

Selected Applications of Homotopy Algebras in Quantum Field Theory and String Field Theory

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Abstract

Homotopy algebras, such as A_∞ and L_∞ algebras, provide a robust mathematical framework that ensures gauge invariance and consistency in Quantum Field Theory (QFT) and String Field Theory (SFT). Leveraging recent developments in the description of correlation functions, this thesis presents three novel applications of homotopy algebras to both QFT and SFT.

First, we reformulate the Ward-Takahashi identity using homotopy algebras. This approach establishes a unified algebraic language for treating quantum symmetries and anomalies, clarifying the connection between algebraic structures and physical conservation laws.

Second, we derive scattering amplitudes through the LSZ reduction formula within the context of homotopy algebras. This formulation provides a consistent and unified formula for calculating amplitudes that remains valid at the loop level, resolving ambiguities present in earlier algebraic constructions.

Third, we derive a "BSFT-like action" via a specific field redefinition. This action exhibits characteristics remarkably similar to Boundary String Field Theory (BSFT), suggesting that the formulation of BSFT—conventionally defined through worldsheet path integrals—can be understood and regularized through the rigorous language of homotopy algebras.

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1 Introduction

Homotopy algebras, such as A_∞ algebras [1, 2, 3, 4, 5, 6] and L_∞ algebras [7, 8], have played a pivotal role in the formulation of string field theory (SFT) [9, 10, 11, 12, 13, 14, 15, 16]. In recent years, homotopy algebras have been utilized not only for constructing actions but also for a variety of other applications. For instance, homotopy structures can be employed to reproduce the Feynman diagram expansion [17, 18, 40, 20, 21, 22, 23, 24, 25] and establish connections between different formulations [26, 27]. Crucially, homotopy algebras can be applied not only to SFT but also to a broad range of quantum field theories (QFTs) [28, 29, 30, 31, 32, 33, 34]. One might argue that introducing homotopy algebras to ordinary QFT is excessive, given that it does not possess the complex gauge redundancies of SFT. Nevertheless, there are compelling reasons to adopt this approach. The use of homotopy algebras enables the manipulation of quantities such as effective actions and observables in closed forms across all perturbative orders. Furthermore, this formalism provides a universal description independent of specific theoretical details. This means that insights gained in QFT through homotopy algebras can be directly translated to SFT. This is particularly valuable as it allows us to apply the well-developed analytical techniques of QFT to SFT, a field where analysis remains technically demanding.

In this thesis, building upon the study of correlation functions based on homotopy algebras [35, 36, 37], we present applications derived from this framework in both QFT and SFT.

First, as a primary application to QFT, we reformulate the Ward-Takahashi identity—a cornerstone of symmetry relations—using the language of homotopy algebras [38]. Symmetry plays a pivotal role in the analysis of quantum field theory; in particular, global symmetry is a fundamental concept intimately related to conserved charges and anomalies. While global symmetry has been discussed within the framework of homotopy algebras in previous works [39, 40], these discussions have largely remained at the classical level. We extend this analysis to the quantum regime. Utilizing the correlation functions proposed in [35, 36, 37], we derive a quantum expression for the Ward-Takahashi identity. Central to this derivation is the introduction of a new identity, which we term the algebraic Schwinger-Dyson equation. Furthermore, this equation refines the proof of the Schwinger-Dyson equation for correlation functions presented in [35, 36, 37], thereby facilitating extensions to more general theories. Crucially, the Ward-Takahashi identity derived in our framework contains a term corresponding to a quantum anomaly. Since this term is generally ill-defined, appropriate regularization is required. To address this, we apply regularization using stubs [7], a technique widely utilized in string field theory and one that has seen significant development in the context of homotopy algebras in recent years [41, 42, 43, 44]. By employing stubs, we can perform heat kernel regularization while preserving the underlying homotopy algebraic structure, making it the optimal choice for our purposes.

As a second application to QFT, we express the LSZ reduction formula—a fundamental formula for obtaining scattering amplitudes—in terms of homotopy algebras [45]. In the context of homotopy algebras, it is well established that tree-level on-shell scattering amplitudes are provided by an algebraic construction known as the minimal model. However, the correct formulation at the loop level remains less understood. While the minimal model can be naively extended to generate the loop-level Feynman diagram expansion, this approach inherently includes tadpole diagrams and diagrams associated with mass renormalization, thereby failing to yield the correct loop-level amplitudes. In standard quantum field theory, the widely accepted method for obtaining physical amplitudes involves using the LSZ reduction formula to amputate external lines from correlation functions. By translating this procedure into the language of homotopy algebras, we derive a formula that yields the correct scattering amplitudes, starting from the expression for correlation functions based on homotopy algebras obtained in previous studies [35, 36, 37]. Furthermore, we investigate the relationship between this LSZ-based formulation and the conventionally known minimal model. Furthermore, the expression for the amplitude derived here is extended to an off-shell amplitude by lifting the restrictions on the input [46].

Utilizing the expression for the off-shell amplitude allows for significant applications in SFT. The off-shell amplitude can be represented using a specific map known as a cohomomorphism, which satisfies distinct algebraic properties. Crucially, this cohomomorphism can be employed to perform a field redefinition. We demonstrate that the action transformed through this field redefinition exhibits properties remarkably similar to those of the formulation known as Boundary String Field Theory (BSFT) [47, 48, 49, 50, 51], particularly within the context of tree-level open SFT. In light of this similarity, we refer to this redefined action as the BSFT-like action [46]. Although the BSFT action is formally defined, it remains a somewhat subtle formulation that inherently contains divergences and requires appropriate regularization. The BSFT-like action derived here serves as a potential key to connecting this subtle BSFT formulation with the well-established framework of open SFT based on A_∞ algebras. Furthermore, leveraging its algebraic construction, the BSFT-like action can be straightforwardly extended to the L_∞ algebra framework and to the loop level. This implies that the original BSFT formulation also holds the potential to be extended to closed string field theory and to higher-loop orders.

The remainder of this thesis is organized as follows. As a preliminary step before discussing specific applications of homotopy algebras, Sections 2 through 5.1 are devoted to a comprehensive review of the necessary background material. In Section 2, we review the definition of homotopy algebras and their fundamental algebraic structures. Section 3 discusses the relationship between homotopy algebras and QFT, specifically from the perspective of the Batalin-Vilkovisky (BV) formalism. In Section 4, we review the homotopy transfer, which

serves as the most fundamental application of these algebraic structures. Finally, Subsection 5.1 reviews the formulation of correlation functions based on homotopy algebras, which form the basis for the applications presented in this thesis. Building upon these foundations, the subsequent sections introduce specific applications of homotopy algebras to QFT and SFT. In Subsection 5.2, within the context of correlation functions based on homotopy algebras, we introduce the algebraic equation corresponding to the Schwinger-Dyson equation. Using this result, we derive the Ward-Takahashi identity in the language of homotopy algebras in Section 6. In Section 7, we formulate scattering amplitudes using the LSZ reduction formula within the homotopy algebra framework. Finally, in Section 8, as an application to SFT, we demonstrate that an action similar to that of BSFT can be obtained through a specific field redefinition. Appendices A and B provide the detailed calculations regarding loop amplitudes that were omitted from the main text.

Note that although most of the discussions in this thesis primarily focus on A_∞ algebras and their extension, quantum A_∞ algebras, the arguments presented here hold equally well for L_∞ algebras and quantum L_∞ algebras unless otherwise specified.

2 Homotopy Algebras

This section introduces homotopy algebras, with a focus on A_∞ algebras, L_∞ algebras, and their extensions. Informally, a homotopy algebra is an algebraic structure in which the defining relations of an ordinary algebra are required to hold only up to homotopy. The discussion mainly concerns A_∞ algebras, which relax the associativity condition of noncommutative algebras, and L_∞ algebras, which relax the Jacobi identity of Lie algebras.

2.1 Differential graded algebras

Before introducing homotopy algebras, differential graded algebras are described. We begin with a $\mathbb{Z}$ -graded vector space (or, more generally, a graded module; this distinction is not emphasized in what follows.). We write

$$V = \bigoplus_{k \in \mathbb{Z}} V_k, \quad (2.1)$$

where V_k denotes the subspace of degree k . Throughout this thesis, only the parity of the degree is relevant; thus, the terms “even degree” and “odd degree” are used frequently.

A bilinear product $m_2 : V \otimes V \rightarrow V$ is introduced on this space. In particular, if

$$m_2 : V_k \otimes V_l \rightarrow V_{k+l+n}, \quad (2.2)$$

then m_2 is called a product of degree n . This thesis is concerned primarily with products of degree 1.

The product of degree 1 is called associative if it satisfies

$$m_2(m_2(a, b), c) + (-1)^{\deg(a)} m_2(a, m_2(b, c)) = 0, \quad (2.3)$$

for all homogeneous $a, b, c \in V$. Here $\deg(a)$ denotes the degree of a . For notational simplicity, when no confusion arises, the degree of a is written simply as a . The product is called graded commutative if it satisfies

$$m_2(a, b) = (-1)^{ab} m_2(b, a), \quad (2.4)$$

for all homogeneous $a, b \in V$. The product is called a graded Lie bracket if it is graded commutative and if it satisfies the graded Jacobi identity

$$m_2(m_2(a, b), c) + (-1)^{a(b+c)} m_2(m_2(b, c), a) + (-1)^{c(a+b)} m_2(m_2(c, a), b) = 0, \quad (2.5)$$

for all homogeneous $a, b, c \in V$.

