \in [\nu, T], \\ Y_T^{\nu,\eta,i} = & g(X_T^{\nu,\eta,i}, i), \quad K_\nu^{\nu,\eta,i} = 0, \quad Y_t^{\nu,\eta,i} \geq \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu,\eta,j} - c_{i,j}(t, X_t^{\nu,\eta,i})\}, \quad t \in [\nu, T], \\ \int_\nu^T & \left(Y_t^{\nu,\eta,i} - \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu,\eta,j} - c_{i,j}(t, X_t^{\nu,\eta,i})\} \right) dK_t^{\nu,\eta,i} = 0, \\ (Y^{\nu,\eta,i}, Z^{\nu,\eta,i}, K^{\nu,\eta,i}) \in & \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}, \end{aligned} \quad (5.3.6)$$

where $X^{\nu,\eta,i} = (X_t^{\nu,\eta,i})_{\nu \leq t \leq T}$ is a strong solution to the following SDE,

$$dX_t = b(t, X_t, i)dt + \sigma(t, X_t, i)dW_t, \quad t \in [\nu, T], \quad X_\nu = \eta. \quad (5.3.7)$$

In the next section, we show that the solution $Y_t^{t,x,i}$ of the multidimensional RBSDE (5.3.6) is a value function of the optimal switching problem under ambiguity. In this section, we first prove the existence of solutions to the multidimensional RBSDE (5.3.6).

Theorem 5.8 Under Assumption 5.1, 5.3, 5.4 and 5.7, the multidimensional RBSDE (5.3.6) has a solution in $(\mathbb{S}_c^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T])^I$ for any $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$.

In the case when the switching costs are non-negative, Theorem 5.8 are proved by Theorem 3.2 in Hamadène and Zhang (2010) and Theorem 2.1 in Hu and Tang (2010). We use the strategy of the proof of Theorem 3.2 in Hamadène and Zhang (2010), but there is a problem for a priori estimates of Picard's iterations of the multidimensional RBSDE (5.3.6). Hamadène and Zhang (2010) define the process in $\mathbb{S}^2[\nu, T]$ that is larger than all Picard's iterations, however, this process may not be larger than Picard's

iterations in our problem since we allow the switching costs to be negative. Therefore, we can not use the results of Hamadène and Zhang (2010) straightforwardly. However, thanks to Proposition 5.5, we can define the other process in $\mathbb{S}^2[\nu, T]$ that is larger than all Picard's iterations in our problem.

Proof of Theorem 5.8. Throughout this proof, we fix an arbitrary $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$.

Step.1 Picard's iterations. Let $(Y^{\nu, \eta, i, 0}, Z^{\nu, \eta, i, 0})$ be a solution to the following BSDE.

$$\begin{aligned} -dY_t^{\nu, \eta, i, 0} &= \left(\psi(t, X_t^{\nu, \eta, i}, i) - \rho(t, X_t^{\nu, \eta, i}, i) Y_t^{\nu, \eta, i, 0} - \varsigma(t, X_t^{\nu, \eta, i}, i, Z_t^{\nu, \eta, i, 0}) \right) dt \\ &\quad - (Z_t^{\nu, \eta, i, 0})' dW_t, \quad t \in [\nu, T], \\ Y_T^{\nu, \eta, i, 0} &= g(X_T^{\nu, \eta, i}, i), \quad (Y^{\nu, \eta, i, 0}, Z^{\nu, \eta, i, 0}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T], \end{aligned}$$

for all $i \in \mathcal{I}$. Then, by Assumption 5.1, 5.3, 5.4 and 5.7, the above BSDE has a unique solution. For any $n \geq 1$, we consider the following RBSDE recursively.

$$\begin{aligned} -dY_t^{\nu, \eta, i, n} &= \left(\psi(t, X_t^{\nu, \eta, i}, i) - \rho(t, X_t^{\nu, \eta, i}, i) Y_t^{\nu, \eta, i, n} - \varsigma(t, X_t^{\nu, \eta, i}, i, Z_t^{\nu, \eta, i, n}) \right) dt \\ &\quad - (Z_t^{\nu, \eta, i, n})' dW_t + dK_t^{\nu, \eta, i, n}, \quad t \in [\nu, T], \\ Y_T^{\nu, \eta, i, n} &= g(X_T^{\nu, \eta, i}, i), \quad K_\nu^{\nu, \eta, i, n} = 0, \\ Y_t^{\nu, \eta, i, n} &\geq \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu, \eta, j, n-1} - c_{i,j}(t, X_t^{\nu, \eta, i})\}, \quad t \in [\nu, T], \\ \int_\nu^T \left(Y_t^{\nu, \eta, i, n} - \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu, \eta, j, n-1} - c_{i,j}(t, X_t^{\nu, \eta, i})\} \right) dK_t^{\nu, \eta, i, n} &= 0, \\ (Y^{\nu, \eta, i, n}, Z^{\nu, \eta, i, n}, K^{\nu, \eta, i, n}) &\in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}. \end{aligned} \tag{5.3.8}$$

Under Assumption 5.1, 5.3, 5.4 and 5.7, by Theorem 5.2 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997a), the RBSDE (5.3.8) has a unique solution for all n and i . Furthermore, by the comparison theorem (Theorem 4.1 in El Karoui et al. (1997a)), we have $Y_t^{\nu, \eta, i, n-1} \leq Y_t^{\nu, \eta, i, n}$, $\mathbb{P}$ -a.s. for all i and n .

Step.2 Non-ambiguity processes. Consider the following BSDE.

$$\begin{aligned} -dU_t^{\nu, \eta, i, 0} &= \left(\psi(t, X_t^{\nu, \eta, i}, i) - \rho(t, X_t^{\nu, \eta, i}, i) U_t^{\nu, \eta, i, 0} \right) dt - (V_t^{\nu, \eta, i, 0})' dW_t, \quad t \in [\nu, T], \\ U_T^{\nu, \eta, i, 0} &= g(X_T^{\nu, \eta, i}, i), \quad (U^{\nu, \eta, i, 0}, V^{\nu, \eta, i, 0}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T], \quad i \in \mathcal{I}. \end{aligned}$$

Then, the above BSDE has a unique solution. Similarly, we consider the following RBSDE for any $n \geq 1$.

$$\begin{aligned}
-dU_t^{\nu,\eta,i,n} &= \left(\psi(t, X_t^{\nu,\eta,i}, i) - \rho(t, X_t^{\nu,\eta,i}, i) U_t^{\nu,\eta,i,n} \right) dt \\
&\quad - (V_t^{\nu,\eta,i,n})' dW_t + dS_t^{\nu,\eta,i,n}, \quad t \in [\nu, T], \\
U_T^{\nu,\eta,i,n} &= g(X_T^{\nu,\eta,i}, i), \quad S_\nu^{\nu,\eta,i,n} = 0, \\
U_t^{\nu,\eta,i,n} &\geq \max_{j \in \mathcal{I} \setminus \{i\}} \{U_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\}, \quad t \in [\nu, T], \\
\int_\nu^T \left(U_t^{\nu,\eta,i,n} - \max_{j \in \mathcal{I} \setminus \{i\}} \{U_t^{\nu,\eta,j,n-1} - c_{i,j}(t, X_t^{\nu,\eta,i})\} \right) dS_t^{\nu,\eta,i,n} &= 0, \\
(U^{\nu,\eta,i,n}, V^{\nu,\eta,i,n}, S^{\nu,\eta,i,n}) &\in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}.
\end{aligned}$$

Then the above RBSDE has a unique solution and we obtain that $U_t^{\nu,\eta,i,n} \geq U_t^{\nu,\eta,i,n-1}$, $\mathbb{P}$ -a.s. for all $(t, i) \in [\nu, T] \times \mathcal{I}$ and $n \geq 1$ by the comparison theorem. By the definition of ς and Assumption 5.3.4, we have

$$\varsigma(t, x, i, z) \geq 0, \quad \forall (t, x, i, z) \in [0, T] \times \mathbb{R}^d \times \mathcal{I} \times \mathbb{R}^d.$$

Hence, applying the comparison theorem again to $U_t^{\nu,\eta,i,n}$ and $Y_t^{\nu,\eta,i,n}$, we obtain that $U_t^{\nu,\eta,i,n} \geq Y_t^{\nu,\eta,i,n}$, $\mathbb{P}$ -a.s. for all $(t, i) \in [\nu, T] \times \mathcal{I}$ and $n \geq 1$. Furthermore, $U^{\nu,\eta,i,n}$ has a Snell envelope representation such that

$$\begin{aligned}
U_t^{\nu,\eta,i,n} &= \operatorname{esssup}_{\tau^* \in \mathcal{T}_t^T} \mathbb{E} \left[\int_t^{\tau^*} \frac{D_s^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} \psi(s, X_s^{\nu,\eta,i}, i) ds + \frac{D_T^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} g(X_T^{\nu,\eta,i}, i) \mathbb{1}_{\{\tau^*=T\}} \right. \\
&\quad \left. + \frac{D_{\tau^*}^{\nu,\eta,i}}{D_t^{\nu,\eta,i}} \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ U_{\tau^*}^{\nu,\eta,j,n-1} - c_{i,j}(\tau^*, X_{\tau^*}^{\nu,\eta,i}) \right\} \mathbb{1}_{\{\tau^* < T\}} \mid \mathcal{F}_t \right], \quad (5.3.9)
\end{aligned}$$

for all $t \in [\nu, T]$ and $n \geq 1$, where

$$D_t^{\nu,\eta,i} = \exp \left\{ - \int_\nu^t \rho(s, X_s^{\nu,\eta,i}, i) ds \right\}, \quad t \in [\nu, T].$$

Step.3 A priori estimates. Fix an arbitrary $t \in [\nu, T]$. Let $(\tau_0, i_0) = (t, i)$ and

$$\begin{aligned}
\tau_k &= \inf \left\{ s \in [\tau_{k-1}, T] \mid U_{\tau_n}^{\nu,\eta,i_{k-1},n-(k-1)} = \max_{j \in \mathcal{I} \setminus \{i_{n-1}\}} \left\{ U_{\tau_n}^{\nu,\eta,j,n-k} - c_{i_{k-1},j}(\tau_k, X_{\tau_k}^{\nu,\eta,i,\alpha}) \right\} \right\}, \\
i_k &\text{ is such that } U_{\tau_n}^{\nu,\eta,i_{k-1},n-(k-1)} = U_{\tau_n}^{\nu,\eta,i_k,n-k} - c_{i_{k-1},i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,i,\alpha}),
\end{aligned}$$

for all $k = 1, \dots, n$. Then, we define $\alpha^n = (\tau_k, i_k)_{k \geq 0}$ and it holds that

$$U_t^{\nu, \eta, i, n} = \mathbb{E} \left[\int_t^T D_s^{t, X_t^{\nu, \eta, i, \alpha^n}} \psi(s, X_s^{\nu, \eta, i, \alpha^n}, \alpha_t^n) ds + D_T^{t, X_t^{\nu, \eta, i, \alpha^n}} g(X_T^{\nu, \eta, i, \alpha^n}, \alpha_T^n) \right. \\ \left. - \sum_{k=1}^n D_{\tau_k}^{t, X_t^{\nu, \eta, i, \alpha^n}} c_{i,j}(\tau_k, X_{\tau_k}^{\nu, \eta, i, \alpha^n}) \mathbb{1}_{\{\tau_k < T\}} \mid \mathcal{F}_t \right], \quad (5.3.10)$$

by Proposition 2.3 in El Karoui et al. (1997a). Furthermore, by the polynomial growth condition for c , it is easy to check that α^n is in $\mathbb{A}_i[\nu, T]$. Thus, by Proposition 5.5, we have

$$\mathbb{E} \left[- \sum_{k=1}^n D_{\tau_k}^{t, X_t^{\nu, \eta, i, \alpha^n}} c_{i,j}(\tau_k, X_{\tau_k}^{\nu, \eta, i, \alpha^n}) \mathbb{1}_{\{\tau_k < T\}} \mid \mathcal{F}_t \right] \leq C_f (1 + C_{q,X} (1 + \|X_t^{\nu, \eta, i}\|^q) e^{C_{2q}T}).$$

On the other hand, by Proposition 5.2, there exists a constant $C_T > 0$ such that

$$\mathbb{E} \left[\int_t^T D_s^{t, X_t^{\nu, \eta, i, \alpha^n}} \psi(s, X_s^{\nu, \eta, i, \alpha^n}, \alpha_t^n) ds + D_T^{t, X_t^{\nu, \eta, i, \alpha^n}} g(X_T^{\nu, \eta, i, \alpha^n}, \alpha_T^n) \mid \mathcal{F}_t \right] \\ \leq \mathbb{E} \left[\int_t^T |\psi(s, X_s^{\nu, \eta, i, \alpha^n}, \alpha_t^n)| ds + |g(X_T^{\nu, \eta, i, \alpha^n}, \alpha_T^n)| \mid \mathcal{F}_t \right] \\ \leq C_T (1 + \|X_t^{\nu, \eta, i}\|^q).$$

Finally, there exists a positive constant $C_M > 0$ such that

$$U_t^{\nu, \eta, i, n} = \mathbb{E} \left[\int_t^T D_s^{t, X_t^{\nu, \eta, i, \alpha^n}} \psi(s, X_s^{\nu, \eta, i, \alpha^n}, \alpha_t^n) ds + D_T^{t, X_t^{\nu, \eta, i, \alpha^n}} g(X_T^{\nu, \eta, i, \alpha^n}, \alpha_T^n) \right. \\ \left. - \sum_{k=1}^n D_{\tau_k}^{t, X_t^{\nu, \eta, i, \alpha^n}} c_{i,j}(\tau_k, X_{\tau_k}^{\nu, \eta, i, \alpha^n}) \mathbb{1}_{\{\tau_k < T\}} \mid \mathcal{F}_t \right] \\ \leq C_M (1 + \|X_t^{\nu, \eta, i}\|^q),$$

for all $n \geq 1$. Note that C_M does not depend on n and t . This implies that

$$U_t^{\nu, \eta, i, n} \leq M_t^{\nu, \eta} := C_M \left(1 + \sum_{j \in \mathcal{I}} \|X_t^{\nu, \eta, j}\|^q \right),$$

for all $t \in [\nu, T]$, $i \in \mathcal{I}$ and $n \geq 1$. By Proposition 5.2, $M^{\nu, \eta}$ is in $\mathbb{S}^2[\nu, T]$. Since $Y_t^{\nu, \eta, i, 0} \leq Y_t^{\nu, \eta, i, n} \leq U_t^{\nu, \eta, i, n} \leq M_t^{\nu, \eta}$ for all $t \in [\nu, T]$, $i \in \mathcal{I}$ and $n \geq 1$ and since $Y^{\nu, \eta, i, 0} \in \mathbb{S}^2[\nu, T]$ for all $i \in \mathcal{I}$, there exists a finitely positive constant C_a such that

$$\sum_{i \in \mathcal{I}} \mathbb{E} \left[\sup_{\nu \leq t \leq T} |Y_t^{\nu, \eta, i, n}|^2 \right] \leq C_a, \quad (5.3.11)$$

for all $n \geq 0$. Furthermore, by the polynomial growth condition for c , Proposition 5.2 and the inequality (5.3.11), there exists a positive constant C_b such that

$$\mathbb{E} \left[\sup_{\nu \leq t \leq T} \left| \left(\max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu, \eta, j, n-1} - c_{i,j}(t, X_t^{\nu, \eta, i})\} \right)^+ \right|^2 \right] \leq C_b,$$

for all $n \geq 0$. Hence, Proposition 3.5 in El Karoui et al. (1997a) leads to that there exists a finitely positive constant C_c such that

$$\mathbb{E} \left[\sup_{\nu \leq t \leq T} |Y_t^{\nu, \eta, i, n}|^2 + \int_{\nu}^T \|Z_t^{\nu, \eta, i, n}\|^2 dt + |K_T^{\nu, \eta, i, n}|^2 \right] \leq C_c, \quad (5.3.12)$$

for all $n \geq 0$ and $i \in \mathcal{I}$.

Step.4 The rest of this proof is exactly the same as step 3-5 in the proof of Theorem 3.2 in Hamadène and Zhang (2010). Thanks to the inequality (5.3.12), we can use the monotone limit theorem by Peng (1999) and show that a limit of $(Y^{\nu, \eta, i, n})_{n \geq 0}$ and associated processes $(Z^{\nu, \eta, i}, K^{\nu, \eta, i})$ satisfy properties of the solution to the multidimensional RBSDE (5.3.6). This limit, denoted by $(Y^{\nu, \eta, i})$, and $(K^{\nu, \eta, i})$ are continuous by the non-free loop condition. By the continuity of $(Y^{\nu, \eta, i})$ and $(K^{\nu, \eta, i})$, we conclude that a triplet $(Y^{\nu, \eta, i}, Z^{\nu, \eta, i}, K^{\nu, \eta, i})$ is a $\mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T]$ limit of the sequence $(Y^{\nu, \eta, i, n}, Z^{\nu, \eta, i, n}, K^{\nu, \eta, i, n})_{n \geq 0}$. $\square$

Remark 5.9 According to Corollary 3.3 in Hamadène and Zhang (2010), the solution $(Y^{\nu, \eta, i})$ constructed in Theorem 5.8 is a minimum solution of the multidimensional RBSDE (5.3.6): For any solution $(\tilde{Y}^{\nu, \eta, i})$ of the multidimensional RBSDE (5.3.6),

$$\tilde{Y}_t^{\nu, \eta, i} \geq Y_t^{\nu, \eta, i}, \mathbb{P}\text{-a.s.},$$

for all $t \in [\nu, T]$ and $i \in \mathcal{I}$.

Theorem 5.8 provides the existence of the multidimensional RBSDE (5.3.6). Other articles prove the uniqueness of the solution after proving the existence. However, we do not prove the uniqueness. Instead, we prove the pathwise uniqueness of the minimal solution of the multidimensional RBSDE (5.3.6) since this is a sufficient condition for the verification of the optimal switching problem under ambiguity.

Proposition 5.10 Suppose that Assumption 5.1, 5.3, 5.4 and 5.7 are satisfied. For any $(\nu, \tilde{\nu}) \in (\mathcal{T}_0^T)^2$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ such that $\nu \leq \tilde{\nu}$ $\mathbb{P}$ -a.s., we consider the minimum solutions of the multidimensional RBSDE (5.3.6) $Y^{\nu, \eta, i}$ and $Y^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}$. Then,

$$Y_t^{\nu, \eta, i} = Y_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i} \quad \mathbb{P}\text{-a.s.}, \quad (5.3.13)$$

for all $i \in \mathcal{I}$ and $t \in [\tilde{\nu}, T]$.

