

Some Two-Person Zero-Sum Dynamic Game

by

Yutaka Kimura¹ and Kensuke Tanaka²

1 A Two-person Zero-Sum Dynamic Game with a Parameter

We give a two-person zero-sum dynamic game with a parameter (DPG_θ) by a sequence of the following objects

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta; n \in N) \quad (1.1)$$

where

1. S_n is the state space at time $n \in N$ and is assumed to be a Borel space, that is, a nonempty Borel subset of a complete separable metric space.
2. A_n and B_n are the action spaces at time $n \in N$ of players I and II, respectively. It is assumed that A_n and B_n are Borel spaces.
3. $\{t_{n+1}\}$ is the law of motion of the system; t_{n+1} is a Borel measurable transition probability from $H_n A_n B_n$ to S_{n+1} , $n \in N$. Here, $H_1 = S_1$, $H_n = S_1 A_1 B_1 \cdots S_{n-1} A_{n-1} B_{n-1} S_n$, $H_\infty = S_1 A_1 B_1 S_2 A_2 B_2 S_3 \cdots$. Then, H_n is the set of histories of the game for horizon $n \in N$, while H_∞ is the set of all infinite histories of the game.
4. $u_n : H_n A_n B_n \rightarrow \mathbb{R}$, is a Borel measurable function and $v_n : H_n A_n B_n \rightarrow \mathbb{R}_+$, is a nonnegative bounded Borel measurable function, where $\mathbb{R}_+ = (0, \infty)$. Of course, u_n and v_n may be recognized as functions on H_∞ . Doing so, we assume that

$$\lim_{n \rightarrow \infty} u_n = u \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} v_n = v \in \mathbb{R}_+.$$

5. $\theta : S_1 \rightarrow \mathbb{R}$ is a real valued function, which is called a parameter function of the game.
6. $T_\theta^n = u_n - \theta v_n : H_n A_n B_n \rightarrow \mathbb{R}$, is a loss function of player I at stage $n \in N$ and $-T_\theta^n$, is a loss function of player II.

Let $F_n(G_n)$ be the set of all universally measurable transition probabilities from $H_n(H_n)$ to $A_n(B_n)$. A universally measurable strategy of player I(II) is a sequence $f = \{f_n\}(g = \{g_n\})$ such that $f_n \in F_n(g_n \in G_n)$ for each $n \in N$. Denote by $F(G)$ the set of all strategies for player I(II).

Let $E_{f_n}, E_{g_n}, E_{t_{n+1}}$ denote the conditional expectation operator with respect to $f_n \in F_n, g_n \in G_n, t_{n+1}$, respectively. Then, each pair of strategies $f =$

¹Department of Mathematics and Information Science, Graduate School of Science and Technology, Niigata University, 950-2181, Niigata, Japan

²Department of Mathematics, Faculty of Science, Niigata University, 950-2181, Niigata, Japan

$\{f_n\}(g = \{g_n\})$, together with the law of motion $\{t_{n+1}\}$, defines uniquely a universally measurable transition probability $P_{fg}(\cdot|\cdot)$ from S_1 to $A_1B_1S_2A_2B_2S_3\cdots$ such that, for two bounded Borel measurable functions u_n, v_n defined on $H_nA_nB_n$ ($n \in N$), we have for $s_1 \in S_1$ and $h \in H_\infty$,

$$\begin{aligned} E(u_n, f, g)(s_1) &= \int u_n(h) P_{fg}(dh|s_1) \\ &= E_{f_1} E_{g_1} E_{t_2} \cdots E_{f_{n-1}} E_{g_{n-1}} E_{t_n} E_{f_n} E_{g_n} u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} E(v_n, f, g)(s_1) &= \int v_n(h) P_{fg}(dh|s_1) \\ &= E_{f_1} E_{g_1} E_{t_2} \cdots E_{f_{n-1}} E_{g_{n-1}} E_{t_n} E_{f_n} E_{g_n} v_n(s_1) \end{aligned}$$

where u_n and v_n are also regarded as functions on H_∞ .