Next, a differential is introduced on this space. Consider a linear map $Q : V \rightarrow V$ of degree 1. The linear map Q is called a differential if it is nilpotent,

$$Q^2 = 0, \quad (2.6)$$

and if it acts as a derivation with respect to the product m_2 , that is, it satisfies the graded Leibniz rule

$$Qm_2(a, b) + m_2(Qa, b) + (-1)^a m_2(a, Qb) = 0, \quad (2.7)$$

for all homogeneous $a, b \in V$.

A graded vector space equipped with a product and a differential is called a differential graded algebra. Homotopy algebras, such as A_∞ and L_∞ algebras, arise as extensions of this structure. These homotopy algebras are discussed in later subsections.

2.2 A_∞ algebras

An A_∞ algebra is obtained by relaxing the associativity condition of a differential graded associative algebra. In an A_∞ algebra, the binary product m_2 is not strictly associative. Instead, there exists a 3-linear product $m_3 : V \otimes V \otimes V \rightarrow V$ of degree 1, which satisfies

$$\begin{aligned} 0 = & Qm_3(a, b, c) + m_3(Qa, b, c) + (-1)^a m_3(a, Qb, c) + (-1)^{a+b} m_3(a, b, Qc) \\ & + m_2(m_2(a, b), c) + (-1)^a m_2(a, m_2(b, c)), \end{aligned} \quad (2.8)$$

for all homogeneous $a, b, c \in V$. The first line of this equation can be interpreted as a graded Leibniz rule for the 3-linear product m_3 . In other words, the equation requires that the failure of associativity of m_2 is compensated by the failure of m_3 to satisfy the Leibniz rule.

Since two types of products, m_2 and m_3 , are now present, one can also inquire about the associativity relations that mix them. In an A_∞ algebra, these relations are not satisfied strictly. Instead, the violations are controlled by higher operations: a 4-linear product $m_4 : V^{\otimes 4} \rightarrow V$ of degree 1 and a 5-linear product $m_5 : V^{\otimes 5} \rightarrow V$ of degree 1. These products satisfy

$$\begin{aligned} 0 = & Qm_4(a, b, c, d) + m_4(Qa, b, c, d) + (-1)^a m_4(a, Qb, c, d) \\ & + (-1)^{a+b} m_4(a, b, Qc, d) + (-1)^{a+b+c} m_4(a, b, c, Qd) \\ & + m_2(m_3(a, b, c), d) + (-1)^a m_2(a, m_3(b, c, d)) \\ & + m_3(m_2(a, b), c, d) + (-1)^a m_3(a, m_2(b, c), d) + (-1)^{a+b} m_3(a, b, m_2(c, d)), \end{aligned} \quad (2.9)$$

for all homogeneous $a, b, c, d \in V$, and

$$\begin{aligned} 0 = & Qm_5(a, b, c, d, e) + m_5(Qa, b, c, d, e) + (-1)^a m_5(a, Qb, c, d, e) + (-1)^{a+b} m_5(a, b, Qc, d, e) \\ & + (-1)^{a+b+c} m_5(a, b, c, Qd, e) + (-1)^{a+b+c+d} m_5(a, b, c, d, Qe) \\ & + m_2(m_4(a, b, c, d), e) + (-1)^a m_2(a, m_4(b, c, d, e)) \\ & + m_3(m_3(a, b, c), d, e) + (-1)^a m_3(a, m_3(b, c, d), e) + (-1)^{a+b} m_3(a, b, m_3(c, d, e)) \\ & + m_4(m_2(a, b), c, d, e) + (-1)^a m_4(a, m_2(b, c), d, e) \\ & + (-1)^{a+b} m_4(a, b, m_2(c, d), e) + (-1)^{a+b+c} m_4(a, b, c, m_2(d, e)), \end{aligned} \quad (2.10)$$

for all homogeneous $a, b, c, d, e \in V$. These relations also impose cancellations between the failure of associativity and the failure of the Leibniz rule, which mutually cancel.

Similarly, higher multilinear products $m_n : V^{\otimes n} \rightarrow V$ of degree 1 can be introduced for all $n \geq 2$. These operations are required to satisfy

$$\begin{aligned} 0 = & Qm_n(a_1, \dots, a_n) + \sum_{i=1}^n (-1)^{a_1+\dots+a_{i-1}} m_n(a_1, \dots, a_{i-1}, Qa_i, a_{i+1}, \dots, a_n) \\ & + \sum_{k=2}^{n-1} \sum_{i=1}^{k-1} (-1)^{a_1+\dots+a_{i-1}} m_k(a_1, \dots, a_{i-1}, m_{n-k+1}(a_i, \dots, a_{n-k+i+1}), a_{n-k+i+2}, \dots, a_n), \end{aligned} \quad (2.11)$$

for all homogeneous $a_1, \dots, a_n \in V$. These relations are called the A_∞ relations, and an algebra equipped with multilinear products $\{m_n\}_{n \geq 2}$ and a differential Q satisfying them is called an A_∞ algebra.

2.3 L_∞ algebras

An L_∞ algebra is obtained by relaxing the Jacobi identity of a differential graded Lie algebra. As in the case of A_∞ algebras, higher products are introduced to control the violation of the Jacobi identity. To distinguish this structure from the A_∞ case, multilinear products of degree 1 in an L_∞ algebra are denoted by $\{l_n\}_{n \geq 2}$. These multilinear products are graded symmetric, and they satisfy

$$0 = Ql_n(a_1, \dots, a_n) + \sum_{i=1}^n (-1)^{a_1 + \dots + a_{i-1}} l_n(a_1, \dots, a_{i-1}, Qa_i, a_{i+1}, \dots, a_n) \\ + \sum_{k=2}^{n-1} \sum_{\sigma} (-1)^{\epsilon(\sigma)} l_k(l_{n-k+1}(a_{\sigma(1)}, \dots, a_{\sigma(n-k+1)}), a_{\sigma(n-k+1)}, \dots, a_{\sigma(n)}), \quad (2.12)$$

for all homogeneous $a_1, \dots, a_n \in V$. Here the sum over σ runs over all permutations that split the set of inputs $\{a_1, \dots, a_n\}$ into two ordered blocks: one block of size $n - k + 1$, labeled by $\sigma(1) < \dots < \sigma(n - k + 1)$, and one block of size $k - 1$, labeled by $\sigma(n - k + 1) < \dots < \sigma(n)$. The sign $\epsilon(\sigma)$ is the graded sign obtained when the homogeneous $a_1, \dots, a_n$ are reordered according to σ . This relation again expresses that the failure of the Jacobi identity is canceled by the failure of the higher multilinear products to satisfy a graded Leibniz rule. These relations are called L_∞ relations, and an algebra equipped with multilinear products $\{l_n\}_{n \geq 2}$ and a differential Q satisfying them is called an L_∞ algebra.

2.4 Symplectic form and cyclicity

On a graded vector space V , introduce a symplectic form $\omega : V \otimes V \rightarrow \mathbb{C}$ of degree -1 . The form ω is assumed to be non-degenerate and closed. Furthermore, the form is assumed to be graded anti-symmetric, in the sense that for all homogeneous $a, b \in V$,

$$\omega(a, b) = -(-1)^{ab} \omega(b, a). \quad (2.13)$$

Let $\{e_s\}$ be a (homogeneous) basis of V . The index s is taken to run from 1 to $\dim V$. With respect to the symplectic form ω , one can define a dual basis $\{\tilde{e}^s\}$ by

$$\omega(e_s, \tilde{e}^t) = (-1)^{e_s} \delta_s^t. \quad (2.14)$$

Since ω has degree -1 , the degrees of e_s and $\tilde{e}^s$ must satisfy

$$\deg(e_s) + \deg(\tilde{e}^s) = 1. \quad (2.15)$$

In particular, this implies that e_a and $\tilde{e}^s$ have opposite parity: if e_s has even degree, then $\tilde{e}^s$ has odd degree, and vice versa. Using the basis $\{e_s\}$ and the dual basis $\{\tilde{e}^s\}$, one can construct

a bivector

$$U = \sum_s \tilde{e}^s \otimes e_s \in V \otimes V. \quad (2.16)$$

This element plays the role of the “inverse” of the symplectic form ω . More precisely, U satisfies

$$(\omega \otimes \mathbb{I})(\mathbb{I} \otimes U) = \sum_s \omega(\mathbb{I}, \tilde{e}^s) \otimes e_s = \mathbb{I}, \quad (2.17)$$

where $\mathbb{I}$ denotes the identity operator on V . This identity is to be understood as an operator acting from the left. In the middle expression, when it acts on a homogeneous $a \in V$, one obtains terms of the form

$$(-1)^a \sum_s \omega(a, \tilde{e}^s) e_s, \quad (2.18)$$

and a sign appears when $\tilde{e}^a$ and e_s are moved past a . In other words, contracting ω with U reproduces the identity on V . From this identity, it follows that U is symmetric, as follows:

$$\begin{aligned} U &= \sum_s \tilde{e}^s \otimes e_s = \sum_s ((-1)^{\tilde{e}^s} \sum_t \omega(\tilde{e}^s, \tilde{e}^t) e_t) \otimes e_s \\ &= \sum_s \sum_t (-1)^{\tilde{e}^s} e_t \otimes \omega(\tilde{e}^s, \tilde{e}^t) e_s = \sum_s \sum_t (-1)^{\tilde{e}^s+1} e_t \otimes \omega(\tilde{e}^t, \tilde{e}^s) e_s \\ &= \sum_t e_t \otimes \tilde{e}^t. \end{aligned} \quad (2.19)$$

Here it is used that the pairing $(\tilde{e}^s, \tilde{e}^t)$ is non-vanishing only when the two entries have opposite parity.