Proof of Proposition 5.10. By Assumption 5.1, the SDE (5.3.7) has a strong solution for all $i \in \mathcal{I}$. This implies that

$$X_t^{\nu, \eta, i} = X_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i} \quad \mathbb{P}\text{-a.s.},$$

for all $i \in \mathcal{I}$ and $t \in [\tilde{\nu}, T]$. Hence, $(Y^{\nu, \eta, i}, Z^{\nu, \eta, i}, \widehat{K}^{\nu, \eta, i} = K^{\nu, \eta, i} - K_{\tilde{\nu}}^{\nu, \eta, i})$ satisfies the following multidimensional RBSDE on $[\tilde{\nu}, T]$.

$$\begin{aligned} -dY_t^{\nu, \eta, i} &= \left(\psi(t, X_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}, i) - \rho(t, X_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}, i) Y_t^{\nu, \eta, i} \right. \\ &\quad \left. - \varsigma(t, X_{\tilde{\nu}}^{\nu, \eta, i}, i, Z_t^{\nu, \eta, i, n}) \right) dt - (Z_t^{\nu, \eta, i})' dW_t + d\widehat{K}_t^{\nu, \eta, i}, \quad t \in [\tilde{\nu}, T], \\ Y_T^{\nu, \eta, i} &= g(X_T^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}, i), \quad \widehat{K}_{\tilde{\nu}}^{\nu, \eta, i} = 0, \\ Y_t^{\nu, \eta, i} &\geq \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu, \eta, j} - c_{i,j}(t, X_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i})\}, \quad t \in [\tilde{\nu}, T], \\ \int_{\tilde{\nu}}^T \left(Y_t^{\nu, \eta, i} - \max_{j \in \mathcal{I} \setminus \{i\}} \{Y_t^{\nu, \eta, j} - c_{i,j}(t, X_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i})\} \right) d\widehat{K}_t^{\nu, \eta, i} &= 0, \\ (Y^{\nu, \eta, i}, Z^{\nu, \eta, i}, \widehat{K}^{\nu, \eta, i}) &\in \mathbb{S}^2[\tilde{\nu}, T] \times \mathbb{H}_d^2[\tilde{\nu}, T] \times \mathbb{K}^2[\tilde{\nu}, T], \quad i \in \mathcal{I}. \end{aligned} \quad (5.3.14)$$

Since for each i , the multidimensional RBSDE (5.3.14) is the same as the multidimensional RBSDE (5.3.6) starting from $(\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i)$, it holds that $Y_t^{\nu, \eta, i} \geq Y_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}$ $\mathbb{P}$ -a.s. for all $i \in \mathcal{I}$ and $t \in [\tilde{\nu}, T]$ because of the minimality of $Y^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}$ (see Remark 5.9).

On the other hand, recursively applying the comparison theorem to the Picard's iterations of $Y_t^{\nu, \eta, i}$ constructed in Theorem 5.8 on $[\tilde{\nu}, T]$ leads to that

$$Y_t^{\nu, \eta, i, n} \leq Y_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i} \quad \mathbb{P}\text{-a.s.},$$

for all $n \geq 0$, $i \in \mathcal{I}$ and $t \in [\tilde{\nu}, T]$. Taking a limit of the above inequality, we obtain that $Y_t^{\nu, \eta, i} \leq Y_t^{\tilde{\nu}, X_{\tilde{\nu}}^{\nu, \eta, i}, i}$ for all $i \in \mathcal{I}$ and $t \in [\tilde{\nu}, T]$. Hence, the equality (5.3.13) holds.

□

5.4 Verification and a Viscosity Solution

In this section, we show that the minimum solution in Theorem 5.8 can be interpreted as the value function of the optimal switching problem under ambiguity. Proposition 5.11 provides a verification of Y . The proof of Proposition 5.11 is standard, so it is in Appendix 5.B.

Proposition 5.11 Suppose that Assumption 5.1, 5.3, 5.4 and 5.7 are satisfied.

1. For an arbitrary $(\nu, \eta, \iota) \in \widetilde{\mathcal{K}_T^{2q}}$, let $Y^{\nu, \eta, \iota}$ be a minimum solution of the multidimensional RBSDE (5.3.6). Then,

$$Y_t^{\nu, \eta, \iota} \geq Y_t^{\nu, \eta, \iota, \alpha}, \quad \forall t \in [\nu, T], \quad (5.4.1)$$

for all $\alpha = (\tau_k, i_k)_{k \geq 0} \in \mathbb{A}_\iota[\nu, T]$.

2. Let $\alpha^* = (\tau_k^*, i_k^*)_{k \geq 0}$ be a control such that $(\tau_0^*, i_0^*) = (\nu, \iota)$ and that for all $n \geq 1$,

$$\tau_n^* := \inf \left\{ s \in [\tau_{n-1}^*, T] \mid Y_s^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_{n-1}^*} = \max_{j \in \mathcal{I} \setminus \{i_{n-1}^*\}} \{Y_s^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, j} - c_{i_{n-1}^*, j}(s, X_s^*)\} \right\},$$

$$i_n^* \text{ is such that } Y_{\tau_n^*}^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_{n-1}^*} = Y_{\tau_n^*}^{\tau_{n-1}^*, X_{\tau_{n-1}^*}^*, i_n^*} - c_{i_{n-1}^*, i_n^*}(\tau_n^*, X_{\tau_n^*}^*),$$

where $X^* = X^{\nu, \eta, \iota, \alpha^*}$. Then, α^* is an admissible control and

$$Y_t^{\nu, \eta, \iota} = Y_t^{\nu, \eta, \iota, \alpha^*}, \quad \forall t \in [\nu, T]. \quad (5.4.2)$$

By Proposition 5.11, we obtain that

$$Y_t^{t, x, i} = \sup_{\alpha \in \mathbb{A}_i[t, T]} Y_t^{t, x, i, \alpha} = \sup_{\alpha \in \mathbb{A}_i[t, T]} J(t, x, i, \alpha), \quad (5.4.3)$$

for all $(t, x, i) \in \overline{\mathcal{K}_T}$. Hence, $Y_t^{t, x, i}$ is the value function of the optimal switching problem under ambiguity. Furthermore, α^* defined in Proposition 5.11.2 is an optimal control of the problem.

We next study a relationship between the multidimensional RBSDE (5.3.6) and partial differential equations (hereafter PDEs). Let $u : [0, T] \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ be a function. Consider the following PDE,

$$\begin{aligned} \min\{-u_t(t, x, i) - \mathcal{L}^i u(t, x, i) - \psi(t, x, i) + \rho(t, x, i)u(t, x, i) + \varsigma(t, x, i, \sigma'(t, x, i))\nabla u(t, x, i), \\ u(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{u(t, x, j) - c_{i,j}(t, x)\}\} = 0, \quad (t, x, i) \in \overline{\mathcal{K}_T}, \end{aligned} \quad (5.4.4)$$

$$u(T, x, i) = g(x, i), \quad (5.4.5)$$

where $u_t(t, x, i) = \frac{\partial u(t, x, i)}{\partial t}$, $\nabla u(t, x, i) = \frac{\partial u(t, x, i)}{\partial x}$ and

$$\mathcal{L}^i f(t, x) = (\nabla f(t, x))' b(t, x, i) + \frac{1}{2} \text{tr} \left(\sigma \sigma'(t, x, i) \frac{\partial^2 f(t, x)}{\partial x \partial x'} \right).$$

If the PDE (5.4.4) has a classical solution, then we can easily show that this solution is a value function of the optimal switching problem under ambiguity. However, the classical solution does not always exist. We shall consider a more general concept of solutions, i.e., a viscosity solution. Let $C^{1,2}([0, T) \times \mathbb{R}^d \times \mathcal{I})$ be a set of functions that are continuously differentiable with respect to t and twice continuously differentiable with respect to x on $[0, T) \times \mathbb{R}^d \times \mathcal{I}$.

Definition 5.12 (Viscosity solution)

1. *Viscosity supersolution.*

A lower semi-continuous function $(u(\cdot, \cdot, 1), \dots, u(\cdot, \cdot, I))$ is a viscosity supersolution of the PDE (5.4.4) if for any $(t, x, i) \in [0, T) \times \mathbb{R}^d \times \mathcal{I}$ and any $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^d \times \mathcal{I})$ such that $v(\cdot, \cdot, i) - \varphi(\cdot, \cdot, i)$ attains a local minimum at (t, x) for all $i \in \mathcal{I}$,

$$\begin{aligned} \min\{-\varphi_t(t, x, i) - \mathcal{L}^i \varphi(t, x, i) \\ - \psi(t, x, i) + \rho(t, x, i)u(t, x, i) + \varsigma(t, x, i, \sigma'(t, x, i))\nabla \varphi(t, x, i), \\ u(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{u(t, x, j) - c_{i,j}(t, x)\}\} \geq 0, \\ u(T, x, i) \geq g(x, i). \end{aligned}$$

2. *Viscosity subsolution.*

A upper semi-continuous function $(u(\cdot, \cdot, 1), \dots, u(\cdot, \cdot, I))$ is a viscosity subsolution of the PDE (5.4.4) if for any $(t, x, i) \in [0, T) \times \mathbb{R}^d \times \mathcal{I}$ and any $\varphi \in C^{1,2}([0, T) \times$

$\mathbb{R}^d \times \mathcal{I}$) such that $v(\cdot, \cdot, i) - \varphi(\cdot, \cdot, i)$ attains a local maximum at (t, x) for all $i \in \mathcal{I}$,

$$\begin{aligned} & \min\{-\varphi_t(t, x, i) - \mathcal{L}^i \varphi(t, x, i) \\ & \quad - \psi(t, x, i) + \rho(t, x, i)u(t, x, i) + \varsigma(t, x, i, \sigma'(t, x, i)\nabla \varphi(t, x, i)), \\ & \quad u(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{u(t, x, j) - c_{i,j}(t, x)\}\} \leq 0, \\ & u(T, x, i) \leq g(x, i). \end{aligned}$$

3. Viscosity solution.

A locally bounded function $(u(\cdot, \cdot, 1), \dots, u(\cdot, \cdot, I))$ is a viscosity solution of the PDE (5.4.4) if its lower semi-continuous envelope is a viscosity supersolution of the PDE (5.4.4), and if its upper semi-continuous envelope is a viscosity subsolution of the PDE (5.4.4).

More details of the viscosity solutions are in Crandall, Ishii, and Lions (1992). We define a set of functions $\mathcal{CP}([0, T] \times \mathbb{R}^d)$ as follows.

$$\begin{aligned} & \mathcal{CP}([0, T] \times \mathbb{R}^d) \\ & := \left\{ f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \mid \begin{array}{l} f \text{ is jointly continuous and} \\ \text{there exist positive constants } C \text{ and } q \text{ such that} \\ |f(t, x)| \leq C(1 + \|x\|^q), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d. \end{array} \right\}. \end{aligned} \quad (5.4.6)$$

Let

$$v(t, x, i) := Y_t^{t, x, i}, \quad (5.4.7)$$

for $(t, x, i) \in \overline{\mathcal{K}_T}$, where $Y_t^{t, x, i}$ is a minimum solution of the multidimensional RBSDE (5.3.6).

Now, we will prove that v is a unique viscosity solution of the PDE (5.4.4) in $\mathcal{CP}([0, T] \times \mathbb{R}^d)$. Hamadène and Morlais (2013) study the viscosity solution of the PDE similar to (5.4.4). Main differences between our model and the model in Hamadène and Morlais (2013) are as follows.

1. Hamadène and Morlais (2013) consider that a generator of RBSDE for Y^i depends on the other Y^j , but we consider the case when it does not depend on the other Y^j .

2. They assume that switching costs are non-negative, but we allow negative switching costs.
3. They assume that a dynamics of the forward variable X does not depend on a control process, but we allow the dynamics of X to depend on the control.

In fact, the results of Hamadène and Morlais (2013) can be applied to our model. Hamadène and Morlais (2013) prove the existence and uniqueness of the viscosity solution without using non-negativity of the switching costs. Furthermore, the controllability of X does not affect to their results. Hence, we can provide the existence and uniqueness of the solution to the PDE (5.3.6) in the viscosity sense and prove that the value function is a unique viscosity solution to (5.3.6).

Proposition 5.13 Suppose that Assumption 5.1, 5.3, 5.4 and 5.7 are satisfied. Let

$$\vec{v} := (v(\cdot, \cdot, 1), \dots, v(\cdot, \cdot, I)).$$

Then, $\vec{v}$ is a unique viscosity solution of the PDE (5.4.4) in $(\mathcal{CP}([0, T] \times \mathbb{R}^d))^I$.

Proof of Proposition 5.13. Let $(t, x, i) \in \overline{\mathcal{K}_T}$. Let $(Y^{t,x,i,n})_{n \geq 0}$ be a sequence of the Picard's iterations defined in Theorem 5.8. Then, by El Karoui et al. (1997b), there exists $v_n(\cdot, \cdot, i) \in \mathcal{CP}([0, T] \times \mathbb{R}^d)$ for all $n \geq 0$ and $i \in \mathcal{I}$ such that

$$Y_s^{t,x,i,n} = v_n(t, X_s^{t,x,i}, i),$$

for all $s \in [t, T]$. Furthermore, we define $\bar{v} \in \mathcal{CP}([0, T] \times \mathbb{R}^d)$ as

$$\bar{v}(t, x) := M_t^{t,x},$$

where $M^{t,x}$ is defined in Theorem 5.8. Recall that $Y^{t,x,i,n} \rightarrow Y^{t,x,i}$ in the mean-square sense. Therefore, $\vec{v}$ is a lower semi-continuous function and it satisfies the polynomial growth condition with respect to x since $v_0 \leq v_n \leq \bar{v}$ and $v_n \leq v_{n+1}$ for all $n \geq 1$.

On the other hand, Corollary 1 in Hamadène and Morlais (2013) provides the continuity and uniqueness of a viscosity solution to the PDE (5.3.6). Furthermore, by Theorem 1 in Hamadène and Morlais (2013), $\vec{v}$ is a viscosity solution of the PDE (5.3.6). Hence, we conclude that $\vec{v}$ is a unique viscosity solution of the PDE (5.3.6) in $(\mathcal{CP}([0, T] \times \mathbb{R}^d))^I$. $\square$

5.5 The Infinite Horizon Problem

In this section, we consider the infinite horizon optimal switching problem under ambiguity. Let $\mathbb{A}_i[\nu, \infty)$ be a set of admissible controls like (5.2.1) but $\tau_k \rightarrow \infty$ $\mathbb{P}$ -almost surely. Furthermore, we assume as follows.

Assumption 5.14

1. *Time-homogeneity.* $b, \sigma, \psi, \phi, \varsigma$, and c do not depend on t . There exists a positive constant ρ such that

$$\rho(t, x, i) = \rho > 0,$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$. Θ_t only depends on the values of X_t and α_t . We denote Θ_t with $X_t = x \in \mathbb{R}^d$ and $\alpha_t = i \in \mathcal{I}$ by $\Theta^{x,i}$.

2. *Sufficiently large discount.* ρ is sufficiently large in the following sense. There exist constants $C \geq 0$ and $c_\infty > 0$ such that

$$\mathbb{E} \left[e^{-\rho t} \zeta_t^{\theta,0} \|X_t^{x,i,\alpha}\|^q \right] \leq C(1 + \|x\|^q) e^{-c_\infty t}, \quad (5.5.1)$$

$$\mathbb{E} \left[\sup_{s \geq t} e^{-\rho s} \|X_s^{x,i,\alpha}\|^q \right] \leq C(1 + \|x\|^q) e^{-c_\infty t}, \quad (5.5.2)$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$, $\theta \in \Theta[0, \infty)$ and $\alpha \in \mathbb{A}_i[0, \infty)$, where $X^{x,i,\alpha}$ is a solution to the SDE (5.2.2) starting at $X_0^{x,i,\alpha} = x$ and controlled by $\alpha \in \mathbb{A}_i[0, \infty)$.

3. *Polynomial growth conditions.* ψ, ϕ and c are continuous and satisfy the polynomial growth condition in Assumption 5.4.2.
4. *Non-negative reward condition.*

$$\psi(x, i) - \varsigma(x, i, 0) \geq 0, \quad (5.5.3)$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$.

5. *Temporary terminal condition.* There exist polynomial growth functions $g(x, 1), \dots, g(x, I)$ such that

(a)

$$g(x, i) \leq 0, \quad (5.5.4)$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$;

(b)

$$g(x, i) \geq \max_{j \in \mathcal{I} \setminus \{i\}} \{g(x, j) - c_{i,j}(x)\}, \quad (5.5.5)$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$;

(c)

$$\inf_{\theta_t \in \Theta[T, \tilde{T})} \mathbb{E} \left[e^{-\rho \tilde{T}} \zeta_{\tilde{T}}^{\theta, T} g(X_{\tilde{T}}^{\nu, \eta, i}, i) \mid \mathcal{F}_T \right] \geq e^{-\rho T} g(X_T^{\nu, \eta, i}, i), \quad (5.5.6)$$

for all $0 \leq T \leq \tilde{T}$, $\nu \in \mathcal{T}_0^T$, $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ and $i \in \mathcal{I}$.

6. *Non-free loop condition in the infinite horizon.* For all finite loop $(i_0, i_1, \dots, i_m) \in \mathcal{I}^{m+1}$ with $i_0 = i_m$ and $i_0 \neq i_1$ and for all $x \in \mathbb{R}^d$, c satisfies

$$c_{i_0, i_1}(x) + \dots + c_{i_{m-1}, i_m}(x) > 0. \quad (5.5.7)$$

7. *Strong triangular condition in the infinite horizon.*

$$c_{k,j}(x) \leq c_{k,i}(x) - C_i(1 + C_{q,X}^\infty(1 + \|x\|^q)), \quad (5.5.8)$$

for all $i \in \mathcal{N}$, $(j, k) \in \mathcal{I}$ and $x \in \mathbb{R}^d$ with $j \neq i$ and $k \neq j$, where $C_i, C_{q,X}^\infty$ and q are defined in Proposition 5.2 and Assumption 5.4.