Under our assumptions, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$, from the dominated convergence theorem and Fubini's theorem

$$\begin{aligned} U(f, g)(s_1) &= \lim_{n \rightarrow \infty} E(u_n, f, g)(s_1) \\ &= \lim_{n \rightarrow \infty} E_{f_1} E_{g_1} E_{t_2} \cdots E_{f_{n-1}} E_{g_{n-1}} E_{t_n} E_{f_n} E_{g_n} u_n(s_1) \\ &= \lim_{n \rightarrow \infty} E_{g_1} E_{f_1} E_{t_2} \cdots E_{g_{n-1}} E_{f_{n-1}} E_{t_n} E_{g_n} E_{f_n} u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} V(f, g)(s_1) &= \lim_{n \rightarrow \infty} E(v_n, f, g)(s_1) \\ &= \lim_{n \rightarrow \infty} E_{f_1} E_{g_1} E_{t_2} \cdots E_{f_{n-1}} E_{g_{n-1}} E_{t_n} E_{f_n} E_{g_n} v_n(s_1) \\ &= \lim_{n \rightarrow \infty} E_{g_1} E_{f_1} E_{t_2} \cdots E_{g_{n-1}} E_{f_{n-1}} E_{t_n} E_{g_n} E_{f_n} v_n(s_1). \end{aligned}$$

For the loss function with the parameter function θ ;

$$T_\theta^n = u_n - \theta v_n,$$

we have for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$,

$$\begin{aligned} T_\theta(f, g)(s_1) &= \lim_{n \rightarrow \infty} E_{fg} T_\theta^n(f, g)(s_1) \\ &= U(f, g)(s_1) - \theta(s_1) V(f, g)(s_1). \end{aligned}$$

We define for initial state $s_1 \in S_1$,

$$\bar{T}_\theta(s_1) = \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1), \quad \underline{T}_\theta(s_1) = \sup_{g \in G} \inf_{f \in F} T_\theta(f, g)(s_1).$$

Then, $\bar{T}_\theta(s_1)(\underline{T}_\theta(s_1))$ is called the upper (the lower) value function of the parametric game. In general, it holds that $\bar{T}_\theta(s_1) \geq \underline{T}_\theta(s_1)$ for all $s_1 \in S_1$. Further, we call the **duality gap** the interval $[\underline{T}_\theta(s_1), \bar{T}_\theta(s_1)]$ for all $s_1 \in S_1$.

Definition 1.1 We shall say that the two-person zero-sum game (DPG_θ) has a saddle value function (in short, a value function), if

$$\bar{T}_\theta(s_1) = \underline{T}_\theta(s_1) = T_\theta^*(s_1)$$

and this common function is called the value function of the game and is denoted by $T_\theta^*(s_1)$.

Definition 1.2 A strategy $\bar{f} \in F$ is said to be a **mini-sup** of the game (DPG_θ) if

$$\sup_{g \in G} T_\theta(\bar{f}, g)(s_1) = \underline{T}_\theta(s_1)$$

and a strategy $\bar{g} \in G$ is said to be a **max-inf** of the game (DPG_θ) if

$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = \overline{T}_\theta(s_1).$$

Definition 1.3 A pair strategies $(\bar{f}, \bar{g}) \in F \times G$ is said to be a **saddle point** of the game (DPG_θ) if

$$\inf_{f \in F} T_\theta(f, \bar{g})(s_1) = T_\theta(\bar{f}, \bar{g})(s_1) = \sup_{g \in G} T_\theta(\bar{f}, g)(s_1).$$

2 A Two-Person Zero-Sum Dynamic Fractional Game

We define a two-person zero-sum dynamic fractional game (DFG) as follows :

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \bar{\theta}, \underline{\theta}; n \in N) \quad (2.1)$$

where S_n is the state space and A_n and B_n are the action spaces at time $n \in N$ of players I and II, respectively. $\{t_{n+1}\}$ is the law of motion of the system. These terms are defined like as the game (DPG_θ) . Further, $u_n : H_n A_n B_n \rightarrow \mathbb{R}$, is a bounded Borel measurable function and $v_n : H_n A_n B_n \rightarrow \mathbb{R}_+$, is a nonnegative bounded Borel measurable function, $\mathbb{R}_+ = (0, \infty)$. We assume that

$$\lim_{n \rightarrow \infty} u_n = u \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} v_n = v \in \mathbb{R}_+.$$