When a homotopy algebra, such as an A_∞ algebra or an L_∞ algebra, is equipped with a symplectic form, one can ask for compatibility conditions between the symplectic form and the multilinear products. Concretely, this means requiring relations of the form

$$\omega(a_1, m_n(a_2, \dots, a_{n+1})) = (-1)^{a_1(a_2+\dots+a_{n+1}+1)+a_2} \omega(a_2, m_n(a_3, \dots, a_{n+1}, a_1)). \quad (2.20)$$

In this subsection, the multilinear products are denoted by m_n , but the above relations are not specific to A_∞ algebras; they apply more generally, including the L_∞ case. One can also impose a compatibility condition between the differential Q and the symplectic form as

$$\omega(a_1, Qa_2) = (-1)^{a_1 a_2 + a_1 + a_2} \omega(a_2, Qa_1). \quad (2.21)$$

This property is called cyclicity. An algebra whose products and differentials satisfy these relations is said to be cyclic. The cyclicity condition can also be expressed in terms of the bivector U . For the multilinear products, the cyclicity can be written in the form

$$\sum_s (-1)^{e_s(a_1+\dots+a_{n-1})} [m_n(\tilde{e}^s, a_1, \dots, a_{n-1}) \otimes e_s + (-1)^{\tilde{e}^s} \tilde{e}^s \otimes m_n(a_1, \dots, a_{n-1}, e_s)] = 0. \quad (2.22)$$

while for the differential Q it can be written as

$$\sum_s Q\tilde{e}^s \otimes e_s + (-1)^{\tilde{e}^s} \tilde{e}^s \otimes Qe_s = 0. \quad (2.23)$$

For a proof that these relations are equivalent to the cyclicity conditions (2.20) and (2.21), see Appendix A of [38].

2.5 Quantum A_∞/L_∞ algebras

In the context of quantum field theory, it is important to consider extensions of A_∞ and L_∞ algebras, referred to as quantum A_∞ and quantum L_∞ algebras. The discussion begins with the quantum A_∞ algebra.

In a quantum A_∞ algebra, the multilinear products acquire an additional label g . We introduce a family of products $m_{g,n} : V^{\otimes n} \rightarrow V$. The indices (g, n) are constrained as follows: for $g = 0$, the multilinear products are defined only for $n \geq 2$, while for $g \geq 1$, the multilinear products are defined for all $n \geq 0$. The label g is commonly interpreted as genus in the context of string field theory. Each $m_{g,n}$ is taken to have degree 1. The differential Q together with the $m_{0,g}$ satisfies the usual A_∞ relations. The products $m_{g,n}$ with $g \geq 1$ satisfy additional relations:

$$\begin{aligned} 0 = & Qm_{g,n}(a_1, \dots, a_n) + \sum_{i=1}^n (-1)^{a_1 + \dots + a_{i-1}} m_{g,n}(a_1, \dots, a_{i-1}, Qa_i, a_{i+1}, \dots, a_n) \\ & + \sum_{h=0}^g \sum_{k=1}^{n+1} \sum_{i=1}^{k-1} (-1)^{a_1 + \dots + a_{i-1}} \\ & \quad \times m_{h,k}(a_1, \dots, a_{i-1}, m_{g-h,n-k+1}(a_i, \dots, a_{n-k+i+1}), a_{n-k+i+2}, \dots, a_n) \\ & - \sum_s \sum_{i=0}^n \sum_{j=0}^{n-i} (-1)^{a_1 + \dots + a_i + e_s(a_{i+1} + \dots + a_{i+j})} \\ & \quad \times m_{g-1,n+2}(a_1, \dots, a_i, \tilde{e}^s, a_{i+1}, \dots, a_{i+j}, e_s, a_{i+j+1}, \dots, a_n), \end{aligned} \quad (2.24)$$

We set $m_{0,1} = 0$. Consequently, the sum in the second line effectively runs over all (h, k) with the exception of $(0, 1)$ and $(g, n+1)$. These relations are called quantum A_∞ relations. An algebra that satisfies the quantum A_∞ relations and whose differential Q and products $m_{g,n}$ are cyclic with respect to ω is called a quantum A_∞ algebra. The natural role of this cyclicity condition will become clear in Section 2.6.

A quantum L_∞ algebra is defined in complete analogy. We introduce a family of degree 1 multilinear products $l_{g,n} : V^{\otimes n} \rightarrow V$, that are graded commutative. As before, the index range is taken to be: for $g = 0$, the products $l_{0,n}$ are defined only for $n \geq 2$, and for $g \geq 1$, the

products $l_{g,n}$ are defined for all $n \geq 0$. The differential Q together with the $l_{0,n}$ satisfy the usual L_∞ relations. For $g \geq 1$, the products $l_{g,n}$ satisfy the relations:

$$\begin{aligned}
0 = & Ql_{g,n}(a_1, \dots, a_n) + \sum_{i=1}^n (-1)^{a_1 + \dots + a_{i-1}} l_{g,n}(a_1, \dots, a_{i-1}, Qa_i, a_{i+1}, \dots, a_n) \\
& + \sum_{h=0}^g \sum_{k=1}^{n+1} \sum_{\sigma} (-1)^{\epsilon(\sigma)} l_{h,k}(l_{g-h,n-k+1}(a_{\sigma(1)}, \dots, a_{\sigma(n-k+1)}), a_{\sigma(n-k+2)}, \dots, a_{\sigma(n)}) \\
& - \sum_s l_{g-1,n+2}(\tilde{e}^s, e_s, a_1, \dots, a_n). \tag{2.25}
\end{aligned}$$

Here, the sum over σ in the second line is understood in the same way as in the L_∞ relations (2.12). These relations are called the quantum L_∞ relations. An algebra that satisfies the quantum L_∞ relations, and in which both the differential Q and the products $l_{g,n}$ are cyclic with respect to ω , is called a quantum L_∞ algebra.

2.6 Coalgebra representation

It is convenient to describe homotopy algebras in terms of coalgebras. For a graded vector space V , the tensor algebra $\mathcal{TV}$ is defined as

$$\mathcal{TV} = \bigoplus_{n=0}^{\infty} V^{\otimes n}. \tag{2.26}$$

Here $V^{\otimes 0}$ is identified with $\mathbb{C}$, and its single basis is denoted by $\mathbf{1}$. This basis $\mathbf{1}$ satisfies

$$\mathbf{1} \otimes A = A \otimes \mathbf{1} = A, \tag{2.27}$$

for $A \in \mathcal{TV}$. The tensor algebra carries both a product ∇ and a coproduct Δ . The product ∇ is defined by

$$\nabla(A \otimes' B) = A \otimes B, \tag{2.28}$$

for all $A, B \in \mathcal{TV}$. Here $\otimes'$ denotes the tensor product between two copies of the tensor algebra $\mathcal{TV}$; it is used simply to distinguish this external tensor product from the internal tensor product $\otimes$ that appears inside elements of $\mathcal{TV}$. The coproduct Δ is defined by

$$\begin{aligned}
\Delta \mathbf{1} &= \mathbf{1} \otimes' \mathbf{1}, \tag{2.29} \\
\Delta(a_1 \otimes \dots \otimes a_n) &= \mathbf{1} \otimes' (a_1 \otimes \dots \otimes a_n) \\
&+ \sum_{i=1}^{n-1} (a_1 \otimes \dots \otimes a_i) \otimes' (a_{i+1} \otimes \dots \otimes a_n) \\
&+ (a_1 \otimes \dots \otimes a_n) \otimes' \mathbf{1}, \tag{2.30}
\end{aligned}$$

for all $a_1, \dots, a_n \in V$. Let $\pi_n : \mathcal{TV} \rightarrow V^{\otimes n}$ denote the projection onto the n -fold tensor component. The projections π_n , together with the product ∇ and the coproduct Δ , satisfy the compatibility relations:

$$\nabla(\pi_k \otimes' \pi_l) \Delta = \pi_{k+l}. \quad (2.31)$$

The key operator in the coalgebra approach is the coderivation. A coderivation is defined as a linear map $\mathbf{C} : \mathcal{TV} \rightarrow \mathcal{TV}$ satisfying

$$\Delta \mathbf{C} = (\mathbf{I} \otimes' \mathbf{C} + \mathbf{I} \otimes' \mathbf{C}) \Delta, \quad (2.32)$$

where $\mathbf{I}$ denotes an identity operator on the tensor algebra $\mathcal{TV}$.