The time-homogeneity (Assumption 5.14.1) is a standard condition. With taking account of the time-homogeneity and the Markov property of X , the starting time does not matter to the optimal switching problem. The sufficiently large discount condition (Assumption 5.14.2) is also standard. If it is not postulated, then the value function can diverge. Therefore, we need this condition to consider meaningful problems. However, the condition (5.5.1) is slightly strong. Indeed, it is sufficient to satisfy (5.5.1) with $\theta = 0$ and (5.5.2) in order to prove the finiteness of the value function (Proposition 5.15). The condition (5.5.1) is needed to prove the convergent property of the value function from the finite horizon to the infinite horizon (Proposition 5.18).

Under the non-negative reward condition (Assumption 5.14.4), the rewards of the optimal switching problem in the infinite horizon is non-negative. Indeed, by the definition of ς , we have

$$\psi(X_t^{x,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{x,i,\alpha}, \alpha_t) \geq \psi(X_t^{x,i,\alpha}, \alpha_t) - \varsigma(X_t^{x,i,\alpha}, \alpha_t, 0) \geq 0,$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$, $\theta_t \in \Theta_t$ and $\alpha \in \mathbb{A}_i[0, \infty)$. The non-negative reward condition guarantees that an optimal switching problem in a longer finite horizon has a large value function. This restriction is needed to exchange the orders of taking limits of Picard's iterations n and time horizons T . This is slightly restrictive, however, it can be replaced to a lower bounded condition (Remark 5.16).

The temporary terminal conditions (Assumption 5.14.5) are assumed for purely technical reasons. However, they are not so restrictive. If all switching costs are non-negative, then we can choose $g(x, i) = 0$ for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ satisfying all the temporary terminal conditions. Once we find the constants $g_1, \dots, g_I$ satisfying the inequality (5.5.5), then $g_1 - \max_{j \in \mathcal{I}} g_j, \dots, g_I - \max_{j \in \mathcal{I}} g_j$ satisfy all the temporary terminal conditions. If $g(x, i)$ satisfies the inequalities (5.5.4) and (5.5.5) and if $g(\cdot, i)$ is twice continuously differentiable for all $i \in \mathcal{I}$, then one of sufficient conditions to satisfy the inequality (5.5.6) is

$$\mathcal{L}^i g(x, i) - \rho g(x, i) - (\nabla g(x, i))' \sigma(x, i) \theta \geq 0, \quad (5.5.9)$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ and $\theta \in \Theta^{x,i}$. The condition (5.5.9) can be derived by applying the Ito's lemma to $e^{-\rho t} \zeta_t^\theta g(X_t, i)$. If the switching costs are constants, we can easily find the constants satisfying the temporary terminal conditions. On the other hand, in the major applications such as the buy low and sell high problem and the pair-trading problem, we can also find the functions satisfying the temporary terminal conditions. The other assumptions are essentially the same as the finite horizon problem.

The objective function in the infinite horizon is

$$J(x, i, \alpha) = \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \zeta_t^\theta \left(\psi(X_t^{x,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{x,i,\alpha}, \alpha_t) \right) dt - \sum_{k=1}^\infty e^{-\rho \tau_k} \zeta_{\tau_k}^\theta c_{i_{k-1}, i_k}(X_{\tau_k}^{x,i,\alpha}) \right], \quad (5.5.10)$$

for $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0, \infty)$. The optimal switching problem under ambiguity in the infinite horizon is

$$v^\infty(x, i) := \sup_{\alpha \in \mathbb{A}_i[0, \infty)} J(x, i, \alpha), \quad (5.5.11)$$

for $(x, i) \in \mathbb{R}^d \times \mathcal{I}$. We can easily show that v^∞ is polynomial growth with respect to x .

Proposition 5.15 Under Assumption 5.1 and 5.14, there exists a positive constant C such that

$$0 \leq v^\infty(x, i) \leq C(1 + \|x\|^q),$$

for all $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Thus, v^∞ is polynomial growth with respect to x .

Proof of Proposition 5.15. It is clear that v^∞ is non-negative by the non-negative reward condition. Fix an arbitrary $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Then, by the polynomial growth condition of ψ and c and the strong triangular condition, we have

$$J(x, i, \alpha) \leq \mathbb{E} \left[\int_0^\infty e^{-\rho t} \psi(X_t^{x, i, \alpha}, \alpha_t) dt - \sum_{k=1}^\infty e^{-\rho \tau_k} c_{i_{k-1}, i_k}(X_{\tau_k}^{x, i, \alpha}) \right] \leq C(1 + \|x\|^q),$$

for all $\alpha \in \mathbb{A}_i[0, \infty)$, where C is the positive constant not depending on x, i and α . Hence, we obtain the desired result. $\square$

Remark 5.16 Assumption 5.14.6 (the inequality (5.5.3)) can be replaced to a lower bounded condition. We assume that there exists some constant $c_{\psi, \varsigma}$ such that

$$\psi(x, i) - \varsigma(x, i, 0) \geq c_{\psi, \varsigma},$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$. Then,

$$\begin{aligned} J(x, i, \alpha) - \frac{c_{\psi, \varsigma}}{\rho} &= J(x, i, \alpha) - \int_0^\infty e^{-\rho t} c_{\psi, \varsigma} dt \\ &= \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \zeta_t^\theta \left(\psi(X_t^{x, i, \alpha}, \alpha_t) - \theta'_t \phi(X_t^{x, i, \alpha}, \alpha_t) - c_{\psi, \varsigma} \right) dt \right. \\ &\quad \left. - \sum_{k=1}^\infty e^{-\rho \tau_k} \zeta_{\tau_k}^\theta c_{i_{k-1}, i_k}(X_{\tau_k}^{x, i, \alpha}) \right], \end{aligned}$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0, \infty)$. By the definition ς , we have

$$\psi(X_t^{x,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{x,i,\alpha}, \alpha_t) - c_{\psi,\varsigma} \geq \psi(X_t^{x,i,\alpha}, \alpha_t) - \varsigma(X_t^{x,i,\alpha}, \alpha_t, 0) - c_{\psi,\varsigma} \geq 0,$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$, $\theta_t \in \Theta_t$ and $\alpha \in \mathbb{A}_i[0, \infty)$. Hence, we can replace the original rewards to non-negative rewards. $c_{\psi,\varsigma}$ may be negative, but it is finite.

Remark 5.17 Similarly to Remark 5.6, the strong triangular condition in the infinite horizon is not necessarily needed. Instead of the strong triangular condition, it is sufficient to hold the following inequality

$$\mathbb{E} \left[- \sum_{k=1}^{\infty} e^{-\rho \tau_k} c_{i_{k-1}, i_k}(X_{\tau_k}^{x,i,\alpha}) \right] \leq C(1 + \|x\|^q),$$

for all $x \in \mathbb{R}^d$, $i \in \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0, \infty)$, where C is a positive constant not depending on (x, i) and α . Furthermore, if the above inequality is satisfied, then we do not also need the inequality (5.5.2).

We consider the following multidimensional RBSDE on $[\nu, T]$ for $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$,

$$\begin{aligned} -d\widehat{Y}_t^{T,\nu,\eta,i} &= \left(\psi(X_t^{\nu,\eta,i}, i) - \rho \widehat{Y}_t^{T,\nu,\eta,i} - \varsigma(X_t^{\nu,\eta,i}, i, \widehat{Z}_t^{T,\nu,\eta,i}) \right) dt \\ &\quad - (\widehat{Z}_t^{T,\nu,\eta,i})' dW_t + d\widehat{K}^{T,\nu,\eta,i}, \quad t \in [\nu, T], \\ \widehat{Y}_T^{T,\nu,\eta,i} &= g(X_T^{\nu,\eta,i}, i), \quad \widehat{K}_\nu^{T,\nu,\eta,i} = 0, \\ \widehat{Y}_t^{T,\nu,\eta,i} &\geq \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \widehat{Y}_t^{T,\nu,\eta,j} - c_{i,j}(X_t^{\nu,\eta,i}) \right\}, \quad t \in [\nu, T], \\ \int_0^T \left(\widehat{Y}_t^{T,\nu,\eta,i} - \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \widehat{Y}_t^{T,\nu,\eta,j} - c_{i,j}(X_t^{\nu,\eta,i}) \right\} \right) dt &= 0, \\ (\widehat{Y}^{T,\nu,\eta,i}, \widehat{Z}^{T,\nu,\eta,i}, \widehat{K}^{T,\nu,\eta,i}) &\in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T] \times \mathbb{K}^2[\nu, T], \quad i \in \mathcal{I}, \end{aligned} \tag{5.5.12}$$

where g is the function satisfying the temporary terminal conditions. By Theorem 5.8 and Proposition 5.10, there exists a unique minimum solution of the multidimensional RBSDE (5.5.12). Now, we show that the solution to the multidimensional RBSDE (5.5.12) converges to the value function (5.5.11) as $T \rightarrow \infty$.

Proposition 5.18 Under Assumption 5.1, 5.3, 5.7 and 5.14, $\widehat{Y}_t^{T,\nu,\eta,\iota} \leq \widehat{Y}_t^{\widetilde{T},\nu,\eta,\iota}$ for all $\nu \in \mathcal{T}_0^T$, $\nu \leq t \leq T \leq \widetilde{T}$, $\eta \in L_\nu^{2q}(\mathbb{R}^d)$ and $\iota \in \widetilde{\mathcal{I}}_\nu$. Furthermore, for all $(t, x, i) \in$

$$[0, \infty) \times \mathbb{R}^d \times \mathcal{I},$$

$$\lim_{T \rightarrow \infty} \widehat{Y}_t^{T,t,x,i} = v^\infty(x, i). \quad (5.5.13)$$

Finally, $v^\infty(\cdot, i)$ is continuous for all $i \in \mathcal{I}$.

Since the proof of Proposition 5.18 is too long, we put it in Appendix 5.C.

We next study the relationships between v^∞ and PDE. Consider the following PDE.

$$\begin{aligned} \min\{-\mathcal{L}^i u(x, i) - \psi(x, i) + \rho u(x, i) + \varsigma(x, i, \sigma'(x, i) \nabla u(x, i)), \\ u(x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{u(x, j) - c_{i,j}(x)\}\} = 0, \quad (x, i) \in \mathbb{R}^d \times \mathcal{I}, \end{aligned} \quad (5.5.14)$$

where

$$\mathcal{L}^i f(x) = (\nabla f(x))' b(x, i) + \frac{1}{2} \text{tr} \left(\sigma \sigma'(x, i) \frac{\partial f(x)}{\partial x \partial x'} \right).$$

Then the following proposition holds.

Proposition 5.19 Under Assumption 5.1, 5.3, 5.7 and 5.14, v^∞ is a viscosity solution of the PDE (5.5.14).

The proof of Proposition 5.19 is in Appendix 5.D. By Proposition 5.19, we can study the optimal switching problem under ambiguity through the PDE (5.5.14). Moreover, we can easily show the uniqueness of the solution to the PDE (5.5.14) using the method of Proposition 3.1 in Hamadène and Morlais (2013), so we omit the proof of the uniqueness.

5.6 Financial Applications

5.6.1 Monotone Conditions

We first prove that under certain conditions, the optimal switching problem under ambiguity can be interpreted as the optimal switching problem with a shift of the drift of X not depending on its value function. We first assume the followings.

Assumption 5.20 *Monotone conditions.* We assume $d = 1$.

1. *κ -ignorance.* There exist non-negative constants $\kappa_1, \dots, \kappa_I$ such that

$$\Theta_t^{x,i} = [-\kappa_i, \kappa_i],$$

for all $i \in \mathcal{I}$, $x \in \mathbb{R}^d$ and $t \in [0, \infty)$.

2. For every $x, y \in \mathbb{R}$, X satisfies,

$$x \leq y \quad \Rightarrow \quad X_s^{t,x,i} \leq X_s^{t,y,i}, \quad \mathbb{P}\text{-a.s.}, \quad (5.6.1)$$

for all $t, s \in [0, T]$, $i \in \mathcal{I}$ with $t \leq s$.

3. ρ does not depend on a value of x .

4. For every $(t, i) \in [0, T] \times \mathcal{I}$, $\psi(t, \cdot, i)$ is non-decreasing.

5. $\phi(t, x, i) = 0$ for every $(t, x, i) \in \overline{\mathcal{K}_T}$.

6. For every $i \in \mathcal{I}$, $g(\cdot, i)$ is non-decreasing.

7. For every $(t, i, j) \in [0, T] \times (\mathcal{I})^2$, $c_{i,j}(t, \cdot)$ is non-increasing.

Chen and Epstein (2002) call Assumption 5.20.1 κ -ignorance. The other conditions guarantee the monotonicity of the value function with respect to the initial value of X . Under Assumption 5.20, we can prove the following result.

Proposition 5.21 Suppose that Assumption 5.1, 5.3, 5.4, 5.7 and 5.20 are satisfied. For all $(t, x, i) \in \overline{\mathcal{K}_T}$ and $\alpha \in \mathbb{A}_i[t, T]$, let ${}^{-\kappa}X^{t,x,i,\alpha}$ be a solution to the following SDE,

$$\begin{aligned} d{}^{-\kappa}X_s^{t,x,i,\alpha} &= \left(b(s, {}^{-\kappa}X_s^{t,x,i,\alpha}, \alpha_s) - \kappa_{\alpha_s} |\sigma(s, {}^{-\kappa}X_s^{t,x,i,\alpha}, \alpha_s)| \right) ds + \sigma(s, {}^{-\kappa}X_s^{t,x,i,\alpha}, \alpha_s) dW_s, \\ {}^{-\kappa}X_t^{t,x,i,\alpha} &= x. \end{aligned}$$

Then, the value function $v(t, x, i)$ satisfies

$$\begin{aligned} v(t, x, i) &= \sup_{\alpha \in \mathbb{A}_i[t, T]} \mathbb{E} \left[\int_t^T {}^{-\kappa}D_s^{t,i,\alpha} \psi(s, {}^{-\kappa}X_s^{t,x,i,\alpha}, \alpha_s) ds + {}^{-\kappa}D_T^{t,i,\alpha} g({}^{-\kappa}X_T^{t,x,i,\alpha}, \alpha_T) \right. \\ &\quad \left. - \sum_{t \leq \tau_k \leq T} {}^{-\kappa}D_{\tau_k}^{t,i,\alpha} c_{i_{k-1}, i_k}(\tau_k, {}^{-\kappa}X_{\tau_k}^{t,x,i,\alpha}) \mid \mathcal{F}_t \right], \quad (5.6.2) \end{aligned}$$

where

$${}^{-\kappa}D_s^{t,i,\alpha} = \exp \left\{ - \int_t^s \rho(u, \alpha_u) du \right\}, \quad s \in [t, T].$$

Furthermore, $x \rightarrow v(t, x, i)$ is non-decreasing for all $(t, i) \in [0, T] \times \mathcal{I}$.

Proof of Proposition 5.21. By the κ -ignorance and $\phi = 0$, we have

$$\varsigma(t, x, i, z) = \kappa_i |z|,$$

for all $(t, x, i, z) \in \overline{\mathcal{K}_T} \times \mathbb{R}$. Now, fix an arbitrary $t \in [0, T]$ and $x, \tilde{x} \in \mathbb{R}$ with $x \leq \tilde{x}$.

Then, by the monotone conditions 2-6, we have

$$\psi(s, X_s^{t,x,i}, i) - \rho(s, i)y - \kappa_i |z| \leq \psi(s, X_s^{t,\tilde{x},i}, i) - \rho(s, i)y - \kappa_i |z|, \quad (5.6.3)$$

$$g(X_T^{t,x,i}, i) \leq g(X_T^{t,\tilde{x},i}, i), \quad (5.6.4)$$

for all $(s, i, y, z) \in [t, T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}$. Furthermore, by the monotone conditions 2 and 7, we have

$$\max_{j \in \mathcal{I} \setminus \{i\}} \left\{ y^j - c_{i,j}(s, X_s^{t,x,i}) \right\} \leq \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \bar{y}^j - c_{i,j}(s, X_s^{t,\tilde{x},i}) \right\}, \quad (5.6.5)$$

for all $(s, i) \in [t, T] \times \mathcal{I}$ and $(y^1, \dots, y^I), (\bar{y}^1, \dots, \bar{y}^I) \in \mathbb{R}^I$ with $y^k \leq \bar{y}^k$ for all $k \in \mathcal{I}$.

Let $(Y^{t,x,i,n})_{i \in \mathcal{I}, n \geq 0}$ and $(Y^{t,\tilde{x},i,n})_{i \in \mathcal{I}, n \geq 0}$ be the Picard's iterations defined in Theorem 5.8 with starting x and $\tilde{x}$, respectively. Then, by the inequalities (5.6.3), (5.6.4) and (5.6.5), recursively applying the comparison theorem leads to that

$$Y_s^{t,x,i,n} \leq Y_s^{t,\tilde{x},i,n},$$

for all $i \in \mathcal{I}$, $s \in [t, T]$ and $n \geq 0$. Taking a limit of the above inequality, we have

$$v(t, x, i) = Y_t^{t,x,i} \leq Y_t^{t,\tilde{x},i} = v(t, \tilde{x}, i), \quad (5.6.6)$$

for all $i \in \mathcal{I}$. Since we arbitrarily choose t, x and $\tilde{x}$ with $x \leq \tilde{x}$, the inequality (5.6.6) implies that a mapping $x \rightarrow v(t, x, i)$ is non-decreasing for all $t \in [0, T]$ and $i \in \mathcal{I}$.