Under our assumptions, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$, $g = \{g_n\} \in G$,

$$U(f, g)(s_1) = \lim_{n \rightarrow \infty} E(u_n, f, g)(s_1), \quad V(f, g)(s_1) = \lim_{n \rightarrow \infty} E(v_n, f, g)(s_1) > 0.$$

Using the notations $U(f, g)(s_1)$ and $V(f, g)(s_1)$, we give

$$W(f, g)(s_1) = \frac{U(f, g)(s_1)}{V(f, g)(s_1)}$$

and we define for an initial state $s_1 \in S_1$,

$$\bar{\theta}(s_1) = \inf_{f \in F} \sup_{g \in G} W(f, g)(s_1), \quad \underline{\theta}(s_1) = \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1).$$

Then, $\bar{\theta}(s_1)(\underline{\theta}(s_1))$ is called the upper (the lower) value function of the game (DFG) . In general, it holds that $\bar{\theta}(s_1) \geq \underline{\theta}(s_1)$ for all $s_1 \in S_1$ and the interval $[\underline{\theta}(s_1), \bar{\theta}(s_1)]$ is called the **duality gap** of the game (DFG) .

Definition 2.1 The game (DFG) is said to have a **value function** if the duality gap is equal to zero. We shall call the value function of the game (DFG) the common value function

$$\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1).$$

Further, $g^* \in G$ is said to be a **max-inf** of the game (DFG) if

$$\bar{\theta}(s_1) = \inf_{f \in F} \sup_{g \in G} W(f, g)(s_1) = \inf_{f \in F} W(f, g^*)(s_1). \quad (2.2)$$

Similarly, $f^* \in F$ is said to be a **mini-sup** of the game (DFG) if

$$\underline{\theta}(s_1) = \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1) = \sup_{g \in G} W(f^*, g)(s_1). \quad (2.3)$$

Lemma 2.1 $\bar{T}_\theta(s_1)$ has the following properties.

- (1) If two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$ satisfy that $\theta_1(s_1) > \theta_2(s_1) \geq 0$, it follows that

$$\bar{T}_{\theta_1}(s_1) \leq \bar{T}_{\theta_2}(s_1).$$

- (2) If $\bar{T}_\theta(s_1) < 0$, it holds that $\theta(s_1) \geq \bar{\theta}(s_1)$.

- (3) If $\bar{T}_\theta(s_1) > 0$, it holds that $\theta(s_1) \leq \bar{\theta}(s_1)$.

- (4) If $\theta(s_1) > \bar{\theta}(s_1)$, it holds that $\bar{T}_\theta(s_1) \leq 0$.

- (5) If $\theta(s_1) < \bar{\theta}(s_1)$, it holds that $\bar{T}_\theta(s_1) \geq 0$.

Proof. (1) If $\theta_1(s_1) > \theta_2(s_1)$, then, we get $\theta_1(s_1)U(f, g)(s_1) > \theta_2(s_1)U(f, g)(s_1)$, because $U(f, g)(s_1)$ is positive for all $(f, g) \in F \times G$. Then, it follows that for all $(f, g) \in F \times G$,

$$T_{\theta_1}(f, g)(s_1) < T_{\theta_2}(f, g)(s_1).$$

Therefore, we get that

$$\begin{aligned} \bar{T}_{\theta_1}(s_1) &= \inf_{f \in F} \sup_{g \in G} T_{\theta_1}(f, g)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta_2}(f, g)(s_1) \\ &= \bar{T}_{\theta_2}(s_1). \end{aligned}$$

Thus, the proof of (1) in the lemma is complete.

(2) Since $\bar{T}_\theta(s_1) < 0$, from the definition of $\bar{T}_\theta(s_1)$, there exists $\bar{f} \in F$ such that $\sup_{g \in G} T_\theta(\bar{f}, g)(s_1) < 0$, that is, for all $g \in G$,

$$T_\theta(\bar{f}, g)(s_1) = U(\bar{f}, g)(s_1) - \theta(s_1)V(\bar{f}, g)(s_1) < 0. \quad (2.4)$$

From (2.4), this shows that for all $g \in G$,

$$W(\bar{f}, g)(s_1) = \frac{U(\bar{f}, g)(s_1)}{V(\bar{f}, g)(s_1)} < \theta(s_1) \quad (2.5)$$

that is,

$$\sup_{g \in G} W(\bar{f}, g)(s_1) \leq \theta(s_1). \quad (2.6)$$

From the definition of $\bar{\theta}(s_1)$ and (2.6), it follows that $\theta(s_1) \geq \bar{\theta}(s_1)$.