A coderivation $\mathbf{Q}$ can be constructed from a differential Q as follow:

$$\mathbf{Q}\pi_0 = 0, \quad \mathbf{Q}\pi_n = \sum_{i=0}^{n-1} \mathbb{I}^{\otimes i} \otimes Q \otimes \mathbb{I}^{\otimes n-i-1} \quad \text{for } n \geq 1. \quad (2.33)$$

A coderivation $\mathbf{m}_n$ can also be constructed from the multilinear products m_n as follow:

$$\mathbf{m}_n \pi_k = 0 \quad \text{for } k < n, \quad \mathbf{m}_n \pi_k = \sum_{i=0}^{k-n} \mathbb{I}^{\otimes i} \otimes m_n \otimes \mathbb{I}^{\otimes k-n-i} \quad \text{for } k \geq n. \quad (2.34)$$

These coderivations inherit the degrees of the original differential and products, and in particular are of degree 1. Using this coderivation, the A_∞ relations (2.11) can be expressed in a dramatically simpler form:

$$\left(Q + \sum_{n=2}^{\infty} \mathbf{m}_n \right)^2 = 0. \quad (2.35)$$

In fact,

$$\pi_1 \left(Q + \sum_{n=2}^{\infty} \mathbf{m}_n \right)^2 \pi_1 = Q^2 = 0, \quad (2.36)$$

encodes the nilpotency of Q ;

$$\pi_1 \left(Q + \sum_{n=2}^{\infty} \mathbf{m}_n \right)^2 \pi_2 = Q m_2(\mathbb{I} \otimes \mathbb{I}) + m_2(Q \otimes \mathbb{I}) + m_2(\mathbb{I} \otimes Q) = 0, \quad (2.37)$$

encodes the Leibniz rule; and $\pi_1 \left(Q + \sum_{n=2}^{\infty} \mathbf{m}_n \right)^2 \pi_n = 0$ for $n \geq 3$ encodes the higher A_∞ relations. By acting from the left with π_n for $n \geq 2$, one obtains equations that are equivalent to the A_∞ relations derived by acting with π_1 . For notational convenience, write

$$\mathbf{m} = \sum_{n=2}^{\infty} \mathbf{m}_n, \quad (2.38)$$

$$\mathbf{M} = \mathbf{Q} + \mathbf{m}. \quad (2.39)$$

In this way, the A_∞ relations can be reformulated as the single statement that the coderivation $\mathbf{M}$ on the tensor algebra $\mathcal{TV}$ is nilpotent.

Using the coalgebra representation, the quantum A_∞ relations (2.24) can likewise be written in a compact form. For this purpose, the multiproducts $m_{g,n}$ are likewise promoted to coderivations $\mathbf{m}_{g,n}$. In addition, the basis elements e_a of V can be regarded as maps $V^{\otimes 0} \rightarrow V$, and they can be extended to coderivations $\mathbf{e}_s$ in the same way:

$$\mathbf{e}_s \pi_0 = e_s, \quad \mathbf{e}_s \pi_n = \sum_{i=0}^{n-1} \mathbb{I}^{\otimes i} \otimes e_s \otimes \mathbb{I}^{\otimes n-i-1} \quad \text{for } n \geq 1. \quad (2.40)$$

The dual basis elements $\tilde{e}^s$ are extended to coderivations $\tilde{\mathbf{e}}^s$ in exactly the same way. Using these coderivations, an operator

$$\mathbf{U} = \frac{1}{2} \sum_s \tilde{\mathbf{e}}^s \mathbf{e}_s \quad (2.41)$$

is constructed. Using these operators and a formal parameter $\hbar$, the quantum A_∞ relations (2.24) can be written in the compact form:

$$\left(\mathbf{Q} + \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \hbar^g \mathbf{m}_{g,n} - \hbar \mathbf{U} \right)^2 = 0. \quad (2.42)$$

Provided that we set $\mathbf{m}_{0,0} = \mathbf{m}_{0,1} = 0$, the coefficient of $\hbar^g$ in the component $\pi_1(\mathbf{Q} + \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \hbar^g \mathbf{m}_{g,n} - \hbar \mathbf{U})^2 \pi_n = 0$ of this equation is precisely the quantum A_∞ relations (2.24). By acting on this equation from the left with π_2 , one obtains, in addition to the parts that are equivalent to those obtained from π_1 , an extra condition of the form

$$\sum_s [\pi_1 \mathbf{m}_{g,n} \tilde{\mathbf{e}}^s \otimes e_s + (-1)^{\tilde{e}^s} \tilde{\mathbf{e}}^s \otimes \pi_1 \mathbf{m}_{g,n} \mathbf{e}_s] = 0. \quad (2.43)$$

This condition is satisfied precisely when $m_{g,n}$ is cyclic. Acting with π_n for $n \geq 3$ yields no additional independent conditions. This explains why it is natural to include the cyclicity condition in the definition of a quantum A_∞ algebra. Note that this condition is necessary for cyclicity, but when the products are not graded commutative, it is not sufficient. Therefore, in an approach that takes (2.43) the defining equation of an A_∞ algebra, cyclicity is not strictly required. In this thesis, however, cyclicity is imposed throughout. For notational simplicity, and as long as no confusion arises, the same conventions as in the A_∞ case are used here as well:

$$\mathbf{m} = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \hbar^g \mathbf{m}_{g,n}, \quad (2.44)$$

$$\mathbf{M} = \mathbf{Q} + \mathbf{m}. \quad (2.45)$$

In the A_∞ case, the condition is that $\mathbf{M}$ be nilpotent. In the quantum A_∞ case, the nilpotency is required only after adding the $-\hbar \mathbf{U}$ correction, i.e.

$$(\mathbf{M} - \hbar \mathbf{U})^2 = 0. \quad (2.46)$$

The correction proportional to $\hbar$ is interpreted, in quantum-field-theoretic terms, as a quantum correction, and $\hbar$ is identified with the Dirac constant. For Euclidean theories, the expression above is used as it is, whereas in the Lorentzian case, one considers instead the replacement

$$-\hbar \mathbf{U} \rightarrow i\hbar \mathbf{U}. \quad (2.47)$$

This can simply be regarded as a redefinition of $\hbar$ by a factor of $-i$, so that the factor $-i$ is absorbed into the definition of $m_{g,n}$.

The L_∞ relations can also be written compactly in the coalgebra representation. Since in an L_∞ algebra the inputs are always graded symmetrized, one considers the graded symmetric tensor product

$$a_1 \wedge \cdots \wedge a_n = \sum_{\sigma} (-1)^{\epsilon(\sigma)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}. \quad (2.48)$$

Here, the sum over σ runs over all permutations of $\{1, \dots, n\}$, and the sign is the graded sign produced when rearranging $a_1, \dots, a_n$ into the order specified by σ . Using this symmetrized tensor product, the symmetrized tensor algebra is defined by

$$SV = \bigoplus_{n=0}^{\infty} V^{\wedge n}. \quad (2.49)$$

The symmetrized tensor algebra inherits the usual coalgebra structure from the ordinary tensor algebra. Using coderivations on this symmetrized tensor algebra, the L_∞ relations can be written in a compact form. From the graded commutative multilinear products l_n , one can construct coderivations $\mathbf{l}_n$ on the symmetrized tensor algebra. The construction is completely analogous to the A_∞ case, but written with the symmetrized tensor product, it takes the form

$$\mathbf{l}_n \pi_k = 0 \quad \text{for } k < n, \quad \mathbf{l}_n \pi_k = \frac{1}{(k-n)!} l_n \wedge \mathbb{I}^{\wedge k-n} \quad \text{for } k \geq n. \quad (2.50)$$

Using these coderivations, the L_∞ relations (2.12) are summarized by

$$(\mathbf{Q} + \sum_{n=2}^{\infty} \mathbf{l}_n)^2 = 0. \quad (2.51)$$

As in the A_∞ case, acting from the left with π_1 reproduces the L_∞ relations (2.12), and no additional independent conditions arise from higher projections. For notational simplicity, write

$$l = \sum_{n=2}^{\infty} l_n, \quad (2.52)$$

$$L = Q + l. \quad (2.53)$$

The L_∞ relations are thus equivalent to the statement that the coderivation L on the symmetrized tensor algebra $\mathcal{S}V$ is nilpotent.

The quantum L_∞ relations can be summarized in exactly the same way. Each graded commutative multilinear product $l_{g,n}$ is promoted to a coderivation in the same way, and together with U , one can write

$$(Q + \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \hbar^g l_{g,n} - \hbar U)^2 = 0. \quad (2.54)$$

Again, provided that we set $l_{0,0} = l_{0,1} = 0$, the coefficient of $\hbar^g$ in the component obtained by acting from the left with π_1 precisely yields the quantum L_∞ relations. By acting from the left with π_2 , one obtains an additional independent condition

$$\sum_s \pi_1 l_{g,n} \tilde{e}^s \wedge e_s = 0. \quad (2.55)$$

In the present case, since the operations $l_{g,n}$ are graded commutative, this condition is equivalent to cyclicity. Acting with π_n for $n \geq 3$ yields no additional independent conditions. For notational simplicity, and as long as no confusion arises, the same convention as in the L_∞ case is used here:

$$l = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \hbar^g l_{g,n}, \quad (2.56)$$

$$L = Q + l. \quad (2.57)$$

The quantum L_∞ relations state that the operator $L - \hbar U$ is nilpotent. In the Lorentzian case, one instead uses the replacement $-\hbar U \rightarrow i\hbar U$.