Now, let us consider the following PDE,

$$\begin{aligned} \min \{ & -w_t(t, x, i) - \mathcal{L}^{-\kappa,i} w(t, x, i) - \psi(t, x, i) + \rho(t, i)w(t, x, i), \\ & w(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{w(t, x, j) - c_{i,j}(t, x)\} \} = 0, \quad (t, x, i) \in \overline{\mathcal{K}_T}, \\ & w(T, x, i) = g(x, i), \end{aligned} \quad (5.6.7)$$

where

$$\mathcal{L}^{-\kappa,i} f(t, x) = (b(t, x, i) - \kappa_i |\sigma(t, x, i)|) \nabla f(t, x) + \frac{1}{2} (\sigma(t, x, i))^2 \frac{\partial^2 f(t, x)}{\partial x^2}.$$

The PDE (5.6.7) has a unique continuous viscosity solution. Let $(t, x) \in [0, T) \times \mathbb{R}$ and let $\varphi \in C^{1,2}([0, T) \times \mathbb{R} \times \mathcal{I})$ be a test function such that $v(\cdot, \cdot, i) - \varphi(\cdot, \cdot, i)$ attains a local minimum at (t, x) for all $i \in \mathcal{I}$. Since $y \rightarrow v(s, y, j)$ is monotone non-decreasing for all $(s, j) \in [0, T) \times \mathcal{I}$, we have $\nabla \varphi(t, x, i) \geq 0$ for all $i \in \mathcal{I}$. Since v is the viscosity supersolution of the PDE (5.4.4) by Proposition 5.13, we have

$$\begin{aligned} & \min\{-\varphi_t(t, x, i) - \mathcal{L}^{-\kappa, i}\varphi(t, x, i) - \psi(t, x, i) + \rho(t, i)v(t, x, i), \\ & \quad v(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{v(t, x, j) - c_{i,j}(t, x)\}\} \\ & = \min\{-\varphi_t(t, x, i) - \mathcal{L}^i\varphi(t, x, i) - \psi(t, x, i) + \rho(t, i)v(t, x, i) + \kappa|\sigma(t, x, i)\nabla\varphi(t, x, i)|, \\ & \quad v(t, x, i) - \max_{j \in \mathcal{I} \setminus \{i\}} \{v(t, x, j) - c_{i,j}(t, x)\}\} \geq 0, \end{aligned}$$

for all $i \in \mathcal{I}$. Hence, v is a viscosity supersolution of the PDE (5.6.7). The comparison theorem of the viscosity solutions gives $v \geq w$. Using the similar argument, we also have $v \leq w$. Thus, $v = w$. Since a value function of the optimal switching problem in the right hand side of our desired equality (5.6.2) is a unique viscosity solution of the PDE (5.6.7), we obtain the equality (5.6.2). $\square$

In the infinite horizon case, Proposition 5.21 also holds under the same conditions as Assumption 5.20. Proposition 5.21 implies that under the monotone conditions, the optimal switching problem under ambiguity can be regarded as usual optimal switching problems. Thus, we can use existing results in the literature of the optimal switching if the monotone conditions are satisfied. In fact, under the monotone conditions, it is sufficient to solve the PDE (5.6.7) instead of the PDE (5.4.4) to derive the value function.

The monotone conditions and Proposition 5.21 are very similar to the results of Cheng and Riedel (2013). Cheng and Riedel (2013) study the optimal stopping problem under ambiguity and show that if a payoff function $f(t, x)$ is non-decreasing in x and κ -ignorance is satisfied, then the optimal stopping problem under ambiguity can be regarded as the standard optimal stopping problem in which the drift of X shifts into $b - \kappa|\sigma|$ (Theorem 4.1 in Cheng and Riedel (2013)). Our result implies that the optimal switching problem under ambiguity holds the same property as the optimal stopping under ambiguity.

In Section 5.6.2 and 5.6.3, we consider two applications of the optimal switching problem under ambiguity in finance. The first application in Section 5.6.2 is a selection of investment funds and it satisfies the monotone conditions. However, the second application (the buy low and sell high problem) in Section 5.6.3 does not satisfy the monotone conditions and it definitely needs negative switching costs.

5.6.2 Selection of Investment Funds

In this section, we consider an optimal selection of two investment funds under ambiguity in the infinite horizon. Let $d = 1$ and $\mathcal{I} = \{1, 2\}$. Assume that X satisfies the following SDE,

$$dX_t = b_{\alpha_t} X_t dt + \sigma_{\alpha_t} X_t dW_t, \quad (5.6.8)$$

where $b_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1, 2$ are constants. The solution to the SDE (5.6.8) is

$$X_t^{x,i,\alpha} = x \exp \left\{ \int_0^t \left(b_{\alpha_s} - \frac{1}{2} \sigma_{\alpha_s}^2 \right) ds + \int_0^t \sigma_{\alpha_s} dW_s \right\},$$

for all $\alpha \in \mathbb{A}_i[0, \infty)$. Assume that $\phi = 0$ and that ψ is

$$\psi(x) = x^p, \quad x \in [0, \infty), \quad 0 < p < 1.$$

The switching costs $c_{1,2}$ and $c_{2,1}$ are constants over x , and they satisfy $c_{1,2} + c_{2,1} > 0$.

The constant discount rate ρ , satisfies

$$\rho > p \max_{i \in \mathcal{I}} \left\{ b_i - \frac{1-p}{2} \sigma_i^2 \right\}.$$

The set of multiple priors is

$$\Theta^{x,i} = [-\kappa_i, \kappa_i], \quad \kappa_i \geq 0,$$

for all $x \in \mathbb{R}$ and $i \in \mathcal{I}$. In the above settings, an optimal switching problem of interest is

$$v^\infty(x, i) = \sup_{\alpha \in \mathbb{A}_i[0, \infty)} \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \zeta_t^{\theta, 0} (X_t^{x,i,\alpha})^p dt - \sum_{k=1}^\infty e^{-\rho \tau_k} \zeta_{\tau_k}^{\theta, 0} c_{i_{k-1}, i_k} \right]. \quad (5.6.9)$$

Since the problem (5.6.9) satisfies Assumption 5.1, 5.3, 5.7 and 5.14, we can use the results in Section 5.5. Furthermore, the problem (5.6.9) also satisfies the monotone conditions (Assumption 5.20).

Without ambiguity (i.e., $\kappa_i = 0$ for all $i \in \mathcal{I}$), the problem (5.6.9) is well studied by Ly Vath and Pham (2007). We shortly summarize their results as follows.

Proposition 5.22 (Theorem 4.1 in Ly Vath and Pham (2007)) Let

$$K_i = \frac{1}{\rho - b_i p + \frac{1}{2} \sigma_i^2 p (1 - p)}, \quad (5.6.10)$$

for all $i \in \mathcal{I}$. Let $i, j \in \mathcal{I}$, $i \neq j$.

1. If $K_i = K_j$, then it is always optimal to switch from regime i to j if the corresponding switching cost is non-positive, and never optimal to switch otherwise.
2. If $K_j > K_i$, then the following switching strategies depending on the switching costs are optimal.
 - (a) $c_{i,j} \leq 0$: it is always optimal to switch from regime i to j if one first stands in i and it is always optimal not to switch from j to i otherwise.
 - (b) $c_{i,j} > 0$:
 - i. $c_{j,i} \geq 0$: there exists $\underline{x}_i^* \in [0, \infty)$ such that if one first stands in regime i , then it is optimal to switch from i to j whenever X exceeds $\underline{x}_i^*$. If one first stands in regime j , then it is optimal not to switch from j to i .
 - ii. $c_{j,i} < 0$: there exist $\underline{x}_i^*, \bar{x}_j^* \in [0, \infty)$ with $\bar{x}_j^* < \underline{x}_i^*$ such that if one first stands in regime i , then it is optimal to switch from i to j whenever X exceeds $\underline{x}_i^*$ and that if one first stands in regime j , then it is optimal to switch from j to i whenever X falls below $\bar{x}_j^*$.

For details of $\underline{x}_i^*$ and $\bar{x}_j^*$ and the functional form of the value function, we refer to Ly Vath and Pham (2007). By Proposition 5.22, the types of the switching strategies are determined by K_i defined in (5.6.10) and the switching costs. The most interesting case is Proposition 5.22.2.(b).ii in which the decision maker continuously switches the regimes.

The problem (5.6.9) can be interpreted as an optimal selection of investment funds. An investor chooses a fund to maximize her expected utility with multiple priors. The switching costs are interpreted as costs or benefits in changing funds.

We now assume $K_2 > K_1$ and $c_{1,2} > 0 > c_{2,1}$. Then, heuristically speaking, the fund 2 (regime 2) is more attractive than the fund 1 (regime 1), but one requires the positive switching cost $c_{1,2}$ to switch from the fund 1 to the fund 2. On the other hand, one gets the switching benefit $-c_{2,1}$ when switching from the fund 2 to the fund 1. We can also interpret the fund 2 as a new fund well performing and the fund 1 as an old fund less performing. To obtain customers, the fund 1 begins the campaign that one switching from the fund 2 to the fund 1 obtains the benefit $-c_{2,1}$. Then, the investor has a motivation switching between the fund 1 and 2.

However, in practice, the investor may doubt the good performance of the fund 2 since the fund 2 is new and less experienced. She therefore considers that the fund 2 has a premium of ambiguity. Mathematically, this implies that $\kappa_2 > 0$ and $\kappa_1 = 0$. We now consider the case that $\kappa_2 > 0$ and $\kappa_1 = 0$.

Since the problem (5.6.9) satisfies the monotone conditions, we can use the results of Ly Vath and Pham (2007). Let

$$K_2^\kappa = \frac{1}{\rho - (b_2 - \kappa_2 \sigma_2)p + \frac{1}{2}\sigma_2^2 p(1-p)} > 0.$$

Then, we have

$$K_2^\kappa - K_1 = \frac{pK_2^\kappa K_1}{2} \left((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2\kappa_2 \sigma_2 \right).$$

Therefore, the sign of $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2\kappa_2 \sigma_2$ determines the type of the switching strategy. On the other hand, we have

$$K_2 - K_1 = \frac{pK_2 K_1}{2} \left((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) \right) > 0.$$

Hence, $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2)$ is positive. However, if κ_2 is sufficiently large such that $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) < 2\kappa_2 \sigma_2$, then $K_2^\kappa < K_1$. Therefore, the large ambiguity with respect to the fund 2 can change the type of the switching strategy.

To illustrate effects of ambiguity, we conduct a numerical simulation. Let $b_1 = 0.03$, $b_2 = 0.07$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $p = 0.5$, $\rho = 0.03$, $c_{1,2} = 30000$, and $c_{2,1} = -1000$. Then,

$$K_1 = 61.53846 \dots < 160 = K_2.$$

Hence, the investor continuously switches between the fund 1 and 2 without ambiguity. On the other hand, we have

$$\frac{(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2)}{2\sigma_2} = \frac{1}{15} = 0.0666\ldots$$

Thus, if $\kappa_2 > 1/15$, then the type of switching strategy changes to that one always chooses the fund 1.

Figure 5.1 displays the switching thresholds $\underline{x}_1^*$ and $\bar{x}_2^*$ with different degrees of ambiguity κ_2 . If one is investing in the fund 1 at time t and if $X_t \geq \underline{x}_1^*$, then she switches from the fund 1 to the fund 2. On the other hand, if one is investing in the fund 2 at time t and if $X_t \leq \bar{x}_2^*$, then she switches from the fund 2 to the fund 1.

According to Figure 5.1, in a higher degree of ambiguity κ_2 , both of the thresholds $\underline{x}_1^*$ and $\bar{x}_2^*$ are large. This implies that if κ_2 is large, then the investor investing in the fund 1 needs sufficiently large wealth X to switch from the fund 1 to the fund 2. On the other hand, if κ_2 is large, then the investor investing in the fund 2 switches to the fund 1 with smaller wealth than that in small κ_2 . Each behavior is well convincing. The large ambiguity makes the fund 2 less attractive, so the investor tends to choose the fund 1.

Remark 5.23 If $\kappa_1 > 0$, then let

$$K_1^\kappa = \frac{1}{\rho - (b_1 - \kappa_1\sigma_1)p + \frac{1}{2}\sigma_1^2p(1-p)} > 0,$$

and

$$K_2^\kappa - K_1^\kappa = \frac{pK_2^\kappa K_1^\kappa}{2} \left((1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) - 2(\kappa_2\sigma_2 - \kappa_1\sigma_1) \right).$$

Hence, in this case, if $(1-p)(\sigma_1^2 - \sigma_2^2) - 2(b_1 - b_2) < 2(\kappa_2\sigma_2 - \kappa_1\sigma_1)$, then $K_2^\kappa < K_1^\kappa$.

5.6.3 Buy Low and Sell High

Next, we consider an optimal trading (buy and sell) rule under ambiguity. Without ambiguity, this problem in trading a mean-reverting asset is well studied by Zhang and Zhang (2008). We adopt their settings and consider an optimal trading rule under ambiguity. Let $d = 1$. A trader concerns with trading of a certain asset. A cumulative

log return of this asset at time t is denoted by X_t and it satisfies the following SDE.

$$dX_t = a(b - X_t)dt + \sigma dW_t, \quad (5.6.11)$$

where $a > 0$, $b \in \mathbb{R}$ and $\sigma > 0$ are constants. Therefore, the asset price at time t is given by $S_t = \exp(X_t)$. We denote the solution to the SDE (5.6.11) starting from $X_0 = x$ by X^x . Furthermore, this asset does not have any dividend and coupon. This implies $\psi = 0$ and $\phi = 0$.

Let $\mathcal{I} = \{1, 2\}$. The regime $i = 1$ means that the trader's position is flat. Hence, she wants to buy the asset at as low a price as possible. The regime $i = 2$ means that the trader's position is long. Hence, she wants to sell the asset at as high a price as possible. If the trader goes from the regime 1 to the regime 2, in other words, if she buys the asset, then the switching cost function is

$$c_{1,2}(x) = e^x(1 + K), \quad (5.6.12)$$

where $K \in (0, 1)$ is a constant percentage of slippage or commission per transaction. On the other hand, if the trader goes from the regime 2 to the regime 1, in other words, if she sells the asset, then the cost (benefit) function is

$$c_{2,1}(x) = -e^x(1 - K). \quad (5.6.13)$$

The set of multiple priors is

$$\Theta^{x,i} = [-\kappa, \kappa], \quad \kappa \geq 0,$$

for all $x \in \mathbb{R}$ and $i \in \mathcal{I}$. Therefore, we assume κ -ignorance.

The buy low and sell high problem under ambiguity can be interpreted as the following optimal switching problem,

$$v(x, i) = \sup_{\alpha \in \mathbb{A}_i[0, \infty)} \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[- \sum_{k=1}^{\infty} e^{-\rho \tau_k} \zeta_{\tau_k}^{\theta, 0} c_{i_{k-1}, i_k}(X_{\tau_k}^x) \right]. \quad (5.6.14)$$

More directly, the problem (5.6.14) can be expressed as

$$\begin{aligned}
& v(x, 1) \\
&= \sup_{\alpha \in \mathbb{A}_i[0, \infty)} \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[\sum_{k=1}^{\infty} \left(e^{-\rho\tau_{2k}} \zeta_{\tau_{2k}}^{\theta, 0} e^{X_{\tau_{2k}}^x} (1 - K) - e^{-\rho\tau_{2k-1}} \zeta_{\tau_{2k-1}}^{\theta, 0} e^{X_{\tau_{2k-1}}^x} (1 + K) \right) \right], \\
& v(x, 2) \\
&= \sup_{\alpha \in \mathbb{A}_i[0, \infty)} \inf_{\theta \in \Theta[0, \infty)} \mathbb{E} \left[e^{-\rho\tau_1} \zeta_{\tau_1}^{\theta, 0} e^{X_{\tau_1}^x} (1 - K) \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \left(e^{-\rho\tau_{2k+1}} \zeta_{\tau_{2k+1}}^{\theta, 0} e^{X_{\tau_{2k+1}}^x} (1 - K) - e^{-\rho\tau_{2k}} \zeta_{\tau_{2k}}^{\theta, 0} e^{X_{\tau_{2k}}^x} (1 + K) \right) \right].
\end{aligned}$$

The cost/benefit functions (5.6.12) and (5.6.13) do not satisfy the polynomial growth condition and the strong triangular condition. However, changing variables from X to S , then these functions satisfy the polynomial growth condition. Furthermore, we can easily prove Proposition 5.5 in the problem (5.6.14) (see Lemma 4 in Zhang and Zhang (2008) and Remark 5.17 in this chapter). Therefore, we can apply the method in Section 5.5. Note that for sufficiently large constant $C \geq 0$, the following function satisfies the temporary terminal conditions:

$$g(x, i) = -\mathbb{1}_{\{i=1\}} e^x (1 - K) - C.$$

It is easy to show that g satisfies the sufficient condition (5.5.9) for sufficiently large C .

According to Proposition 5.19, the value function v is a viscosity solution of the following system of PDEs.

$$\min\{-\mathcal{L}v(x, 1) + \rho v(x, 1) + \kappa\sigma|\nabla v(x, 1)|, v(x, 1) - v(x, 2) + e^x(1 + K)\} = 0, \quad (5.6.15)$$

$$\min\{-\mathcal{L}v(x, 2) + \rho v(x, 2) + \kappa\sigma|\nabla v(x, 2)|, v(x, 2) - v(x, 1) - e^x(1 - K)\} = 0, \quad (5.6.16)$$

where

$$\mathcal{L}f(x) = a(b - x)\nabla f(x) + \frac{\sigma^2}{2} \frac{\partial^2 f(x)}{\partial x^2}.$$

Unfortunately, the problem (5.6.14) does not satisfy the monotone conditions, therefore we need to solve the system of PDEs (5.6.15) and (5.6.16). It seems to be difficult to solve this system since it contains the absolute values of the first derivatives of v . However, we can find a continuous solution to the system of PDEs (5.6.15) and (5.6.16)

using the smooth-fit techniques (details of the smooth-fit techniques are in Chapter 5 in Pham (2009)).