(3) Since $\bar{T}_\theta(s_1) > 0$, that is, for all $f \in F$, $\sup_{g \in G} T_\theta(f, g)(s_1) > 0$, there exists $g_f \in G$, which depends on f , such that

$$T_\theta(f, g_f)(s_1) = U(f, g_f)(s_1) - \theta(s_1)V(f, g_f)(s_1) > 0. \quad (2.7)$$

From (2.7), it follows that for all $f \in F$, $W(f, g_f)(s_1) = U(f, g_f)(s_1)/V(f, g_f)(s_1) > \theta(s_1)$. This shows that $\bar{\theta}(s_1) \geq \theta(s_1)$.

(4) Since $\theta(s_1) > \bar{\theta}(s_1)$, from the definition of $\bar{\theta}(s_1)$, there exists $\bar{f} \in F$ such that for all $g \in G$,

$$\theta(s_1) > \sup_{g \in G} W(\bar{f}, g)(s_1).$$

This shows that for all $g \in G$, $T_\theta(\bar{f}, g)(s_1) < 0$. Hence, we get that

$$\begin{aligned} 0 &\geq \sup_{g \in G} T_\theta(\bar{f}, g)(s_1) \\ &\geq \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1) \\ &= \bar{T}_\theta(s_1). \end{aligned}$$

(5) Since $\bar{\theta}(s_1) > \theta(s_1)$, from the definition of $\bar{\theta}(s_1)$, it follows that for all $f \in F$,

$$\sup_{g \in G} W(f, g)(s_1) > \theta(s_1).$$

Thus, there exists $g_f \in G$, which depends on f , such that $W(f, g_f)(s_1) > \theta(s_1)$, that is, for all $f \in F$,

$$\begin{aligned} \sup_{g \in G} T_\theta(f, g)(s_1) &\geq T_\theta(f, g_f)(s_1) \\ &> 0. \end{aligned}$$

Hence, we get that

$$\bar{T}_\theta(s_1) = \inf_{f \in F} \sup_{g \in G} T_\theta(f, g)(s_1) \geq 0.$$

□

Lemma 2.2 $\underline{T}_\theta(s_1)$ has the following properties.

(1) If two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$ satisfy that $\theta_1(s_1) > \theta_2(s_1) \geq 0$, it follows that

$$\underline{T}_{\theta_1}(s_1) \leq \underline{T}_{\theta_2}(s_1).$$

(2) If $\underline{T}_\theta(s_1) < 0$, it holds that $\theta(s_1) \geq \underline{\theta}(s_1)$.

(3) If $\underline{T}_\theta(s_1) > 0$, it holds that $\theta(s_1) \leq \underline{\theta}(s_1)$.

(4) If $\theta(s_1) > \underline{\theta}(s_1)$, it holds that $\underline{T}_\theta(s_1) \leq 0$.

(5) If $\theta(s_1) < \underline{\theta}(s_1)$, it holds that $\underline{T}_\theta(s_1) \geq 0$.

Proof. Using $\underline{T}_\theta(s_1)$ and $\underline{\theta}(s_1)$ instead of $\bar{T}_\theta(s_1)$ and $\bar{\theta}(s_1)$, respectively. we can prove this lemma by similar arguments to the previous one. □

We have the following relations between the game (DFG) and (DPG_θ) .

Theorem 2.1 Suppose that $g^* \in G$ is a max-inf of the game (DFG) . Then, it holds that

(1) $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$.

(2) If $\bar{T}_{\theta^*}(s_1) \leq 0$, g^* is a max-inf of the game (DPG_{θ^*}) .