Whether A_∞ or L_∞ , and whether in the classical or quantum case, all of these structures are defined by a nilpotency condition. For this reason, they share a common overall framework and many parallel properties.

3 Quantum Field Theory Based on Homotopy Algebras

One of the important applications of homotopy algebras is their application to quantum field theory. In particular, they can be applied to the Batalin-Vilkovisky (BV) formalism [52, 53].

n this section, it is explained how to construct QFT actions from homotopy algebras within the BV formalism, and how the corresponding equations of motion and gauge symmetry are encoded.

3.1 Relation between the BV formalism and homotopy algebras

The BV formalism is a powerful framework for treating reducible gauge theories. In addition to the ordinary fields (denoted here by ϕ^a), antifields with opposite statistics (denoted $\tilde{\phi}_a$) are introduced, and the action is given as a function $S[\phi, \tilde{\phi}]$ that satisfies the master equation

$$\frac{1}{2}\{S, S\} - \hbar\Delta S = 0. \quad (3.1)$$

Here the anti-bracket $\{, \}$ and the BV Laplacian Δ are defined by

$$\{F, G\} = \sum_a \left[\frac{\partial_r F}{\partial \phi^a} \frac{\partial_l G}{\partial \tilde{\phi}_a} - \frac{\partial_r F}{\partial \tilde{\phi}_a} \frac{\partial_l G}{\partial \phi^a} \right], \quad (3.2)$$

$$\Delta F = (-1)^{\phi^a} \frac{\partial_l^2 F}{\partial \phi^a \partial \tilde{\phi}_a}, \quad (3.3)$$

for all function F, G . Here ∂_l and ∂_r denote the left and right derivatives, respectively. Gauge fixing is performed by eliminating the antifields, and the master equation is precisely the condition that the partition function does not depend on how this elimination is carried out. When $\hbar = 0$, the master equation reduces to

$$\{S, S\} = 0, \quad (3.4)$$

which is called the classical master equation. The original equation, with the $\hbar$ -term included, is referred to as the quantum master equation. Using homotopy algebras, one can construct actions that satisfy the master equation. Since the constructions for (quantum) A_∞ and (quantum) L_∞ algebras are almost parallel, the discussion below is given in terms of (quantum) A_∞ algebras.

To apply homotopy algebras to the BV formalism, consider a graded vector space V whose dimension matches that of the field–antifield space. Let its homogeneous basis be $\{c_a | d^a\}$, and assume that the symplectic form satisfies

$$\omega(c_a, d^b) = (-1)^{c_a} \delta_a^b. \quad (3.5)$$

The index a is taken to run from 1 to $(\dim V)/2$. In terms of the previous notation, we identify

$$e_s = \begin{cases} c_s & \text{for } 1 \leq s \leq (\dim V)/2, \\ d^s & \text{for } (\dim V)/2 < s \leq \dim V, \end{cases} \quad (3.6)$$

and the dual basis is defined by

$$\tilde{e}^s = \begin{cases} d^s & \text{for } 1 \leq s \leq (\dim V)/2, \\ c_s & \text{for } (\dim V)/2 < s \leq \dim V, \end{cases} \quad (3.7)$$

On this basis, the identity (2.17) can be written as

$$\sum_a [\omega(\mathbb{I}, d^a) \otimes c_a + \omega(\mathbb{I}, c_a) \otimes d^a] = \mathbb{I}. \quad (3.8)$$

Coupling this basis to the fields and antifields, define

$$\Phi = \sum_a [\phi^a c_a + (-1)^{\tilde{\phi}_a} \tilde{\phi}_a d^a]. \quad (3.9)$$

Here, the degrees are chosen so that the Grassmann parity of the fields and antifields coincides with the parity of the degree. Moreover, the degrees of ϕ^a and c_a are matched, and likewise the degrees of $\tilde{\phi}_a$ and d^a are matched, so that Φ has even degree.

We first construct an action that satisfies the classical master equation. It is given in terms of a cyclic A_∞ algebra by

$$S = \sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Phi, \pi_1 \mathbf{M} \pi_n \frac{1}{1-\Phi}\right). \quad (3.10)$$

Here $\frac{1}{1-\Phi}$ is understood as the formal series

$$\frac{1}{1-\Phi} = \mathbf{1} + \Phi + \Phi^{\otimes 2} + \dots, \quad (3.11)$$

and ω and $\mathbf{M}$ act nontrivially only on the elements of V ; the fields and antifields merely commute past them, producing the signs determined by their degrees. It is straightforward to verify that this action satisfies the classical master equation, as follows. By cyclicity, the derivatives of S can be written as

$$\frac{\partial_l S}{\partial \phi^a} = (-1)^{\phi^a} \frac{\partial_r S}{\partial \phi^a} = (-1)^{\phi^a} \omega\left(c_a, \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right), \quad (3.12)$$

$$\frac{\partial_l S}{\partial \tilde{\phi}_a} = (-1)^{\tilde{\phi}_a} \frac{\partial_r S}{\partial \tilde{\phi}_a} = \omega\left(d^a, \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right). \quad (3.13)$$

Then substituting into (3.4) gives

$$\{S, S\} = 2 \sum_a \omega\left(c_a, \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right) \omega\left(d^a, \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right), \quad (3.14)$$

$$= \omega\left(\pi_1 \mathbf{M} \frac{1}{1-\Phi}, \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right). \quad (3.15)$$

Here, in the second equality, the identity (3.8) has been used. Using cyclicity, this can be rewritten as

$$\{S, S\} = -\omega\left(\Phi, \pi_1 \mathbf{M}^2 \frac{1}{1-\Phi}\right), \quad (3.16)$$

which vanishes by the A_∞ relations.

Next, an action satisfying the quantum master equation is constructed. Using a quantum A_∞ algebra (cyclicity is implicitly assumed), it is given in the same form

$$S = \sum_{n=0}^{\infty} \frac{1}{n+1} \omega\left(\Phi, \pi_1 \mathbf{M} \pi_n \frac{1}{1-\Phi}\right). \quad (3.17)$$

Here $\mathbf{M}$ contains higher-order contributions in $\hbar$. It is also straightforward to check that this action satisfies the quantum master equation. The part $\{S, S\}$ is exactly the same as in (3.16). Since $\mathbf{M}$ now satisfies the quantum A_∞ relations, it is not nilpotent by itself. The contribution of the BV Laplacian is computed as

$$\Delta S = - \sum_a \omega\left(\Phi, \pi_1 \mathbf{M} \mathbf{c}_a \mathbf{d}^a \frac{1}{1-\Phi}\right). \quad (3.18)$$

In the present basis, the operator $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{c}_a \mathbf{d}^a, \quad (3.19)$$

so combining the anti-bracket term $\{S, S\}$ with the BV Laplacian term ΔS gives precisely

$$\frac{1}{2} \{S, S\} - \hbar \Delta S = -\frac{1}{2} \omega\left(\Phi, \pi_1 (\mathbf{M}^2 - \hbar \mathbf{M} \mathbf{U}) \frac{1}{1-\Phi}\right) = -\frac{1}{2} \omega\left(\Phi, \pi_1 (\mathbf{M} - \hbar \mathbf{U})^2 \frac{1}{1-\Phi}\right). \quad (3.20)$$

The right-hand side vanishes as a consequence of the quantum A_∞ relations. In the Lorentzian case, as noted in the previous section, the nilpotent operator is $\mathbf{M} + i\hbar \mathbf{U}$, and the whole action is multiplied by -1 , so it is given by

$$S = - \sum_{n=0}^{\infty} \frac{1}{n+1} \omega\left(\Phi, \pi_1 \mathbf{M} \pi_n \frac{1}{1-\Phi}\right). \quad (3.21)$$

3.2 Equation of motion and gauge symmetry

In this subsection, we focus on the classical action and describe the equations of motion and gauge symmetries using the language of homotopy algebras derived from an action based on an A_∞ algebra. The equation of motion derived from this action is given by

$$\mathbf{M} \frac{1}{1-\Phi} = 0. \quad (3.22)$$

Acting on this equation from the left with π_1 gives

$$Q\Phi + m_2(\Phi, \Phi) + \cdots = 0, \quad (3.23)$$

and the coefficients of this expression in the basis of V provide the equations of motion for the fields (and antifields). Higher projections π_n do not yield independent equations.

We now turn to gauge symmetries. Consider the transformation

$$\delta_\Lambda \Phi = Q\Lambda + \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} m_n(\Phi^{\otimes k}, \Lambda, \Phi^{n-k-1}), \quad (3.24)$$

where Λ is gauge parameter of degree -1 . In the coalgebra representation, this transformation is written as

$$\delta_\Lambda \frac{1}{1-\Phi} = \mathbf{M}\Lambda \frac{1}{1-\Phi}, \quad (3.25)$$

where $\mathbf{\Lambda}$ is the coderivation obtained from Λ in the same way as in (2.40). Here the variation of $\frac{1}{1-\Phi}$ is defined by

$$\delta \frac{1}{1-\Phi} = \frac{1}{1-\Phi} \otimes \delta\Phi \otimes \frac{1}{1-\Phi}. \quad (3.26)$$

This transformation corresponds to a gauge transformation. Under this transformation, the action changes as

$$\delta_\Lambda S = \omega(\Lambda, \pi_1 \mathbf{M}^2 \frac{1}{1-\Phi}). \quad (3.27)$$

This vanishes immediately as a consequence of the A_∞ relations. Combining the equations of motion with this gauge transformation shows that the physical space is given by the cohomology of $\mathbf{M}$ on $\mathcal{TV}$.