First, let $\mathcal{C}_1$ be a continuation region of the regime 1 such that

$$\mathcal{C}_1 = (x_1, \infty),$$

for some x_1 . Thus, the trader in the flat position buys the asset whenever the asset price falls below e^{x_1} . Also let $\mathcal{C}_2$ be a continuation region of the regime 2 such that

$$\mathcal{C}_2 = (-\infty, x_2),$$

for some x_2 . Thus, the trader in the long position sells the asset whenever the asset price exceeds e^{x_2} . Naturally we impose $x_1 \leq x_2$. We assume that

$$\nabla v(x, 1) \leq 0, \quad \forall x \in \mathcal{C}_1, \quad \text{and} \quad \nabla v(x, 2) \geq 0, \quad \forall x \in \mathcal{C}_2. \quad (5.6.17)$$

By Zhang and Zhang (2008), the PDE,

$$-\mathcal{L}V(x, 1) + \rho V(x, 1) - \kappa\sigma \nabla V(x, 1) = 0,$$

on $\mathcal{C}_1$ has a solution such that

$$V(x, 1) = C_1 \varphi_1(x),$$

where C_1 , $m = \sqrt{2a}/\sigma$, and $\lambda = \rho/a$ are constants, and

$$\varphi_1(x) = \int_0^\infty t^{\lambda-1} e^{-0.5t^2 + m(b+\kappa\sigma/a-x)t} dt,$$

Similarly, the PDE,

$$-\mathcal{L}V(x, 2) + \rho V(x, 2) + \kappa\sigma \nabla V(x, 2) = 0,$$

on $\mathcal{C}_2$ has a solution such that

$$V(x, 2) = C_2 \varphi_2(x),$$

where C_2 is a constant and

$$\varphi_2(x) = \int_0^\infty t^{\lambda-1} e^{-0.5t^2 - m(b-\kappa\sigma/a-x)t} dt.$$

Now, let us guess that candidates of the solution to the PDEs (5.6.15) and (5.6.16) are

$$\begin{aligned} v(x, 1) &= V(x, 1), \quad \text{on } \mathcal{C}_1, \\ &= V(x, 2) - e^x(1 + K), \quad \text{not on } \mathcal{C}_1, \end{aligned} \quad (5.6.18)$$

$$\begin{aligned} v(x, 2) &= V(x, 2), \quad \text{on } \mathcal{C}_2, \\ &= V(x, 1) + e^x(1 - K), \quad \text{not on } \mathcal{C}_2. \end{aligned} \quad (5.6.19)$$

Let

$$\varphi_1^*(x) = \int_0^\infty t^\lambda e^{-0.5t^2 + m(b + \kappa\sigma/a - x)t} dt, \quad \varphi_2^*(x) = \int_0^\infty t^\lambda e^{-0.5t^2 - m(b - \kappa\sigma/a - x)t} dt.$$

Then, $\nabla V(x, 1) = -mC_1\varphi_1^*(x)$ and $\nabla V(x, 2) = mC_2\varphi_2^*(x)$. Hence by the conditions (5.6.17), we need $C_1 \geq 0$ and $C_2 \geq 0$. By the smooth-fit conditions, we need

$$\begin{cases} V(x_1, 1) = V(x_1, 2) - e^{x_1}(1 + K), \\ \nabla V(x_1, 1) = \nabla V(x_1, 2) - e^{x_1}(1 + K), \\ V(x_2, 2) = V(x_2, 1) + e^{x_2}(1 - K), \\ \nabla V(x_2, 2) = \nabla V(x_2, 1) + e^{x_2}(1 - K), \end{cases} \quad (5.6.20)$$

$$\begin{cases} v(x, 1) \geq v(x, 2) - e^x(1 + K), \quad \text{on } (x_1, \infty), \\ v(x, 2) \geq v(x, 1) + e^x(1 - K), \quad \text{on } (-\infty, x_2), \end{cases} \quad (5.6.21)$$

$$\begin{cases} (-\mathcal{L} + \rho + \kappa\sigma|\nabla|)(V(x, 2) - e^x(1 + K)) \geq 0, \quad \text{on } (-\infty, x_1), \\ (-\mathcal{L} + \rho + \kappa\sigma|\nabla|)(V(x, 1) + e^x(1 - K)) \geq 0, \quad \text{on } (x_2, \infty). \end{cases} \quad (5.6.22)$$

After simple algebraic computation, the equalities (5.6.20) can be expressed as

$$\begin{aligned} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= e^{x_1}(1 + K) \begin{pmatrix} -\varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_1^*(x_1) & \varphi_2^*(x_1) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1/m \end{pmatrix} \\ &= e^{x_2}(1 - K) \begin{pmatrix} -\varphi_1(x_2) & \varphi_2(x_2) \\ \varphi_1^*(x_2) & \varphi_2^*(x_2) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1/m \end{pmatrix} \geq 0. \end{aligned} \quad (5.6.23)$$

By the definitions of v , the inequalities (5.6.21) are equivalent to

$$V(x, 1) \geq V(x, 2) - e^x(1 + K), \quad V(x, 2) \geq V(x, 1) + e^x(1 - K), \quad (5.6.24)$$

on (x_1, x_2) . For the first inequality of (5.6.22), we have

$$\begin{aligned} (-\mathcal{L} + \rho + \kappa\sigma|\nabla|)(V(x, 2) - e^x(1 + K)) &= (-\mathcal{L} + \rho + \kappa\sigma|\nabla|)(-e^x(1 + K)) \\ &= -\left(\rho - a(b - x) - \frac{\sigma^2}{2} - \kappa\sigma\right)e^x(1 + K) \geq 0 \end{aligned}$$

on $(-\infty, x_1)$ since $(-\infty, x_1) \subseteq \mathcal{C}_2$. Thus, the condition expressed by the first inequality is equivalent to

$$x_1 \leq \frac{1}{a} \left(\frac{\sigma^2}{2} + ab + \kappa\sigma - \rho \right). \quad (5.6.25)$$

Similarly, the condition expressed by the second inequality of (5.6.22) is equivalent to

$$x_2 \geq \frac{1}{a} \left(\frac{\sigma^2}{2} + ab - \kappa\sigma - \rho \right). \quad (5.6.26)$$

Finally, we need

$$e^{x_2}(1 - K) > e^{x_1}(1 + K) \Leftrightarrow x_2 - x_1 > \log(1 + K) - \log(1 - K). \quad (5.6.27)$$

Hence, if C_1, C_2, x_1 and x_2 satisfy the conditions (5.6.23)-(5.6.27), then the candidates of the solutions (5.6.18) and (5.6.19) are true viscosity solutions of the PDEs (5.6.15) and (5.6.16).

To illustrate effects of ambiguity, we conduct a numerical simulation. Let $a = 0.8$, $b = 2$, $\sigma = 0.5$, $\rho = 0.5$, and $K = 0.01$. The values of these parameters are the same as Zhang and Zhang (2008). We compute thresholds (x_1, x_2) with different degrees of ambiguity κ .

Figure 5.2 displays the thresholds. According to Figure 5.2, in a larger degree of ambiguity, both of the optimal thresholds become small. The long position trader (that is, the initial regime is 2) considers the worst case that the steady mean of X is smaller than that without ambiguity. Therefore, she sells the asset at a lower price than that without ambiguity.

On the other hand, in the flat position case (that is, the initial regime is 1), the trader also buys the asset at a lower price than that without ambiguity. That is because a gain of the trader in the flat position does not realize until she sells the asset. Now, we assume that the trader considers the case when the steady mean of X is larger than that without ambiguity. Then, she can expect a bigger profit in her belief than that in the true probability measure. This is a contradiction since she considers the worst case. Therefore, even if the trader has the flat position, she considers the case that the steady mean of X is smaller than that without ambiguity. Hence, the optimal thresholds of buying the asset under ambiguity is lower than that without ambiguity.

Zhang and Zhang (2008) conduct the comparative statics with varying the steady mean of X , i.e., b . Their results are that in a small b , both of the optimal thresholds are also small. These are similar to the results in large ambiguity. However, the results under large ambiguity can not be reproduced by a small b . By the equality (5.6.23)

with $\kappa = 0$, the optimal thresholds under the steady mean b are equal to the optimal thresholds under the steady mean $\tilde{b}$ plus $b - \tilde{b}$ for all $b, \tilde{b} \in \mathbb{R}$ if the other parameters are the same. Therefore, the optimal thresholds are linear in the steady mean b .

On the other hand, Figure 5.3 displays equal differences of the optimal thresholds with different degrees of ambiguity. According to Figure 5.3, the equal differences are not constant, therefore the optimal thresholds are not linear in the degree of ambiguity κ . Our PDEs (5.6.15) and (5.6.16) cause these non-linearities. The PDEs (5.6.15) and (5.6.16) can not be expressed as any variational inequality of an optimal switching problem without ambiguity since these do not satisfy the monotone conditions. Indeed, the difference $x_2 - x_1$ without ambiguity is constant over b , whereas $x_2 - x_1$ is small with large κ . Thus, the optimal switching problem under ambiguity can generate this interesting result which can not be reproduced by the problem without ambiguity.

Appendix 5.A The Moment Estimates of X

Proof of Proposition 5.2. Since $x \rightarrow \|x\|^q$ is twice continuously differentiable for all $q \geq 4$, we can apply the Ito's lemma to $\|X_s^{t,x,i,\alpha}\|^q$. Then, for all $s \in [t, T]$, using the quadratic growth condition for b and σ , we have

$$\begin{aligned} \|X_s^{t,x,i,\alpha}\|^q &= \|x\|^q + \int_t^s q \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' b(r, X_r^{t,x,i,\alpha}, \alpha_r) dr \\ &\quad + \frac{1}{2} \int_t^s \left(q(q-2) \|X_r^{t,x,i,\alpha}\|^{q-4} \|\sigma'(r, X_r^{t,x,i,\alpha}, \alpha_r) X_r^{t,x,i,\alpha}\|^2 \right. \\ &\quad \left. + q \|X_r^{t,x,i,\alpha}\|^{q-2} \|\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\|^2 \right) dr \\ &\quad + \int_t^s q \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) dW_r \\ &\leq \|x\|^q + \widehat{C}_q \int_t^s \left(1 + \|X_r^{t,x,i,\alpha}\|^q \right) dr \\ &\quad + q \int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) dW_r, \end{aligned}$$

where $\widehat{C}_q$ is the constant only depending on q and L . The above stochastic integral in the right hand side is a local martingale. Hence, there exists an increasing sequence of stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n \rightarrow \infty$ and

$$\mathbb{E}[\|X_{s \wedge \tau_n}^{t,x,i,\alpha}\|^q] \leq \|x\|^2 + \widehat{C}_q \mathbb{E} \left[\int_t^{s \wedge \tau_n} \left(1 + \|X_r^{t,x,i,\alpha}\|^q \right) dr \right],$$

for all $s \in [t, T]$ and $n \geq 1$, where $a \wedge b = \min\{a, b\}$. By the Fatou lemma, the monotone convergence theorem and the continuity of $X^{t,x,i,\alpha}$, taking a limit, we have

$$\begin{aligned} 1 + \mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] &\leq 1 + \liminf_{n \rightarrow \infty} \mathbb{E}[\|X_{s \wedge \tau_n}^{t,x,i,\alpha}\|^q] \\ &\leq 1 + \|x\|^2 + \widehat{C}_q \mathbb{E} \left[\int_t^s (1 + \|X_r^{t,x,i,\alpha}\|^q) dr \right] \\ &= 1 + \|x\|^2 + \widehat{C}_q \int_t^s \mathbb{E} [1 + \|X_r^{t,x,i,\alpha}\|^q] dr. \end{aligned}$$

By the Gronwall lemma, we have

$$\mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] \leq 1 + \mathbb{E}[\|X_s^{t,x,i,\alpha}\|^q] \leq (1 + \|x\|^q) e^{\widehat{C}_q(s-t)}, \quad (5.A.1)$$

for all $0 \leq t \leq s$ and $x \in \mathbb{R}^d$. Similarly, we have

$$\begin{aligned} \max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q &\leq \|x\|^q + \widehat{C}_q \int_t^T (1 + \|X_r^{t,x,i,\alpha}\|^q) dr \\ &\quad + q \max_{t \leq s \leq T} \int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) dW_r. \end{aligned}$$

By the Burkholder-Davis-Gundy inequality and Jensen inequality, we have

$$\begin{aligned} &\mathbb{E} \left[\max_{t \leq s \leq T} \int_t^s \|X_r^{t,x,i,\alpha}\|^{q-2} (X_r^{t,x,i,\alpha})' \sigma(r, X_r^{t,x,i,\alpha}, \alpha_r) dW_r \right] \\ &\leq \mathbb{E} \left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-4} \|\sigma'(r, X_r^{t,x,i,\alpha}, \alpha_r) X_r^{t,x,i,\alpha}\|^2 dr \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-2} \|\sigma(r, X_r^{t,x,i,\alpha}, \alpha_r)\|^2 dr \right)^{1/2} \right] \\ &\leq L \mathbb{E} \left[\left(\int_t^T \|X_r^{t,x,i,\alpha}\|^{2q-2} (1 + \|X_r^{t,x,i,\alpha}\|^2) dr \right)^{1/2} \right] \\ &\leq \sqrt{2}L \left(\int_t^T \mathbb{E} [1 + \|X_r^{t,x,i,\alpha}\|^{2q}] dr \right)^{1/2}. \end{aligned}$$

Furthermore, using the inequality (5.A.1), we have

$$\begin{aligned} \left(\int_t^T \mathbb{E} [1 + \|X_r^{t,x,i,\alpha}\|^{2q}] dr \right)^{1/2} &\leq \left(\int_t^T (1 + \|x\|^{2q}) e^{\widehat{C}_{2q}(r-t)} dr \right)^{1/2} \\ &\leq \frac{1}{\widehat{C}_{2q}^{1/2}} (1 + \|x\|^q) e^{\widehat{C}_{2q}(T-t)/2}. \end{aligned}$$

Thus, we obtain

$$\mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q \right] \leq 1 + \mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q \right] \leq C_{q,X} (1 + \|x\|^q) e^{C_q(T-t)},$$

where

$$C_{q,X} = \max \left\{ 1, \widehat{C}_q, \sqrt{\frac{2}{\widehat{C}_q}} qL \right\}, \quad C_q = \frac{\widehat{C}_{2q}}{2}.$$

If $q \in (0, 4)$, then by the Jensen inequality, we have

$$\begin{aligned} \mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^q \right] &= \mathbb{E} \left[\left(\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^4 \right)^{q/4} \right] \leq \left(\mathbb{E} \left[\max_{t \leq s \leq T} \|X_s^{t,x,i,\alpha}\|^4 \right] \right)^{q/4} \\ &\leq C_{4,X}^{q/4} (1 + \|x\|^4)^{q/4} e^{(qC_4/4)(T-t)} \\ &\leq C_{4,X}^{q/4} (1 + \|x\|^q) e^{(qC_4/4)(T-t)}. \end{aligned}$$

It is easy to show the inequality (5.2.8) applying the Ito's lemma to $e^{-\rho s}(1 + \|X_s^{t,x,i,\alpha}\|^q)$.

□

Appendix 5.B Verification of Y

Proof of Proposition 5.11. Step.1 Y is at least as large as any objective function. We define a sequence of random variables as follows.

$$X^0 := \eta, \quad X^k := X_{\tau_k}^{\tau_{k-1}, X^{k-1}, i_{k-1}}, \quad k \geq 1.$$

By the definition, $X^k \in L_{\tau_k}^{2q}(\mathbb{R}^d)$ for all k . Furthermore, for all $k \geq 1$ and $t \in [\tau_{k-1}, \tau_k)$, the strong uniqueness of X leads to that

$$X_t^{\nu, \eta, \iota, \alpha} = X_t^{\tau_{k-1}, X^{k-1}, i_{k-1}}, \quad (5.B.1)$$

$\mathbb{P}$ -almost surely.

Let $N = \inf\{k \mid \tau_k \geq T\}$ and $\tau_0 = \nu$. By the admissibility of $\alpha = (\tau_k, i_k)_{k \geq 0}$, N is finite $\mathbb{P}$ -almost surely. Let $\overline{Z}^{\nu, \eta, \iota, \alpha}$ be a stochastic process such that

$$\overline{Z}_t^{\nu, \eta, \iota, \alpha} = \sum_{k=1}^N Z_t^{\tau_{k-1}, X^{k-1}, i_{k-1}} \mathbb{1}_{[\tau_{k-1}, \tau_k)}(t), \quad t \in [0, T], \quad (5.B.2)$$

Let D^k be a stochastic process on $[\tau_{k-1}, \tau_k]$ such that

$$D_t^k = \exp \left\{ - \int_{\tau_{k-1}}^t \rho(s, X_s^{\tau_{k-1}, X^{k-1}, i_{k-1}}, i_{k-1}) ds \right\}, \quad t \in [\tau_{k-1}, \tau_k].$$

By the equality (5.B.1), we have

$$\begin{aligned} D_t^{\nu, \eta, \iota, \alpha} &= D_t^1, \quad t \in [\tau_0, \tau_1], \\ D_t^{\nu, \eta, \iota, \alpha} &= D_{\tau_{k-1}}^{\nu, \eta, \iota, \alpha} D_t^k, \quad t \in [\tau_{k-1}, \tau_k], \quad k \geq 2. \end{aligned} \quad (5.B.3)$$

Then, for any $k \geq 1$, applying the Ito's lemma to $D_t^k Y_t^{\tau_{k-1}, X^{k-1}, i_{k-1}}$ leads to

$$\begin{aligned} Y_{\tau_{k-1}}^{\tau_{k-1}, X^{k-1}, i_{k-1}} &\geq D_{\tau_k}^k Y_{\tau_k}^{\tau_{k-1}, X^{k-1}, i_{k-1}} + \int_{\tau_{k-1}}^{\tau_k} D_s^k \left(\psi(s, X_s^{\tau_{k-1}, X^{k-1}, i_{k-1}}, i_{k-1}) \right. \\ &\quad \left. - \varsigma(s, X_s^{\tau_{k-1}, X^{k-1}, i_{k-1}}, i_{k-1}, Z_s^{\tau_{k-1}, X^{k-1}, i_{k-1}}) \right) ds \\ &\quad - \int_{\tau_{k-1}}^{\tau_k} D_s^k (Z_s^{\tau_{k-1}, X^{k-1}, i_{k-1}})' dW_s, \end{aligned}$$

where we have used the non-negativity of D_t^k and monotonicity of $K_t^{\tau_{k-1}, X^{k-1}, i_{k-1}}$. Furthermore, by the pathwise uniqueness of X and Y (see Proposition 5.10, (5.B.1) and (5.B.2)), we have

$$\begin{aligned} Y_{\tau_{k-1}}^{\tau_{k-1}, X^{k-1}, i_{k-1}} &\geq D_{\tau_k}^k Y_{\tau_k}^{\tau_k, X^k, i_{k-1}} + \int_{\tau_{k-1}}^{\tau_k} D_s^k \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\ &\quad - \int_{\tau_{k-1}}^{\tau_k} D_s^k (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s, \end{aligned} \quad (5.B.4)$$