Proof. (1) From the definition of $\bar{\theta}(s_1)$ and $\underline{\theta}(s_1)$, in general it holds that $\bar{\theta}(s_1) \geq \underline{\theta}(s_1)$.

On the other hand, since $g^* \in G$ is a max-inf of the game (DFG) , it follows that

$$\begin{aligned}\bar{\theta}(s_1) &= \inf_{f \in F} W(f, g^*)(s_1) \\ &\leq \sup_{g \in G} \inf_{f \in F} W(f, g)(s_1) \\ &= \underline{\theta}(s_1).\end{aligned}$$

Thus, the game (DFG) has a value function, that is, $\bar{\theta} = \underline{\theta}$ on S_1 .

(2) Since $g^* \in G$ is a max-inf of the game (DFG) , it holds that for all $f \in F$,

$$\theta^*(s_1) = \inf_{f \in F} W(f, g^*)(s_1) \leq W(f, g^*)(s_1)$$

that is, for all $f \in F$,

$$0 \leq T_{\theta^*}(f, g^*)(s_1) \leq \sup_{g \in G} T_{\theta^*}(f, g)(s_1). \quad (2.8)$$

Thus, from (2.8) and (2) of the theorem, we get the following :

$$\begin{aligned}0 &\leq \inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1) \\ &= \bar{T}_{\theta^*}(s_1) \leq 0.\end{aligned}$$

This shows that

$$\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1).$$

That is, g^* is a max-inf of the game (DPG_{θ^*}) . □

Corollary 2.1 Suppose that $(f^*, g^*) \in F \times G$ is a saddle point of the game (DFG) . Then, it holds that

(1) $T_{\theta^*}(f^*, g^*)(s_1) = 0$.

(2) (f^*, g^*) is a saddle point of the game (DPG_{θ^*}) .

The proof of the corollary is easily given by Theorem 2.1.

Theorem 2.2 Under $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, suppose that $g^* \in G$ is a max-inf of the game (DPG_{θ^*}) and

$$\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \bar{T}_{\theta^*}(s_1) \geq 0.$$

Then, g^* is a max-inf of the game (DFG) .

Proof. Since $\bar{T}_{\theta^*}(s_1) \geq 0$ and g^* is a max-inf of the game (DPG_{θ^*}) , it follows that

$$\begin{aligned} 0 &\leq \inf_{f \in F} \sup_{g \in G} T_{\theta^*}(f, g)(s_1) \\ &= \inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) \\ &\leq T_{\theta^*}(f, g^*)(s_1) \text{ for all } f \in F, \end{aligned}$$

which implies that for all $f \in F$,

$$\theta^*(s_1) \leq W(f, g^*)(s_1) \leq \sup_{g \in G} W(f, g)(s_1).$$

Therefore, we get that

$$\begin{aligned} \theta^*(s_1) &\leq \inf_{f \in F} W(f, g^*)(s_1) \\ &\leq \inf_{f \in F} \sup_{g \in G} W(f, g)(s_1) \\ &= \theta^*(s_1). \end{aligned}$$

This shows that g^* is a max-inf of the game (DFG) . $\square$

Corollary 2.2 Under $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, suppose that $(f^*, g^*) \in F \times G$ is a saddle point of the game (DPG_{θ^*}) and $T_{\theta^*}(f^*, g^*)(s_1) = 0$ holds. Then, (f^*, g^*) is a saddle point of the game (DFG) .

The proof of the corollary is easily given by Theorem 2.2.

References

- [1] J.P.Aubin, Optima and Equilibria (Springer-Verlag, New York, 1993).
- [2] J.P. Crouzeix, J.A. Ferland and S. Schaible, An Algorithm for Generalized Fractional Programs, Jour.Optim.The.Appl., 47, No.1, (1985), 35–49.
- [3] K. Fan, Minimax theorems, Proc. Nat. Acad. Sci. USA, 39 (1953), 42–47.
- [4] Y.Kimura, Y.Sawasaki and K.Tanaka, A Perturbation on Two-Person Zero-Sum Games, to appear in Annals of Dynamic Games.
- [5] K.Tanaka and K.Yokoyama, On ε -Equilibrium Point in a Noncooperative n -Person Game, J. Math. Anal. Appl., 160 (1991), 413–423.