As a natural quantum extension, one expects the physical space to be determined by the cohomology of $\mathbf{M} - \hbar \mathbf{U}$. This is confirmed in Section 5 in the discussion of correlation functions; there, due to gauge fixing, the physical space is identified with the $\mathbf{M} - \hbar \mathbf{U}$ -closed subspace.

4 Homotopy Transfer

This section provides an overview of homotopy transfer [54, 55]. Roughly speaking, homotopy transfer is a procedure that transports a homotopy algebra structure (such as an A_∞ or L_∞ structure) from a “big” complex to a quasi-isomorphic smaller complex. In physics, it can be used to integrate out unphysical or heavy degrees of freedom to obtain an effective theory and to compute on-shell amplitudes. The concrete constructions—based on the homological perturbation lemma (HPL)—and their physical applications are explained in the following subsections.

4.1 Homological perturbation lemma

The homological perturbation lemma is a key tool for carrying out homotopy transfer explicitly. To use it, consider the following setting. Let V and W be graded vector spaces, and let $p : V \rightarrow W$ and $i : W \rightarrow V$ be maps between them. Furthermore, let V and W each be equipped with degree-odd differentials $Q : V \rightarrow V$ and $Q' : W \rightarrow W$ (so $Q^2 = 0$ and $(Q')^2 = 0$), and assume that the maps satisfy the chain-map conditions

$$Q'p = pQ, \quad iQ' = Qp. \quad (4.1)$$

Assume moreover that the maps p and i satisfy

$$pi = \mathbb{I}_W, \quad (4.2)$$

where $\mathbb{I}_W$ denotes the identity operator on W . Under this assumption, the composite map $ip : V \rightarrow V$ satisfies

$$ipip = ip. \quad (4.3)$$

So it can be regarded as a projection from V onto $iW \subset V$. This projection is denoted by P . Assume also that there exists a degree-odd map $h : V \rightarrow V$ (called a contracting homotopy) satisfying the Hodge-Kodaira decomposition

$$Qh + hQ + P = \mathbb{I}_V, \quad (4.4)$$

where $\mathbb{I}_V$ denotes the identity operator on V . Since p and i induce isomorphisms on the cohomology of V and W , they are quasi-isomorphisms. This structure is called a deformation retract (DR). If, in addition, one assumes side conditions

$$h^2 = 0, \quad (4.5)$$

$$hi = 0, \quad (4.6)$$

$$ph = 0, \quad (4.7)$$

the structure is called a strong deformation retract (SDR). Schematically, this data is written as

$$h \circlearrowleft (V, Q) \overset{p}{\underset{i}{\rightleftarrows}} (W, Q'). \quad (4.8)$$

Homotopy transfer can be carried out for a DR, not only for an SDR. However, in order for condition (4.2) to remain valid after introducing the perturbation, the data must form an SDR. An SDR can always be obtained from a DR [54]. Concretely, replacing $h \rightarrow hQh$ makes (4.5)

hold; replacing $h \rightarrow h(Qh + hQ)$ makes (4.6) hold; and replacing $h \rightarrow (Qh + hQ)h$ makes (4.7) hold. In the end,

$$(QhQh(QhQh + hQhQ) + hQh(QhQh + hQhQ)Q)hQh(QhQh + hQhQ), \quad (4.9)$$

gives a homotopy satisfying all side conditions.

The statement of the HPL is as follows. Given an SDR as in (4.8), and a perturbation $\delta Q : V \rightarrow V$ of the differential on V such that

$$\tilde{Q} = Q + \delta Q, \quad \tilde{Q}^2 = 0, \quad (4.10)$$

we can define perturbed data

$$\delta Q' = p\delta Q \frac{1}{\mathbb{I}_V + h\delta Q} i, \quad (4.11)$$

$$\tilde{p} = p \frac{1}{\mathbb{I}_V + \delta Q h}, \quad (4.12)$$

$$\tilde{i} = \frac{1}{\mathbb{I}_V + h\delta Q} i, \quad (4.13)$$

$$\tilde{h} = \frac{1}{\mathbb{I}_V + h\delta Q} h, \quad (4.14)$$

such that these maps again form an SDR

$$\tilde{h} \circ (V, \tilde{Q}) \xrightleftharpoons[\tilde{i}]{\tilde{p}} (W, \tilde{Q}' = Q' + \delta Q'). \quad (4.15)$$

In other words, $\tilde{Q}'$ is nilpotent, the maps satisfy the chain-map conditions

$$\tilde{Q}' \tilde{p} = \tilde{p} \tilde{Q}, \quad \tilde{i} \tilde{Q}' = \tilde{Q} \tilde{p}, \quad (4.16)$$

the identity

$$\tilde{p} \tilde{i} = \mathbb{I}_W, \quad (4.17)$$

Hodge-Kodaira decomposition

$$\tilde{Q}' \tilde{h} + \tilde{h} \tilde{Q} + \tilde{P} = \mathbb{I}_V, \quad (4.18)$$

where $\tilde{P} = \tilde{i} \tilde{p}$, and the side conditions

$$\tilde{h}^2 = 0, \quad \tilde{h} \tilde{i} = 0, \quad \tilde{p} \tilde{h} = 0, \quad (4.19)$$

are all satisfied. Here, for an operator $\mathbf{A}$, the expression $\frac{1}{\mathbb{I} - \mathbf{A}}$ is understood as the formal series

$$\frac{1}{\mathbb{I} - A} = \mathbb{I} + A + A^2 + \cdots, \quad (4.20)$$

and issues of convergence are not discussed here. A clear review of the proof is given in Appendix A of [40].

To apply the HPL to homotopy algebras, the data of SDR (4.8) is lifted on the tensor algebra. The differential Q and Q' are promoted to coderivations $\mathbf{Q}$ and $\mathbf{Q}'$, and the other maps are lifted as

$$\mathbf{p}\pi_0 = \pi_0, \quad \mathbf{p}\pi_n = p^{\otimes n} \quad \text{for } n \geq 1, \quad (4.21)$$

$$\mathbf{i}\pi_0 = \pi_0, \quad \mathbf{i}\pi_n = i^{\otimes n} \quad \text{for } n \geq 1, \quad (4.22)$$

$$\mathbf{h}\pi_0 = 0, \quad \mathbf{h}\pi_n = \sum_{k=0}^{n-1} \mathbb{I}_V^{\otimes k} \otimes h \otimes P^{\otimes n-k-1} \quad \text{for } n \geq 1. \quad (4.23)$$

We write $\mathbf{P} = \mathbf{i}\mathbf{p}$. These data satisfy all the required identities and form an SDR

$$\mathbf{h} \odot (\mathcal{TV}, \mathbf{Q}) \underset{i}{\overset{p}{\rightleftharpoons}} (\mathcal{TW}, \mathbf{Q}'). \quad (4.24)$$

Adding the coderivation $\mathbf{m}$ promoted from the multilinear products as a perturbation, and assuming that $\mathbf{M} = \mathbf{Q} + \mathbf{m}$ is nilpotent (i.e. defines an A_∞ structure), the HPL guarantees that we can construct the perturbed SDR

$$\tilde{\mathbf{h}} \odot (\mathcal{TV}, \mathbf{M}) \underset{\tilde{i}}{\overset{\tilde{p}}{\rightleftharpoons}} (\mathcal{TW}, \mathbf{M}' = \mathbf{Q}' + \mathbf{m}'). \quad (4.25)$$

Perturbed data are defined as

$$\mathbf{m}' = \mathbf{p}\mathbf{m} \frac{1}{\mathbf{I} + \mathbf{h}\mathbf{m}} \mathbf{i}, \quad (4.26)$$

$$\tilde{\mathbf{p}} = \mathbf{p} \frac{1}{\mathbf{I} + \mathbf{m}\mathbf{h}}, \quad (4.27)$$

$$\tilde{\mathbf{i}} = \frac{1}{\mathbf{I} + \mathbf{h}\mathbf{m}} \mathbf{i}, \quad (4.28)$$

$$\tilde{\mathbf{h}} = \frac{1}{\mathbf{I} + \mathbf{h}\mathbf{m}} \mathbf{h}. \quad (4.29)$$

Furthermore, from the chain-map condition, one obtains

$$\mathbf{Q}' = \mathbf{p}\mathbf{Q}\mathbf{i}. \quad (4.30)$$

In particular, it is straightforward to verify that $\mathbf{m}'$ is again a coderivation. Since $\mathbf{M}' = \mathbf{Q}' + \mathbf{m}'$ is also nilpotent by the HPL, this means that $\mathcal{W}$ inherits an A_∞ structure, with a differential $Q' = pQi$ and multilinear products

$$m'_n = \pi_1 \mathbf{m}' \pi_n. \quad (4.31)$$

Next, include, in addition to the perturbation $\mathbf{m}$, the operator $\mathbf{U}$ defined in (2.41), and assume that $\mathbf{M} - \hbar \mathbf{U} = \mathbf{Q} + \mathbf{m} - \hbar \mathbf{U}$ is nilpotent (i.e., it defines a quantum A_∞ structure). In this case as well, the HPL applies, and a perturbed SDR can be constructed in exactly the same way as

$$\tilde{\mathbf{h}} \circ (\mathcal{TV}, \mathbf{M} - \hbar \mathbf{U}) \xleftrightarrow[\tilde{\mathbf{i}}]{\tilde{\mathbf{p}}} (\mathcal{TW}, \mathbf{M}' - \hbar \mathbf{U}' = \mathbf{Q}' + \mathbf{m}' - \hbar \mathbf{U}'). \quad (4.32)$$