Since each $Y_{\tau_k}^{\tau_k, X^k, i_{k-1}}$ dominates the lower barrier, we obtain

$$\begin{aligned}
Y_{\nu}^{\nu, \eta, \iota} &\geq D_{\tau_1}^1 Y_{\tau_1}^{\tau_1, X^1, i_0} + \int_{\tau_0}^{\tau_1} D_s^1 \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_1} D_s^1 (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s \\
&\geq D_{\tau_1}^1 \left(Y_{\tau_1}^{\tau_1, X^1, i_1} - c_{i_0, i_1}(\tau_1, X_{\tau_1}^{\tau_1, X^1, i_0}) \right) \\
&\quad + \int_{\tau_0}^{\tau_1} D_s^1 \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_1} D_s^1 (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s \\
&= D_{\tau_1}^1 Y_{\tau_1}^{\tau_1, X^1, i_1} - D_{\tau_1}^1 c_{i_0, i_1}(\tau_1, X_{\tau_1}^{\nu, \eta, \iota, \alpha}) \\
&\quad + \int_{\tau_0}^{\tau_1} D_s^1 \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_1} D_s^1 (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s \\
&\geq D_{\tau_1}^1 Y_{\tau_2}^{\tau_1, X^1, i_1} - D_{\tau_1}^1 c_{i_0, i_1}(\tau_1, X_{\tau_1}^{\nu, \eta, \iota, \alpha}) \\
&\quad + \int_{\tau_0}^{\tau_1} D_s^1 \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad + D_{\tau_1}^1 \int_{\tau_1}^{\tau_2} D_s^2 \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_1} D_s^1 (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s - D_{\tau_1}^1 \int_{\tau_1}^{\tau_2} D_s^2 (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s \\
&= D_{\tau_1}^1 Y_{\tau_2}^{\tau_2, X^2, i_1} - D_{\tau_1}^1 c_{i_0, i_1}(\tau_1, X_{\tau_1}^{\nu, \eta, \iota, \alpha}) \\
&\quad + \int_{\tau_0}^{\tau_2} D_s^{\nu, \eta, \iota, \alpha} \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_2} D_s^{\nu, \eta, \iota, \alpha} (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s,
\end{aligned}$$

where we have used Proposition 5.10 and the relationship (5.B.1). By repeating this up to $n \geq 1$, we have

$$\begin{aligned}
Y_{\nu}^{\nu, \eta, \iota} &\geq D_{\tau_n}^{\nu, \eta, \iota, \alpha} Y_{\tau_n}^{\tau_n, X^n, i_{n-1}} - \sum_{k=1}^{n-1} D_{\tau_k}^{\nu, \eta, \iota, \alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu, \eta, \iota, \alpha}) \\
&\quad + \int_{\tau_0}^{\tau_n} D_s^{\nu, \eta, \iota, \alpha} \left(\psi(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s) - \varsigma(s, X_s^{\nu, \eta, \iota, \alpha}, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha}) \right) ds \\
&\quad - \int_{\tau_0}^{\tau_n} D_s^{\nu, \eta, \iota, \alpha} (\bar{Z}_s^{\nu, \eta, \iota, \alpha})' dW_s,
\end{aligned}$$

for all n . Since $\tau_n \rightarrow T$ $\mathbb{P}$ -a.s. and $Y^{\nu,\eta,\iota}$ is continuous, taking a limit, we have

$$\begin{aligned} Y_\nu^{\nu,\eta,\iota} &\geq D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{\nu \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha}) \\ &\quad + \int_\nu^T D_s^{\nu,\eta,\iota,\alpha} \left(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \bar{Z}_s^{\nu,\eta,\iota,\alpha}) \right) ds \\ &\quad - \int_\nu^T D_s^{\nu,\eta,\iota,\alpha} (\bar{Z}_s^{\nu,\eta,\iota,\alpha})' dW_s. \end{aligned}$$

Similarly to the above, we have

$$\begin{aligned} D_t^{\nu,\eta,\iota,\alpha} Y_t^{\nu,\eta,\iota} &\geq D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha}) \\ &\quad + \int_t^T D_s^{\nu,\eta,\iota,\alpha} \left(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \bar{Z}_s^{\nu,\eta,\iota,\alpha}) \right) ds \\ &\quad - \int_t^T D_s^{\nu,\eta,\iota,\alpha} (\bar{Z}_s^{\nu,\eta,\iota,\alpha})' dW_s, \end{aligned} \quad (5.B.5)$$

for all $t \in [\nu, T]$. On the other hand, we have

$$\begin{aligned} D_t^{\nu,\eta,\iota,\alpha} Y_t^{\nu,\eta,\iota,\alpha} &= D_T^{\nu,\eta,\iota,\alpha} g(X_T^{\nu,\eta,\iota,\alpha}, \alpha_T) - \sum_{t \leq \tau_k \leq T} D_{\tau_k}^{\nu,\eta,\iota,\alpha} c_{i_{k-1}, i_k}(\tau_k, X_{\tau_k}^{\nu,\eta,\iota,\alpha}) \\ &\quad + \int_t^T D_s^{\nu,\eta,\iota,\alpha} \left(\psi(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha}) \right) ds \\ &\quad - \int_t^T D_s^{\nu,\eta,\iota,\alpha} (Z_s^{\nu,\eta,\iota,\alpha})' dW_s, \end{aligned}$$

for all $t \in [\nu, T]$. Hence, it holds that

$$\begin{aligned} &D_t^{\nu,\eta,\iota,\alpha} \left(Y_t^{\nu,\eta,\iota} - Y_t^{\nu,\eta,\iota,\alpha} \right) \\ &\geq - \int_t^T D_s^{\nu,\eta,\iota,\alpha} \left(\varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \bar{Z}_s^{\nu,\eta,\iota,\alpha}) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha}) \right) ds \\ &\quad - \int_t^T D_s^{\nu,\eta,\iota,\alpha} (\bar{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})' dW_s \\ &= \int_t^T D_s^{\nu,\eta,\iota,\alpha} \Delta'_s (\bar{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha}) ds - \int_t^T D_s^{\nu,\eta,\iota,\alpha} (\bar{Z}_s^{\nu,\eta,\iota,\alpha} - Z_s^{\nu,\eta,\iota,\alpha})' dW_s, \end{aligned} \quad (5.B.6)$$

where $(\Delta_s)_{\nu \leq s \leq T}$ is a d -dimensional adapted process as follows: Now, we denote by $x_{i,s}$ the i th component of a random vector process $(x_u)_{u \geq 0}$ at time s .

Let $\bar{Z}_s^{\nu,\eta,\iota,\alpha, i} = (\bar{Z}_{1,s}^{\nu,\eta,\iota,\alpha}, \dots, \bar{Z}_{i-1,s}^{\nu,\eta,\iota,\alpha}, \bar{Z}_{i,s}^{\nu,\eta,\iota,\alpha}, Z_{i+1,s}^{\nu,\eta,\iota,\alpha}, \dots, Z_{d,s}^{\nu,\eta,\iota,\alpha})'$ and

let $Z_s^{\nu,\eta,\iota,\alpha, i} = (\bar{Z}_{1,s}^{\nu,\eta,\iota,\alpha}, \dots, \bar{Z}_{i-1,s}^{\nu,\eta,\iota,\alpha}, Z_{i,s}^{\nu,\eta,\iota,\alpha}, Z_{i+1,s}^{\nu,\eta,\iota,\alpha}, \dots, Z_{d,s}^{\nu,\eta,\iota,\alpha})'$. $\Delta_{i,s}$ is

$$\Delta_{i,s} = - \frac{\varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, \bar{Z}_s^{\nu,\eta,\iota,\alpha, i}) - \varsigma(s, X_s^{\nu,\eta,\iota,\alpha}, \alpha_s, Z_s^{\nu,\eta,\iota,\alpha, i})}{\bar{Z}_{i,s}^{\nu,\eta,\iota,\alpha} - Z_{i,s}^{\nu,\eta,\iota,\alpha}},$$

if $\bar{Z}_{i,s}^{\nu,\eta,\ell,\alpha} \neq Z_{i,s}^{\nu,\eta,\ell,\alpha}$ and $\Delta_{i,s} = 0$ otherwise. Then, $(\Delta_s)_{\nu \leq s \leq T}$ is uniformly bounded since $z \rightarrow \varsigma(s, X_s^{\nu,\eta,\ell,\alpha}, \alpha_s, z)$ is uniformly Lipschitz for all s . This implies that the following process,

$$\zeta_s^\Delta = \exp \left\{ \int_\nu^s \Delta'_u dW_u - \frac{1}{2} \int_\nu^s \|\Delta_u\|^2 du \right\}, \quad s \geq \nu$$

is a martingale. Hence, we can define a new probability measure such that

$$\mathbb{P}_T^\Delta(A) := \mathbb{E}[\mathbf{1}_A \zeta_T^\Delta], \quad A \in \mathcal{F}_T.$$

Furthermore, by the Girsanov theorem, the following process,

$$W_t^\Delta := \int_\nu^t \Delta_s ds - W_t, \quad t \in [\nu, T],$$

is a d -dimensional Brownian motion under $\mathbb{P}_T^\Delta$. We denote by $\mathbb{E}_T^\Delta$ an expectation operator under $\mathbb{P}_T^\Delta$. Since $\bar{Z}^{\nu,\eta,\ell,\alpha}$ and $Z^{\nu,\eta,\ell,\alpha}$ are in $\mathbb{H}_d^2[\nu, T]$, it holds that

$$\mathbb{E}_T^\Delta \left[\int_\nu^T (D_s^{\nu,\eta,\ell,\alpha})^2 \|\bar{Z}_s^{\nu,\eta,\ell,\alpha} - Z_s^{\nu,\eta,\ell,\alpha}\|^2 ds \right] < \infty.$$

This implies that the stochastic integral

$$\int_\nu^u D_s^{\nu,\eta,\ell,\alpha} (\bar{Z}_s^{\nu,\eta,\ell,\alpha} - Z_s^{\nu,\eta,\ell,\alpha})' dW_s^\Delta, \quad u \in [\nu, T],$$

is a martingale under $\mathbb{P}_T^\Delta$. Hence, taking conditional expectation of the inequality (5.B.6) under the probability measure $\mathbb{P}_T^\Delta$ given by $\mathcal{F}_t$, we obtain

$$Y_t^{\nu,\eta,\ell} - Y_t^{\nu,\eta,\ell,\alpha} \geq 0,$$

$\mathbb{P}$ -almost surely for all $t \in [\nu, T]$.

Step.2 Optimality of Y . We first prove the admissibility of α^* . Let $\bar{Z}_s^{\nu,\eta,\ell,\alpha^*}$ be a stochastic process defined as (5.B.2). Then, by the definition α^* , $K_s^{\tau_{k-1}^*, X_{\tau_{k-1}^*}^*, i_{k-1}^*} = 0$ for all $k \geq 1$ and $s \in [\tau_{k-1}^*, \tau_k^*]$. Furthermore, it holds that

$$Y_{\tau_k^*}^{\tau_{k-1}^*, X_{\tau_{k-1}^*}^*, i_{k-1}^*} = Y_{\tau_k^*}^{\tau_{k-1}^*, X_{\tau_{k-1}^*}^*, i_k^*} - c_{i_{k-1}^*, i_k^*}(\tau_k^*, X_{\tau_k^*}^*),$$

for all $k \geq 1$. Hence, the following equality holds.

$$\begin{aligned} D_t^{\nu,\eta,\ell,\alpha^*} Y_t^{\nu,\eta,\ell} &= D_{t \vee \tau_n^*}^{\nu,\eta,\ell,\alpha^*} Y_{t \vee \tau_n^*}^{\tau_n^*, X_{\tau_n^*}^*, i_{n-1}^*} - \sum_{k=1}^n D_{\tau_k^*}^{\nu,\eta,\ell,\alpha^*} c_{i_{k-1}^*, i_k^*}(\tau_k^*, X_{\tau_k^*}^*) \mathbf{1}_{[\nu, \tau_k^*]}(t) \\ &\quad + \int_t^{t \vee \tau_n^*} D_s^{\nu,\eta,\ell,\alpha^*} \left(\psi(s, X_s^*, \alpha_s^*) - \varsigma(s, X_s^*, \alpha_s, \bar{Z}_s^{\nu,\eta,\ell,\alpha^*}) \right) ds \\ &\quad - \int_t^{t \vee \tau_n^*} D_s^{\nu,\eta,\ell,\alpha^*} (\bar{Z}_s^{\nu,\eta,\ell,\alpha^*})' dW_s, \end{aligned} \tag{5.B.7}$$

for all $n \geq 1$, where $a \vee b = \max\{a, b\}$. Let $N^* = \inf\{k \mid \tau_k^* \geq T\}$ and $B = \{N^* = +\infty\}$. Suppose that $\mathbb{P}(B) > 0$. Then, as $\mathcal{I}$ is a finite set, there exists a finite loop $i_0, i_1, \dots, i_m, i_0, i_0 \in \mathcal{I}, i_0 \neq i_1$ such that

$$Y_{\tau_{k_{q+l}}^*}^{\nu, \eta, i_{l-1}} = Y_{\tau_{k_{q+l}}^*}^{\nu, \eta, i_l} - c_{i_{l-1}, i_l}(\tau_{k_{q+l}}^*, X_{\tau_{k_{q+l}}^*}^*) \text{ on } B,$$

for all $l = 1, \dots, m+1$, $q \geq 0$ and $i_{m+1} = i_0$, where $(\tau_{k_q}^*)_{q \geq 1}$ is a subsequence of $(\tau_k^*)_{k \geq 0}$.

Let $\bar{\tau} = \lim_{q \rightarrow \infty} \tau_{k_q}^*$. Then $\bar{\tau} < T$ on B and

$$Y_{\bar{\tau}}^{\nu, \eta, i_{l-1}} = Y_{\bar{\tau}}^{\nu, \eta, i_l} - c_{i_{l-1}, i_l}(\bar{\tau}, X_{\bar{\tau}}^*) \text{ on } B,$$

for all $l = 1, \dots, m+1$. This implies that

$$\sum_{l=1}^{m+1} c_{i_{l-1}, i_l}(\bar{\tau}, X_{\bar{\tau}}^*) = 0 \text{ on } B,$$

which is contradiction to Assumption 5.4.3. Therefore, $\mathbb{P}(B) = 0$ and N^* is finite $\mathbb{P}$ -almost surely. Hence, taking the limit of (5.B.7), we have

$$\begin{aligned} D_t^{\nu, \eta, \iota, \alpha^*} Y_t^{\nu, \eta, \iota} &= D_T^{\nu, \eta, \iota, \alpha^*} g(X_T^*, \alpha_T^*) - \sum_{t \leq \tau_k^* \leq T} D_{\tau_k^*}^{\nu, \eta, \iota, \alpha^*} c_{i_{k-1}^*, i_k^*}(\tau_k^*, X_{\tau_k^*}^*) \\ &\quad + \int_t^T D_s^{\nu, \eta, \iota, \alpha^*} \left(\psi(s, X_s^*, \alpha_s^*) - \varsigma(s, X_s^*, \alpha_s, \bar{Z}_s^{\nu, \eta, \iota, \alpha^*}) \right) ds \\ &\quad - \int_t^T D_s^{\nu, \eta, \iota, \alpha^*} (\bar{Z}_s^{\nu, \eta, \iota, \alpha^*})' dW_s. \end{aligned} \quad (5.B.8)$$

Since $(Y^{\nu, \eta, \iota}, \bar{Z}^{\nu, \eta, \iota, \alpha^*}) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]$ and since Assumption 5.1, 5.3, 5.4 and 5.7 are satisfied, $\sum_{t \leq \tau_k^* \leq T} c_{i_{k-1}^*, i_k^*}(\tau_k^*, X_{\tau_k^*}^*)$ is quadratic integrable under $\mathbb{P}$. Hence, α^* is admissible.

We consider the solution of the BSDE (5.3.5) at $(\nu, \eta, \iota, \alpha^*)$, denoted by $(Y^{\nu, \eta, \iota, \alpha^*}, Z^{\nu, \eta, \iota, \alpha^*})$. Then, combining (5.B.8) and $(Y^{\nu, \eta, \iota, \alpha^*}, Z^{\nu, \eta, \iota, \alpha^*})$, we obtain that

$$\begin{aligned} &D_t^{\nu, \eta, \iota, \alpha^*} \left(Y_t^{\nu, \eta, \iota} - Y_t^{\nu, \eta, \iota, \alpha^*} \right) \\ &= \int_t^T D_s^{\nu, \eta, \iota, \alpha^*} \Delta'_s(\bar{Z}_s^{\nu, \eta, \iota, \alpha^*} - Z_s^{\nu, \eta, \iota, \alpha^*}) ds - \int_t^T D_s^{\nu, \eta, \iota, \alpha^*} (\bar{Z}_s^{\nu, \eta, \iota, \alpha^*} - Z_s^{\nu, \eta, \iota, \alpha^*})' dW_s, \end{aligned}$$

where $(\Delta_s)_{0 \leq s \leq T}$ is the stochastic process defined in Step.1. As well as Step.1, we conclude that

$$Y_t^{\nu, \eta, \iota} = Y_t^{\nu, \eta, \iota, \alpha^*},$$

$\mathbb{P}$ -almost surely for all $t \in [\nu, T]$. $\square$

Appendix 5.C Verification in the Infinite Horizon

Proof of Proposition 5.18. Step.1 Monotonicity of $\hat{Y}$. Fix an arbitrary $0 \leq T \leq \tilde{T}$, $\nu \in \mathcal{T}_0^T$ and $\eta \in L_\nu^{2q}(\mathbb{R}^d)$. Let $(\hat{Y}^{T,\nu,\eta,i,n}, \hat{Z}^{T,\nu,\eta,i,n}, \hat{K}^{T,\nu,\eta,i,n})_{n \geq 0}$ be the Picard's iterations of $(\hat{Y}^{T,\nu,\eta,i}, \hat{Z}^{T,\nu,\eta,i}, \hat{K}^{T,\nu,\eta,i})$ constructed in Theorem 5.8.

Also let $(\hat{Y}^{\tilde{T},\nu,\eta,i,n}, \hat{Z}^{\tilde{T},\nu,\eta,i,n}, \hat{K}^{\tilde{T},\nu,\eta,i,n})_{n \geq 0}$ be the Picard's iterations of $(\hat{Y}^{\tilde{T},\nu,\eta,i}, \hat{Z}^{\tilde{T},\nu,\eta,i}, \hat{K}^{\tilde{T},\nu,\eta,i})$ constructed in Theorem 5.8. Then, by the non-negative reward condition, temporary terminal condition and Proposition 2.2 in El Karoui et al. (1997b), we have

$$\begin{aligned} e^{-\rho T} \hat{Y}_T^{\tilde{T},\nu,\eta,i,0} &= \inf_{\theta \in \Theta[T, \tilde{T}]} \mathbb{E} \left[e^{-\rho \tilde{T}} \zeta_{\tilde{T}}^{\theta, T} g(X_{\tilde{T}}^{\nu,\eta,i}, i) \right. \\ &\quad \left. + \int_T^{\tilde{T}} e^{-\rho t} \zeta_t^{\theta, T} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_T \right] \\ &\geq \inf_{\theta \in \Theta[T, \tilde{T}]} \mathbb{E} \left[e^{-\rho \tilde{T}} \zeta_{\tilde{T}}^{\theta, T} g(X_{\tilde{T}}^{\nu,\eta,i}, i) \mid \mathcal{F}_T \right] \geq e^{-\rho T} g(X_T^{\nu,\eta,i}, i). \end{aligned}$$

Hence, $\hat{Y}_T^{\tilde{T},\nu,\eta,i,0} \geq g(X_T^{\nu,\eta,i}, i)$ for all $i \in \mathcal{I}$. On the other hand, for all $i \in \mathcal{I}$, $(\hat{Y}_t^{\tilde{T},\nu,\eta,i,0}, \hat{Z}_t^{\tilde{T},\nu,\eta,i,0})$ is the solution to the following BSDE on $[\nu, T]$,

$$\begin{aligned} -dy_t &= \left(\psi(X_t^{\nu,\eta,i}, i) - \rho y_t - \varsigma(X_t^{\nu,\eta,i}, i, z_t) \right) dt - z'_t dW_t, \\ y_T &= \hat{Y}_T^{\tilde{T},\nu,\eta,i,0}, \quad (y, z) \in \mathbb{S}^2[\nu, T] \times \mathbb{H}_d^2[\nu, T]. \end{aligned}$$

By the comparison theorem, $\hat{Y}_t^{\tilde{T},\nu,\eta,i,0} \geq \hat{Y}_t^{T,\nu,\eta,i,0}$ for all $t \in [\nu, T]$ and $i \in \mathcal{I}$. Similarly, by the non-negative reward condition, temporary terminal condition and Proposition 7.1 in El Karoui et al. (1997a), we have

$$\hat{Y}_T^{\tilde{T},\nu,\eta,i,n} \geq g(X_T^{\nu,\eta,i}, i),$$

for all $n \geq 1$. Hence, recursively applying the comparison theorem, we obtain that $\hat{Y}_t^{\tilde{T},\nu,\eta,i,n} \geq \hat{Y}_t^{T,\nu,\eta,i,n}$ for all $t \in [\nu, T]$, $i \in \mathcal{I}$ and $n \geq 1$. Taking a limit, we also have $\hat{Y}_t^{\tilde{T},\nu,\eta,i} \geq \hat{Y}_t^{T,\nu,\eta,i}$ for all $t \in [\nu, T]$ and $i \in \mathcal{I}$.

Step.2 n-step dominated. Since $T \rightarrow \hat{Y}_t^{T,\nu,\eta,i,n}$ is increasing by Step.1 and since $n \rightarrow \hat{Y}_t^{T,\nu,\eta,i,n}$ is also increasing, we can exchange the orders of taking the limits such that

$$\lim_{T \rightarrow \infty} \hat{Y}_t^{T,\nu,\eta,i} = \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \hat{Y}_t^{T,\nu,\eta,i,n} = \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} \hat{Y}_t^{T,\nu,\eta,i,n} = \lim_{n \rightarrow \infty} \hat{Y}_t^{\infty,\nu,\eta,i,n},$$

where

$$\hat{Y}_t^{\infty,\nu,\eta,i,n} = \lim_{T \rightarrow \infty} \hat{Y}_t^{T,\nu,\eta,i,n}, \quad n \geq 1.$$

By Proposition 2.2 in El Karoui et al. (1997b) and the comparison theorem, it holds that

$$e^{-\rho\nu} \hat{Y}_\nu^{T,\nu,\eta,i,0} = \inf_{\theta \in \Theta[\nu,T]} \mathbb{E} \left[\zeta_T^{\theta,\nu} e^{-\rho T} g(X_T^{\nu,\eta,i}, i) + \int_\nu^T e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_\nu \right], \quad (5.C.1)$$

for all $T \geq \nu$. Now, we choose an arbitrary $\theta \in \Theta[\nu, \infty)$. Then, by the equality (5.C.1) and the temporary terminal condition, we have

$$e^{-\rho\nu} \hat{Y}_\nu^{T,\nu,\eta,i,0} \leq \mathbb{E} \left[\int_\nu^T e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_\nu \right],$$

for all $T \geq \nu$. By the Lebesgue dominated convergence theorem, we have

$$e^{-\rho\nu} \hat{Y}_\nu^{\infty,\nu,\eta,i,0} \leq \mathbb{E} \left[\int_\nu^\infty e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_\nu \right].$$

Since θ is arbitrary, we obtain that

$$e^{-\rho\nu} \hat{Y}_\nu^{\infty,\nu,\eta,i,0} \leq \inf_{\theta \in \Theta[\nu, \infty)} \mathbb{E} \left[\int_\nu^\infty e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_\nu \right]. \quad (5.C.2)$$

Now, we assume that for some $n \geq 1$,

$$e^{-\rho\tilde{\tau}} \hat{Y}_{\tilde{\tau}}^{\infty,\tilde{\tau},\tilde{\eta},j,n-1} \leq \sup_{\alpha \in \mathbb{A}_{j,n-1}[\tilde{\tau}, \infty)} \inf_{\theta \in \Theta[\nu, \infty)} \mathbb{E} \left[\int_{\tilde{\tau}}^\infty e^{-\rho t} \zeta_t^{\theta,\tilde{\tau}} \left(\psi(X_t^{\tilde{\tau},\tilde{\eta},j,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{\tilde{\tau},\tilde{\eta},j,\alpha}, \alpha_t) \right) dt - \sum_{k=1}^{n-1} e^{-\rho\tau_k} \zeta_{\tau_k}^{\theta,\tilde{\tau}} c_{i_{k-1},i_k}(X_{\tau_k}^{\tilde{\tau},\tilde{\eta},j,\alpha}) \mid \mathcal{F}_{\tilde{\tau}} \right],$$

where $\tilde{\tau} \in \mathcal{T}_\nu$ and $\tilde{\eta} \in L_{\tilde{\tau}}^{2q}(\mathbb{R}^d)$, and $\alpha \in \mathbb{A}_{j,n-1}[\tilde{\tau}, \infty)$ is a set of the admissible controls on $[\tilde{\tau}, \infty)$ changing the regimes at most $n-1$ times. On the other hand, by Proposition

7.1 in El Karoui et al. (1997a) and the uniqueness of $\widehat{Y}$, it holds that

$$\begin{aligned} e^{-\rho\nu}\widehat{Y}_\nu^{T,\nu,\eta,i,n} &= \sup_{\tilde{\tau} \in \mathcal{T}_\tau^T} \inf_{\theta \in \Theta[\nu,T]} \mathbb{E} \left[e^{-\rho T} \zeta_T^{\theta,\nu} g(X_T^{\nu,\eta,i}, i) \mathbb{1}_{\{\tilde{\tau}=T\}} \right. \\ &\quad + e^{-\rho\tilde{\tau}} \zeta_{\tilde{\tau}}^{\theta,\nu} \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \widehat{Y}_{\tilde{\tau}}^{T,\tilde{\tau},X_{\tilde{\tau}}^{\nu,\eta,i},j,n-1} - c_{i,j}(X_{\tilde{\tau}}^{\nu,\eta,i}) \right\} \mathbb{1}_{\{\tilde{\tau}<T\}} \\ &\quad \left. + \int_\nu^{\tilde{\tau}} e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \mid \mathcal{F}_\nu \right]. \end{aligned} \quad (5.C.3)$$

Let τ^* be an optimal stopping time of the maximization problem in the right hand side of (5.C.3). Then, by Proposition 2.3 in El Karoui et al. (1997a), we have

$$\tau^* = \inf \left\{ t \in [\nu, T] \mid \widehat{Y}_t^{T,t,X_t^{\nu,\eta,i},i,n} = \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \widehat{Y}_t^{T,t,X_t^{\nu,\eta,i},j,n-1} - c_{i,j}(X_t^{\nu,\eta,i}) \right\} \right\}.$$

Hence,

$$\begin{aligned} &e^{-\rho T} \zeta_T^{\theta,\nu} g(X_T^{\nu,\eta,i}, i) \mathbb{1}_{\{\tau^*=T\}} + e^{-\rho\tau^*} \zeta_{\tau^*}^{\theta,\nu} \max_{j \in \mathcal{I} \setminus \{i\}} \left\{ \widehat{Y}_{\tau^*}^{T,\tau^*,X_{\tau^*}^{\nu,\eta,i},j,n-1} - c_{i,j}(X_{\tau^*}^{\nu,\eta,i}) \right\} \mathbb{1}_{\{\tau^*<T\}} \\ &= e^{-\rho\tau^*} \zeta_{\tau^*}^{\theta,\nu} \left(\widehat{Y}_{\tau^*}^{T,\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,n-1} - c_{i,j^*}(X_{\tau^*}^{\nu,\eta,i}) \mathbb{1}_{\{\tau^*<T\}} \right), \end{aligned}$$

where j^* satisfies

$$\widehat{Y}_{\tau^*}^{T,\tau^*,X_{\tau^*}^{\nu,\eta,i},i,n} = \widehat{Y}_{\tau^*}^{T,\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,n-1} - c_{i,j^*}(X_{\tau^*}^{\nu,\eta,i}),$$

if $\tau^* < T$, and $j^* = i$ otherwise. By the monotonicity of $\widehat{Y}$, we have

$$\widehat{Y}_{\tau^*}^{T,\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,n-1} \leq \widehat{Y}_{\tau^*}^{\infty,\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,n-1}.$$

Hence, we obtain

$$\begin{aligned}
e^{-\rho\nu}\widehat{Y}_\nu^{T,\nu,\eta,i,n} &\leq \inf_{\theta \in \Theta[0,T]} \mathbb{E} \left[\int_\nu^{\tau^*} e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \right. \\
&\quad \left. + e^{-\rho\tau^*} \zeta_{\tau^*}^{\theta,\nu} \widehat{Y}_{\tau^*}^{\infty,\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,n-1} - e^{-\rho\tau^*} \zeta_{\tau^*}^{\theta,\nu} c_{i,j^*}(X_{\tau^*}^{\nu,\eta,i}) \mathbb{1}_{\{\tau^* < T\}} \mid \mathcal{F}_\nu \right] \\
&\leq \inf_{\theta \in \Theta[\nu,\tau^*)} \mathbb{E} \left[\int_\nu^{\tau^*} e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i}, i) - \theta'_t \phi(X_t^{\nu,\eta,i}, i) \right) dt \right. \\
&\quad - e^{-\rho\tau^*} \zeta_{\tau^*}^{\theta,\nu} c_{i,j^*}(X_{\tau^*}^{\nu,\eta,i}) \mathbb{1}_{\{\tau^* < T\}} \\
&\quad + \zeta_{\tau^*}^{\theta,\nu} \sup_{\alpha \in \mathbb{A}_{j^*,n-1}[\tau^*,\infty)} \inf_{\theta \in \Theta[\tau^*,\infty)} \mathbb{E} \left[\int_{\tau^*}^\infty e^{-\rho t} \zeta_t^{\theta,\tau^*} \left(\psi(X_t^{\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,\alpha}, \alpha_t) \right. \right. \\
&\quad \left. \left. - \theta'_t \phi(X_t^{\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,\alpha}, \alpha_t) \right) dt \right. \\
&\quad \left. \left. - \sum_{k=1}^{n-1} e^{-\rho\tau_k} \zeta_{\tau_k}^{\theta,\tau^*} c_{i_{k-1},i_k}(X_{\tau_k}^{\tau^*,X_{\tau^*}^{\nu,\eta,i},j^*,\alpha}) \mid \mathcal{F}_{\tau^*} \right] \mid \mathcal{F}_\nu \right] \\
&\leq \sup_{\alpha \in \mathbb{A}_{i,n}[\nu,\infty)} \inf_{\theta \in \Theta[\nu,\infty)} \mathbb{E} \left[\int_\nu^\infty e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{\nu,\eta,i,\alpha}, \alpha_t) \right) dt \right. \\
&\quad \left. - \sum_{k=1}^n e^{-\rho\tau_k} \zeta_{\tau_k}^{\theta,\nu} c_{i_{k-1},i_k}(X_{\tau_k}^{\nu,\eta,i,\alpha}) \mid \mathcal{F}_\nu \right],
\end{aligned}$$

where we used the uniqueness of the strong solution of X . Taking a limit, we have

$$\begin{aligned}
e^{-\rho\nu}\widehat{Y}_\nu^{\infty,\nu,\eta,i,n} &\leq \sup_{\alpha \in \mathbb{A}_{i,n}[\nu,\infty)} \inf_{\theta \in \Theta[\nu,\infty)} \mathbb{E} \left[\int_\nu^\infty e^{-\rho t} \zeta_t^{\theta,\nu} \left(\psi(X_t^{\nu,\eta,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{\nu,\eta,i,\alpha}, \alpha_t) \right) dt \right. \\
&\quad \left. - \sum_{k=1}^n e^{-\rho\tau_k} \zeta_{\tau_k}^{\theta,\nu} c_{i_{k-1},i_k}(X_{\tau_k}^{\nu,\eta,i,\alpha}) \mid \mathcal{F}_\nu \right]. \tag{5.C.4}
\end{aligned}$$

By the inequalities (5.C.2) and (5.C.4), we can prove that the inequality (5.C.4) holds for all $n \geq 1$ using the induction method. Since $\mathbb{A}_{i,n}[t, \infty) \subseteq \mathbb{A}_i[t, \infty)$ for all $n \geq 1$, the inequality (5.C.4) leads to

$$\lim_{T \rightarrow \infty} \widehat{Y}_t^{T,t,x,i} = \lim_{n \rightarrow \infty} \widehat{Y}_t^{\infty,t,x,i,n} \leq v^\infty(x, i), \tag{5.C.5}$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$. By the monotonicity of $\widehat{Y}$ and the inequality (5.C.5), we have

$$\widehat{Y}_t^{T,t,x,i} \leq v^\infty(x, i), \tag{5.C.6}$$

for all $(t, T, x, i) \in [0, \infty)^2 \times \mathbb{R}^d \times \mathcal{I}$.

Step.3 Convergence. To prove the opposite inequality of (5.C.5), we use the ϵ -optimal argument such as Corollary 2.1 in Bayraktar and Egami (2010). Fix any

$(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$. Let $J^T(t, x, i, \alpha)$ be an objective function in the finite horizon $[0, T]$. Then, by the time-homogeneity, we have

$$Y_t^{T,t,x,i} = Y_0^{T-t,0,x,i} \geq J^{T-t}(0, x, i, \alpha),$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$, $i \in \mathcal{I}$ and $\alpha \in \mathbb{A}_i[0, T-t]$. Now, we fix an arbitrary $t \geq 0$ and $x \in \mathbb{R}^d$. For any $\epsilon > 0$, we choose a control $\alpha^\epsilon = (\tau_k^\epsilon, i_k^\epsilon)_{k \geq 0} \in \mathbb{A}_i[0, \infty)$ such that

$$J(x, i, \alpha^\epsilon) \geq v^\infty(x, i) - \epsilon.$$

For all $T \geq t$, define

$$\alpha_s^{\epsilon, T-t} := \alpha_s, \quad s \in [0, T-t].$$

Then, $\alpha^{\epsilon, T-t} \in \mathbb{A}_i[0, T-t]$ for all $T \geq t$. For all $T \geq t$, let

$$\begin{aligned} \theta^{T-t} := & \arg \inf_{\theta \in \Theta[0, T-t]} \mathbb{E} \left[\int_0^{T-t} e^{-\rho s} \zeta_s^{\theta, 0} \left(\psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) - \theta'_s \phi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) \right) ds \right. \\ & \left. - \sum_{k=1}^{\infty} e^{-\rho \tau_k^\epsilon} \zeta_{\tau_k^\epsilon}^{\theta, 0} c_{i_{k-1}^\epsilon, i_k^\epsilon} (X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon}) \mathbb{1}_{\{\tau_k < T-t\}} + e^{-\rho(T-t)} \zeta_{T-t}^{\theta, 0} g(X_{T-t}^{0,x,i,\alpha^\epsilon}, \alpha_{T-t}^\epsilon) \right]. \end{aligned}$$

Also let

$$\theta_s^{\infty, T-t} := \begin{cases} \theta_s^{T-t}, & \text{if } s < T-t, \\ 0, & \text{otherwise,} \end{cases}$$

for all $T \geq t$. It is easy to check $\theta^{\infty, T-t} \in \Theta[0, \infty)$. Then, we have

$$\begin{aligned} & J(x, i, \alpha^\epsilon) \\ & \leq \mathbb{E} \left[\int_0^\infty e^{-\rho s} \zeta_s^{\theta^{\infty, T-t}, 0} \left(\psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) - (\theta_s^{\infty, T-t})' \phi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) \right) ds \right. \\ & \quad \left. - \sum_{k=1}^{\infty} e^{-\rho \tau_k^\epsilon} \zeta_{\tau_k^\epsilon}^{\theta^{\infty, T-t}, 0} c_{i_{k-1}^\epsilon, i_k^\epsilon} (X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon}) \right] \\ & = J^{T-t}(0, x, i, \alpha^{\epsilon, T-t}) + \mathbb{E} \left[\zeta_{T-t}^{\theta^{\infty, T-t}, 0} \int_{T-t}^\infty e^{-\rho s} \psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) ds \right. \\ & \quad \left. - \zeta_{T-t}^{\theta^{\infty, T-t}, 0} \sum_{k=1}^{\infty} e^{-\rho \tau_k^\epsilon} c_{i_{k-1}^\epsilon, i_k^\epsilon} (X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon}) \mathbb{1}_{\{\tau_k^\epsilon > T-t\}} - e^{-\rho(T-t)} \zeta_{T-t}^{\theta^{\infty, T-t}, 0} g(X_{T-t}^{0,x,i,\alpha^\epsilon}, \alpha_{T-t}^\epsilon) \right], \end{aligned}$$

for all $T \geq t$. By the polynomial growth condition and the strong triangular condition, we have

$$\begin{aligned} & \mathbb{E} \left[\int_{T-t}^\infty e^{-\rho s} \psi(X_s^{0,x,i,\alpha^\epsilon}, \alpha_s^\epsilon) ds - \sum_{k=1}^{\infty} e^{-\rho \tau_k^\epsilon} c_{i_{k-1}^\epsilon, i_k^\epsilon} (X_{\tau_k^\epsilon}^{0,x,i,\alpha^\epsilon}) \mathbb{1}_{\{\tau_k^\epsilon > T-t\}} \mid \mathcal{F}_{T-t} \right] \\ & \leq C_1(1 + \|X_{T-t}^{0,x,i,\alpha^\epsilon}\|^q) e^{-\rho(T-t)}, \end{aligned}$$

for all $T \geq t$, where C_1 is a positive constant not depending on T, t and x . Thus, we have

$$\begin{aligned} & \mathbb{E} \left[\zeta_{T-t}^{\theta^\infty, T-t, 0} \int_{T-t}^{\infty} e^{-\rho s} \psi(X_s^{0, x, i, \alpha^\epsilon}, \alpha_s^\epsilon) ds \right. \\ & \quad \left. - \zeta_{T-t}^{\theta^\infty, T-t, 0} \sum_{k=1}^{\infty} e^{-\rho \tau_k^\epsilon} c_{i_{k-1}^\epsilon, i_k^\epsilon}(X_{\tau_k^\epsilon}^{0, x, i, \alpha^\epsilon}) \mathbb{1}_{\{\tau_k^\epsilon > T-t\}} - e^{-\rho(T-t)} \zeta_{T-t}^{\theta^\infty, T-t, 0} g(X_{T-t}^{0, x, i, \alpha^\epsilon}, \alpha_{T-t}^\epsilon) \right] \\ & \leq C_2 \mathbb{E} \left[\zeta_{T-t}^{\theta^\infty, T-t, 0} \left(1 + \|X_{T-t}^{0, x, i, \alpha^\epsilon}\|^q \right) e^{-\rho(T-t)} \right] \\ & \leq C_3 (1 + \|x\|^q) e^{-c_\infty(T-t)}, \end{aligned}$$

for all $T \geq t$, where C_2, C_3 and c_∞ are positive constants not depending on T, t and x .