Perturbed data are defined as

$$\mathbf{m}' = p\mathbf{m} \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} - \hbar \mathbf{U})} \mathbf{i}, \quad (4.33)$$

$$\mathbf{U}' = p\mathbf{U} \mathbf{i}, \quad (4.34)$$

$$\tilde{\mathbf{p}} = p \frac{1}{\mathbf{I} + (\mathbf{m} - \hbar \mathbf{U}) \mathbf{h}}, \quad (4.35)$$

$$\tilde{\mathbf{i}} = \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} - \hbar \mathbf{U})} \mathbf{i}, \quad (4.36)$$

$$\tilde{\mathbf{h}} = \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} - \hbar \mathbf{U})} \mathbf{h}. \quad (4.37)$$

The map $\mathbf{m}'$ is again a coderivation. This is slightly less obvious than in the A_∞ case: when one checks the coderivation condition (2.32) using the coproduct Δ , the extra term coming from $\mathbf{U}$ that splits as $\sum_s \tilde{e}^s \otimes' e_s$ seems to produce a deviation. However, this contribution vanishes due to the fact that $p\tilde{e}^s \otimes h e_s = 0$. This identity holds because if the projection $p\tilde{e}^s$ is non-zero (implying $ip\tilde{e}^s = \tilde{e}^s$), the corresponding dual basis element e_s must satisfy $ipe_s = e_s$. Consequently, the condition $hi = 0$ ensures that $he_s = 0$. Therefore, the nilpotency of $\mathbf{M}' - \hbar \mathbf{U}'$ obtained from the HPL means that a quantum A_∞ structure is transferred to $\mathcal{TV}$, with multilinear products given by

$$m_{g,n} = [\pi_1 \mathbf{m}' \pi_n]_{\text{coefficient of } \hbar^g}. \quad (4.38)$$

The case of L_∞ and quantum L_∞ algebras is completely analogous; one only replaces the tensor algebra with its symmetrized version.

4.2 Effective theory and on-shell amplitude

A representative application of homotopy transfer in QFT is integrating out degrees of freedom—for example, in the construction of effective theories [40] or in the computation of on-shell amplitudes [17]. As seen in Section 3, homotopy algebras correspond to QFT data. In particular, since the graded vector space V was coupled to the field (and antifield) degrees of freedom, it is here identified with the space of fields (and antifields) itself.

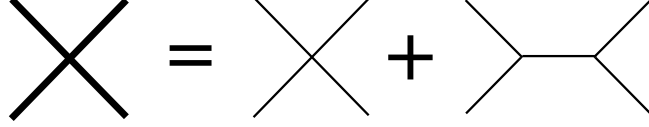


Figure 4.1: The thick diagram represents the four-point vertex of the effective theory. By contrast, the thin diagram on the right is the tree-level Feynman diagram of the original theory, written in terms of its vertices and propagators. The first term on the right-hand side of (4.40) corresponds to the four-point vertex shown on the right of the figure, whereas the second and third terms correspond to the diagram in which two three-point vertices are connected by a propagator.

As an SDR, take W to be a subspace of V , with $p : V \rightarrow W$ the projection and $i : W \rightarrow V$ the inclusion. In this setting, the contracting homotopy h has two physical roles. (i) Gauge fixing: the condition $h\Phi = 0$ specifies a gauge. (ii) Propagator for the eliminated modes: for a field $\Phi = (1 - ip)\Phi = (1 - P)\Phi$ outside W , the Hodge–Kodaira (4.4) decomposition and the gauge fixing condition give $hQ\Phi = \Phi$, so h acts as the inverse of the kinetic operator Q on the complement of W . In this situation, the homotopy algebra induced on $\mathcal{T}W$ by the HPL corresponds to a QFT defined on the restricted field space. Since the maps are quasi-isomorphisms, the cohomology is preserved, and the restricted theory is physically equivalent to the original theory. In this sense, the theory constructed from the transferred homotopy algebra

$$S' = \sum_{n=1}^{\infty} \frac{1}{n+1} \omega(ip\Phi, i\pi_1 \mathbf{M}' \frac{1}{1-p\Phi}), \quad (4.39)$$

is the effective theory of the original one.

Let us look concretely at the 3-linear product m'_3 of the A_∞ algebra induced on W . It can be written in terms of the original products m_2 and m_3 on V as

$$m'_3 = pm_3(i, i, i) - pm_2(hm_2(i, i), i) - pm_2(i, hm_2(i, i, i)). \quad (4.40)$$

This precisely describes the interaction in the effective action: the new four-point vertex is given by the original four-point interaction together with the diagram in which two original three-point interactions are connected by a propagator (see Fig. 4.1). Now we are working with an A_∞ algebra, so only tree-level diagrams appear. If the same computation is carried out for a quantum A_∞ algebra, loop diagrams are also reproduced, giving the full effective interaction.

The special case where W is chosen to be the Q -cohomology, i.e. when $pQ = 0$ and $Qi = 0$, is of particular physical importance. In this case, the induced homotopy algebra on W is called the minimal model; the differential vanishes $Q' = pQi = 0$. This minimal model is uniquely determined up to isomorphisms that preserve the homotopy algebra structure.

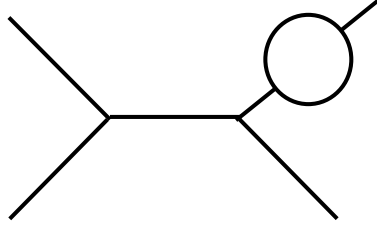


Figure 4.2: Diagrams of this type contribute to mass renormalization. Because an on-shell state necessarily propagates along the internal line, they generate IR divergences. In ordinary quantum field theory, such diagrams are removed.

The most important physical role of the minimal model is that it yields the on-shell amplitudes. In an A_∞ algebra, using the minimal model

$$M_{\min} = pm \frac{1}{I + hm} i, \quad (4.41)$$

the tree-level on-shell $n + 1$ -point amplitudes are given by

$$\omega(ip\Phi_1, i\pi_1 M_{\min} \pi_n(p\Phi_2 \otimes \cdots \otimes p\Phi_{n+1})), \quad (4.42)$$

where $\Phi_1, \dots, \Phi_{n+1}$ are the fields corresponding to external lines, and each of them is restricted to the Q -cohomology —i.e., to on-shell states—by the projection p . This reproduces the tree-level Feynman diagrams for exactly the same reason as in the explicit computation of the effective interaction above, where the same pattern of vertices and propagators appeared. We assume that the external lines, $\Phi_1, \dots, \Phi_{n+1}$, have general momenta. This means that on-shell states do not propagate through the internal propagators (or propagate only on a measure-zero set). The necessity of this assumption is related to the property $hP = Ph = 0$ satisfied by the contracting homotopy h , which corresponds to the propagator. This condition precisely signifies the absence of on-shell state propagation and is equivalent to isolating the principal value part of the propagator. In conventional quantum field theory, the internal propagator is given by the Feynman propagator, which differs from the principal value part only by the on-shell propagating component. We are currently operating under the assumption that this difference can be neglected.

For a quantum A_∞ algebra, the minimal model is

$$M_{\min} = pm \frac{1}{I + h(m - \hbar U)} i. \quad (4.43)$$

This construction successfully reproduces loop diagrams. However, it does not yet yield the correct on-shell amplitude, as it also generates unwanted terms, such as tadpole diagrams indicating a mismatch between the chosen background and the true vacuum, and diagrams

associated with mass renormalization, for example, the type illustrated in Fig. 4.2. This diagram, despite the external lines having general momenta, suffers from internal on-shell state propagation, which causes divergence. In ordinary quantum field theory, these diagrams are removed. Within the homotopy-algebra formalism, a procedure to remove such contributions and obtain the correct loop-level amplitudes is introduced in Section 7 in the discussion of the LSZ reduction formula.

5 Correlation Functions from Homotopy Algebras

This section explains correlation functions described in terms of quantum A_∞ algebras. The formula was first obtained for scalar fields in [35], then extended to Dirac fields in [36, 37], and finally generalized to arbitrary gauge-fixed fields in [38].

Throughout this section, we work in the Lorentzian signature. Thus, the action is given by

$$S = - \sum_{n=0}^{\infty} \frac{1}{n+1} \omega(\Phi, \pi_1 \mathbf{M} \pi_n \frac{1}{1-\Phi}), \quad (5.1)$$

and the operator $\mathbf{M} + i\hbar \mathbf{U}$ is nilpotent. We further assume that the projection $P = ip$ vanishes. In this case, the Hodge-Kodaira decomposition reduces to

$$Qh + hQ = \mathbb{I}. \quad (5.2)$$

This assumption implies that the Q -cohomology is trivial, which corresponds to considering a kinetic operator and propagator whose mass is shifted by the $i\epsilon$ prescription. For the purpose of discussing correlation functions, this causes no problem; however, but in situations where external lines play an essential role, such as in the LSZ reduction formula discussed later, the assumption has to be reconsidered.