This implies that for sufficiently large $\tilde{T}$, it holds that

$$J(x, i, \alpha^\epsilon) \leq J^{T-t}(0, x, i, \alpha^{\epsilon, T-t}) + C_3(1 + \|x\|^q) e^{-c_\infty(T-t)} \leq J^{T-t}(0, x, i, \alpha^{\epsilon, T-t}) + \epsilon, \quad (5.C.7)$$

for all $T \geq \tilde{T}$. Hence, we have

$$\liminf_{T \rightarrow \infty} Y_t^{T, t, x, i} \geq \liminf_{T \rightarrow \infty} J^{T-t}(0, x, i, \alpha^{\epsilon, T-t}) \geq J(x, i, \alpha^\epsilon) - \epsilon \geq v^\infty(x, i) - 2\epsilon.$$

Since ϵ is arbitrarily chosen, we obtain

$$\liminf_{T \rightarrow \infty} \hat{Y}_t^{T, t, x, i} \geq v^\infty(x, i), \quad (5.C.8)$$

for all $(t, x, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{I}$. Thus, we obtain the desired equality (5.5.13). For all $i \in \mathcal{I}$, the convergence of (5.C.8) is locally uniform with respect to t and x by the inequality (5.C.7). Furthermore, $Y_t^{T, t, x, i}$ is continuous in t and x for all $T \geq 0$ and $i \in \mathcal{I}$. Therefore, $v^\infty(x, i)$ is continuous in x for all $i \in \mathcal{I}$. $\square$

Appendix 5.D A Viscosity Solution in the Infinite Horizon

Proof of Proposition 5.19. Let $v^T(t, x, i) = \hat{Y}_t^{T, t, x, i}$ for $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. By the definition, we have

$$v^T(t, x, i) \geq \max_{j \in \mathcal{I} \setminus \{i\}} \{v^T(t, x, j) - c_{i, j}(x)\},$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Hence, taking a limit, we have

$$v^\infty(x, i) \geq \max_{j \in \mathcal{I} \setminus \{i\}} \{v^\infty(x, j) - c_{i,j}(x)\},$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$.

Furthermore, we can show the followings.

Lemma 5.24 For all $T > 0$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$, $v^T(\cdot, x, i)$ is non-increasing. Furthermore, there exists a positive constant C such that

$$|v^T(t, x, i) - v^T(s, x, i)| \leq C(1 + \|x\|^q), \quad (5.D.1)$$

for all $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$.

We will show Lemma 5.24 after the proof of Proposition 5.19.

Let $C^2(\mathbb{R}^d)$ be a set of twice continuously differentiable functions from $\mathbb{R}^d$ onto $\mathbb{R}$. Let $B(x) = \{y \in \mathbb{R}^d \mid \|y - x\| \leq 1\}$ be a unit ball on $\mathbb{R}^d$ centered on x . Now, let us show the viscosity solution property of v^∞ .

Step.1 Viscosity subsolution. We arbitrarily choose $\varphi \in C^2(\mathbb{R}^d)$ and $\bar{x} \in \mathbb{R}^d$ such that $\max\{v^\infty(\cdot, i) - \varphi\} = v^\infty(\bar{x}, i) - \varphi(\bar{x}) = 0$. Let

$$\widehat{\varphi}(x) := \varphi(x) + \|x - \bar{x}\|^4.$$

Let $(t_k, x_k) \in [0, k] \times B(\bar{x})$ for all $k = 1, 2, 3, \dots$ such that

$$\max\{v^k(\cdot, \cdot, i) - \widehat{\varphi}\} = v^k(t_k, x_k, i) - \widehat{\varphi}(x_k).$$

Since $v^k(\cdot, x, i)$ is non-increasing for all $x \in \mathbb{R}^d$ by Lemma 5.24, we have $t_k = 0$ for all k . We choose a subsequence of $(x_k)_{k \geq 1}$ which converges to some $x_0 \in \mathbb{R}^d$. For convenience, we also denote this subsequence by $(x_k)_{k \geq 1}$. Then, since $(x_k)_{k \geq 1} \subseteq B(\bar{x})$, the Dini theorem leads to

$$\lim_{k \rightarrow \infty} v^k(0, x_k, i) = v^\infty(x_0, i).$$

Thus, we have

$$\begin{aligned} 0 &\leq v^\infty(\bar{x}, i) - \varphi(\bar{x}) - (v^\infty(x_0, i) - \varphi(x_0)) \\ &\leq \lim_{k \rightarrow \infty} \left(v^k(0, \bar{x}, i) - \widehat{\varphi}(\bar{x}) - (v^k(0, x_k, i) - \widehat{\varphi}(x_k)) - \|x_k - \bar{x}\|^4 \right) \\ &\leq \lim_{k \rightarrow \infty} \left(-\|x_k - \bar{x}\|^4 \right) = -\|x_0 - \bar{x}\|^4. \end{aligned}$$

Hence, $x_0 = \bar{x}$.

Now, by Proposition 5.13, for all $k \geq 1$, we have

$$\begin{aligned} 0 &\geq -\frac{\partial \widehat{\varphi}(x_k)}{\partial t} - \mathcal{L}^i \widehat{\varphi}(x_k) - \psi(x_k, i) + \rho v^k(0, x_k, i) + \varsigma(x_k, i, \sigma'(x_k, i) \nabla \widehat{\varphi}(x_k)) \\ &= -\mathcal{L}^i \widehat{\varphi}(x_k) - \psi(x_k, i) + \rho v^k(0, x_k, i) + \varsigma(x_k, i, \sigma'(x_k, i) \nabla \widehat{\varphi}(x_k)). \end{aligned}$$

Hence, by the Dini theorem, taking a limit of the above inequality, we have

$$0 \geq -\mathcal{L}^i \varphi(\bar{x}) - \psi(\bar{x}, i) + \rho v^\infty(\bar{x}, i) + \varsigma(\bar{x}, i, \sigma'(\bar{x}, i) \nabla \varphi(\bar{x})).$$

This implies that v^∞ is a viscosity subsolution of the PDE (5.5.14).

Step.2 Viscosity supersolution. We arbitrarily choose $\varphi \in C^2(\mathbb{R}^d)$ and $\underline{x} \in \mathbb{R}^d$ such that $\min\{v^\infty(\cdot, i) - \varphi\} = v^\infty(\underline{x}, i) - \varphi(\underline{x}) = 0$. For $m = 1, 2, 3, \dots$, let

$$\varphi_m(t, x) := \varphi(x) - \|x - \underline{x}\|^4 - \frac{t}{m}$$

Now, fix an arbitrary m temporarily. Let $(t_k, x_k) \in [0, k] \times B(\underline{x})$ for all $k = 1, 2, 3, \dots$ such that

$$\min\{v^k(\cdot, \cdot, i) - \varphi_m\} = v^k(t_k, x_k, i) - \varphi_m(t_k, x_k).$$

For any $k \geq 1$, $t \in [0, k]$ and $x \in B(\underline{x})$, by Lemma 5.24, we have

$$\begin{aligned} v^k(0, x, i) - \varphi_m(0, x) - (v^k(t, x, i) - \varphi_m(t, x)) &\leq -\frac{t}{m} + C(1 + \|x\|^q) \\ &\leq -\frac{t}{m} + C(1 + \max_{y \in B(\underline{x})} \|y\|^q). \end{aligned}$$

We now suppose that

$$t > mC(1 + \max_{y \in B(\underline{x})} \|y\|^q). \quad (5.D.2)$$

Then,

$$\begin{aligned} v^k(0, x, i) - \varphi_m(0, x) - (v^k(t, x, i) - \varphi_m(t, x)) &\leq -\frac{t}{m} + C(1 + \max_{y \in B(\underline{x})} \|y\|^q) \\ &< 0, \end{aligned}$$

for all t satisfying the inequality (5.D.2). This implies that for sufficient large $\widetilde{k}$, all t_k with $k \geq \widetilde{k}$ are in the following compact subset.

$$\left[0, mC(1 + \max_{y \in B(\underline{x})} \|y\|^q)\right].$$

Now, we choose a subsequence of $(t_k, x_k)_{k \geq \tilde{k}}$ converging some (t_0, x_0) . We also write this subsequence as $(t_k, x_k)_{k \geq 1}$ for convenience. Then, by the Dini theorem, we have

$$\lim_{k \rightarrow \infty} v^k(t_k, x_k, i) = v^\infty(x_0, i).$$

Hence, we have

$$\begin{aligned} 0 &\leq v^\infty(x_0, i) - \varphi(x_0) - (v^\infty(\underline{x}, i) - \varphi(\underline{x})) \\ &\leq \lim_{k \rightarrow \infty} \left(v^k(t_k, x_k, i) - \varphi_m(t_k, x_k) - (v^k(t_k, \underline{x}, i) - \varphi_m(t_k, \underline{x})) - \|x_k - \underline{x}\|^4 \right) \\ &\leq \lim_{k \rightarrow \infty} \left(-\|x_k - \underline{x}\|^4 \right) = -\|x_0 - \underline{x}\|^4, \end{aligned}$$

so $x_0 = \underline{x}$.

Now, by Proposition 5.13, we have

$$\begin{aligned} 0 &\leq -\frac{\partial \varphi_m(t_k, x_k)}{\partial t} - \mathcal{L}^i \varphi_m(t_k, x_k) - \psi(x_k, i) + \rho v^k(t_k, x_k, i) \\ &\quad + \varsigma(x_k, i, \sigma'(x_k, i) \nabla \varphi_m(t_k, x_k)) \\ &= \frac{1}{m} - \mathcal{L}^i \varphi_m(t_k, x_k) - \psi(x_k, i) + \rho v^k(t_k, x_k, i) + \varsigma(x_k, i, \sigma'(x_k, i) \nabla \varphi_m(t_k, x_k)), \end{aligned}$$

for all $k \geq 1$. Thus, by the Dini theorem, taking a limit with respect to k , we have

$$0 \leq \frac{1}{m} - \mathcal{L}^i \varphi(\underline{x}) - \psi(\underline{x}, i) + \rho v^\infty(\underline{x}, i) + \varsigma(\underline{x}, i, \sigma'(\underline{x}, i) \nabla \varphi(\underline{x})).$$

Since m is arbitrarily chosen, tending m to infinity, we have

$$0 \leq -\mathcal{L}^i \varphi(\underline{x}) - \psi(\underline{x}, i) + \rho v^\infty(\underline{x}, i) + \varsigma(\underline{x}, i, \sigma'(\underline{x}, i) \nabla \varphi(\underline{x})).$$

This implies that v^∞ is a viscosity supersolution of the PDE (5.5.14). $\square$

Proof of Lemma 5.24. For all $0 \leq h \leq t \leq T$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$, we have

$$\begin{aligned} v^T(t, x, i) &= \widehat{Y}_t^{T, t, x, i} = \widehat{Y}_{t-h}^{T-h, t-h, x, i} \quad (\text{time-homogeneous Markov property}) \\ &\leq \widehat{Y}_{t-h}^{T, t-h, x, i} \quad (\text{monotonicity of } \widehat{Y}) \\ &= v^T(t-h, x, i). \end{aligned}$$

Hence, $v^T(\cdot, x, i)$ is non-increasing for all $T > 0$, $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$.

Now, we prove the inequality (5.D.1). Since $t \rightarrow v^T(t, x, i)$ is non-increasing for all T, x and i , it suffices to derive an upper boundary of $v^T(0, x, i) - v^T(T, x, i)$. Then, by the polynomial growth conditions for ϕ, ψ, g , and c and Propositions 5.2 and 5.5, it is easy to show that

$$\begin{aligned}
v^T(0, x, i) - v^T(T, x, i) &= \widehat{Y}_0^{T,0,x,i} - g(x, i) \\
&\leq \sup_{\alpha \in \mathbb{A}_i[0,T]} \inf_{\theta \in \Theta[0,T]} \mathbb{E} \left[\int_0^T \zeta_t^{\theta,0} e^{-\rho t} \left(\psi(X_t^{0,x,i,\alpha}, \alpha_t) - \theta'_t \phi(X_t^{0,x,i,\alpha}, \alpha_t) \right) dt \right. \\
&\quad \left. + \zeta_T^{\theta,0} e^{-\rho T} g(X_T^{0,x,i,\alpha}, i) - \sum_{0 \leq \tau_k \leq T} \zeta_{\tau_k}^{\theta,0} e^{-\rho \tau_k} c_{i_{k-1}, i_k}(X_{\tau_k}^{0,x,i,\alpha}) \right] - g(x, i) \\
&\leq \sup_{\alpha \in \mathbb{A}_i[0,T]} \mathbb{E} \left[e^{-\rho T} g(X_T^{0,x,i,\alpha}, i) - g(x, i) \right. \\
&\quad \left. + \int_0^T e^{-\rho t} \psi(X_t^{0,x,i,\alpha}, \alpha_t) dt - \sum_{0 \leq \tau_k \leq T} e^{-\rho \tau_k} c_{i_{k-1}, i_k}(X_{\tau_k}^{0,x,i,\alpha}) \right] \\
&\leq C(1 + \|x\|^q),
\end{aligned}$$

where C is a positive constant not depending on T and x . □

Appendix 5.E Figures

Figure 5.1. The Optimal Switching Thresholds in the Selection of Investment Funds. The vertical axis is a logarithmic scale. $\underline{x}_1^*$ under different κ_2 is plotted in the solid line. $\bar{x}_2^*$ under different κ_2 is plotted in the dashed line.

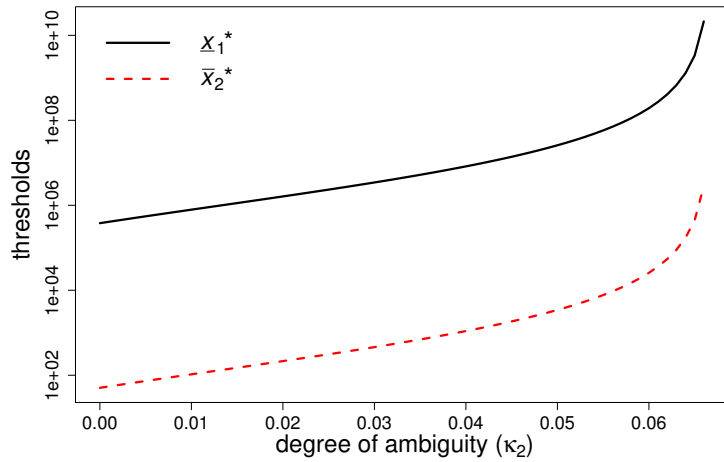


Figure 5.2. The Optimal Switching Thresholds in the Buy Low and Sell High Problem. x_1 under different κ is plotted in the solid line. x_2 under different κ is plotted in the dashed line.

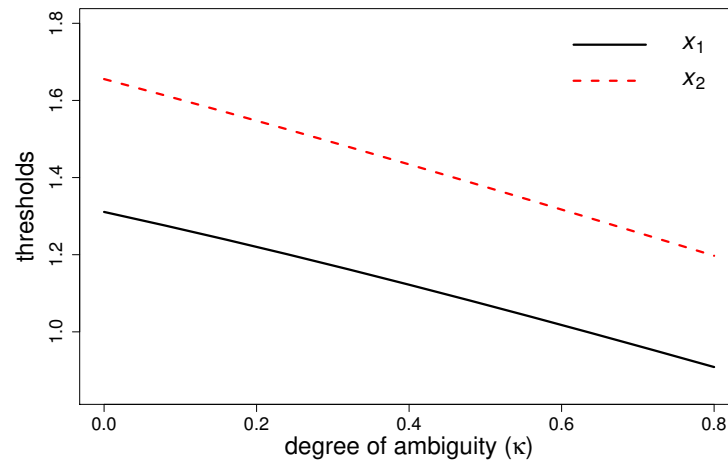
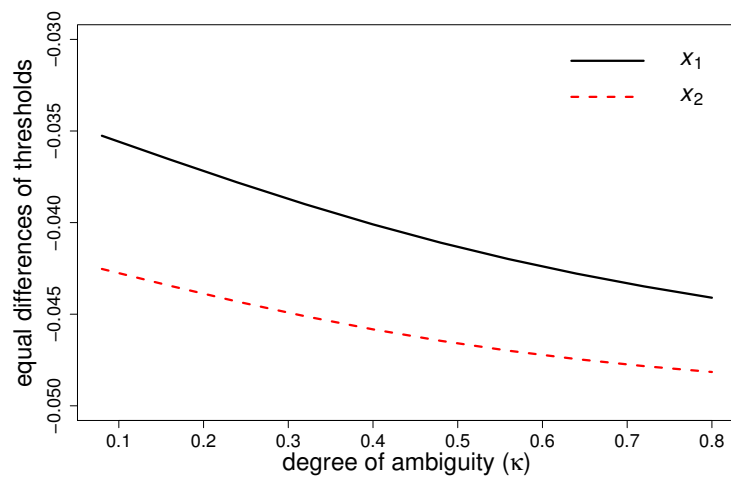


Figure 5.3. The Equal Differences of the Optimal Switching Thresholds in the Buy Low and Sell High Problem. x_1 under different κ is plotted in the solid line. x_2 under different κ is plotted in the dashed line. Each interval of κ is 0.08.



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