In this section, among the labels of the basis e_s , the d -dimensional continuous variable corresponding to spacetime coordinates is explicitly written as x , and the basis is denoted by $e_a(x)$. The dual basis is denoted by $\tilde{e}^a(x)$, and the normalization is given by

$$\omega(e_a(x), \tilde{e}^b(y)) = (-1)^{e_a} \delta_a^b \delta^d(x - y). \quad (5.3)$$

The degree of $e_a(x)$ is assumed to be independent of x , so the sign factor is simply written as $(-1)^{e_a}$.

5.1 Formula for correlation functions

First, we discuss the vector space V corresponding to the space of fields. When $P = 0$, the Hodge-Kodaira decomposition (5.2) implies that V splits into the subspace projected by Qh ,

denoted $V^{(Qh)}$, and the subspace projected by hQ , denoted $V^{(hQ)}$:

$$V = V^{(Qh)} \oplus V^{(hQ)}. \quad (5.4)$$

Here, we assume that h is compatible with ω in the sense that

$$\omega(ha, b) = (-1)^a \omega(a, hb), \quad (5.5)$$

for all homogeneous $a, b \in V$. This assumption is adopted throughout the remainder of this text. Under this assumption, if a basis element $e_a(x)$ belongs to $V^{(Qh)}$, then its dual $\tilde{e}^a(x)$ belongs to $V^{(hQ)}$, and vice versa. Assume that Φ is gauge fixed by the condition $h\Phi = 0$; in other words, $\Phi \in V^{(hQ)}$. Accordingly, the field can be expanded as

$$\Phi = \sum_{e_a \in V^{(hQ)}} \int d^d x \phi^a(x) e_a(x). \quad (5.6)$$

For computing correlation functions, the quantity

$$\mathbf{f} = \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U})}, \quad (5.7)$$

plays a central role. It is defined as a formal series:

$$\mathbf{f} = \mathbf{I} - \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U}) + (\mathbf{h}(\mathbf{m} + i\hbar \mathbf{U}))^2 - \dots. \quad (5.8)$$

Using this $\mathbf{f}$, the correlation function is given by

$$\langle \phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n) \rangle = \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \mathbf{f} \mathbf{1}, \quad (5.9)$$

where

$$\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | = (-1)^n \langle \omega | (\tilde{e}^{a_1}(x_1) \otimes \pi_1) \otimes \cdots \otimes \langle \omega | (\tilde{e}^{a_n}(x_n) \otimes \pi_1). \quad (5.10)$$

Here, $\tilde{e}^{a_i}(x_i)$ is the dual of the basis vector $e_{a_i}(x_i)$ associated with the field $\phi^{a_i}(x_i)$. For gauge-fixed fields we have $e_{a_i}(x_i) \in V^{(hQ)}$, hence the dual satisfies $\tilde{e}^{a_i}(x_i) \in V^{(Qh)}$. The bra $\langle \omega |$ corresponding to ω is defined by

$$\langle \omega | (a \otimes b) = \omega(a, b), \quad (5.11)$$

for all $a, b \in V$. We comment here on the operator $\langle \omega | (\tilde{e}^a \otimes \pi_1)$. It acts as a map $V \rightarrow \mathbb{C}$. The tensor product of such maps, $\langle \omega | (\tilde{e}^{a_1} \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^{a_2} \otimes \pi_1)$, maps $V \otimes V \rightarrow \mathbb{C}$ and is defined for inputs $a, b \in V$ by

$$\langle \omega | (\tilde{e}^{a_1} \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^{a_2} \otimes \pi_1) a \otimes b = (-1)^{a(\tilde{e}^{a_2+1})} \langle \omega | (\tilde{e}^{a_1} \otimes a) \otimes \langle \omega | (\tilde{e}^{a_2} \otimes b). \quad (5.12)$$

Here, the sign factor on the right-hand side originates from the inputs moving past $\langle \omega |$ and $\tilde{e}^{a_2}$. We recall that $\langle \omega |$ is degree-odd. Since

$$\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \frac{1}{1 - \Phi} = \phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n), \quad (5.13)$$

the formula (5.9) may be regarded as its quantum version, and we write this concisely as

$$\mathbf{f} \mathbf{1} = \left\langle \frac{1}{1 - \Phi} \right\rangle. \quad (5.14)$$

The meaning of this expression is that, in the formal series $\frac{1}{1 - \Phi}$, each monomial $\phi^{a_1} \cdots \phi^{a_n}$ is replaced by the corresponding correlation function $\langle \phi^{a_1} \cdots \phi^{a_n} \rangle$.

Let us confirm that this formula (5.9) indeed gives correlation functions by working out a simple scalar-field example. First, choose a basis $c(x)$ of $V^{(hQ)}$ of even degree and a basis $d(x)$ of $V^{(Qh)}$ of odd degree, and define the symplectic form so that

$$\omega(c(x), d(y)) = \delta^d(x - y). \quad (5.15)$$

In other words, $d(x)$ is the dual of $c(x)$ with respect to ω . On this basis, the bivector U is

$$U = \int d^d x [d(x) \otimes c(x) + c(x) \otimes d(x)]. \quad (5.16)$$

Define Φ by

$$\Phi = \int d^d x \phi(x) c(x). \quad (5.17)$$

The kinetic operator Q acts by

$$Qc(x) = (-\partial^2 + m^2)d(x), \quad Qd(x) = 0. \quad (5.18)$$

Define the multilinear products m_n by

$$m_2(c(x), c(y)) = -\frac{g}{2} \delta^d(x - y) d(x), \quad m_2(\text{others}) = 0, \quad (5.19)$$

and take all higher products to vanish. With these choices, it is straightforward to check that the resulting data define a quantum A_∞ algebra. In this case, the action is a simple ϕ^3 theory:

$$S = - \int d^d x \left(\frac{1}{2} \phi(x) (-\partial^2 + m^2) \phi(x) - \frac{g}{6} \phi(x)^3 \right). \quad (5.20)$$

The contracting homotopy h corresponding to the propagator is given by

$$hd(x) = \int d^d y \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{k^2 + m^2} c(y) \quad hc(x) = 0. \quad (5.21)$$

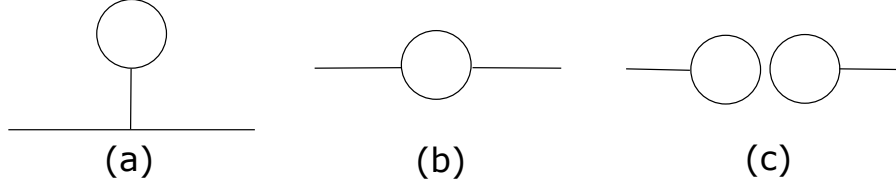


Figure 5.1: From left to right, (a) the diagrams correspond to the tadpole contribution, (b) the 1PI contribution, and (c) the disconnected contribution.

More precisely, the $i\epsilon$ prescription is implemented by shifting $m^2 \rightarrow m^2 - i\epsilon$ in both the kinetic operator and the propagator. With this choice, Q has no cohomology and the projector P vanishes. However, for notational simplicity, this shift is left implicit in what follows. Let us verify the correlation-function formula (5.9) in this setup. Consider the two-point correlation function

$$\langle \phi(x)\phi(y) \rangle = \langle \omega_2(x, y) | \pi_2 \mathbf{f}_1 \rangle. \quad (5.22)$$

$\pi_1 \mathbf{f}_2$ can be expanded as follows,

$$\begin{aligned} \pi_2 \mathbf{f}_1 = & -i\hbar \int d^d x c(x) \otimes hd(x) \\ & + (i\hbar)^2 \int d^d x d^d y \left[c(x) \otimes hm_2(hd(x), hm_2(c(y), hd(y))) \right. \\ & \quad + c(x) \otimes hm_2(hm_2(c(y), hd(y)), hd(x)) \\ & \quad + c(x) \otimes hm_2(c(y), hm_2(hd(x), hd(y))) \\ & \quad + c(x) \otimes hm_2(c(y), hm_2(hd(y), hd(x))) \\ & \quad \left. + hm_2(c(x), hd(x)) \otimes hm_2(c(y) \otimes hd(y)) \right] \\ & + \mathcal{O}(\hbar^3). \end{aligned} \quad (5.23)$$

The first term gives the tree-level two-point correlation function:

$$\begin{aligned} \langle \omega_2(x, y) | \int d^d z c(z) \otimes hd(z) &= \int d^d z \omega(d(x), c(z)) \omega(d(y), hd(z)) \\ &= \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{k^2 + m^2}. \end{aligned} \quad (5.24)$$

Among the contributions at order $\hbar^2$, the first four terms give the one-loop correction to the two-point correlation function. In particular, the top two correspond to tadpole contributions

(Fig.5.1 (a)),

$$\begin{aligned}
& \langle \omega_2(x, y) | \int d^d z d^d w [c(z) \otimes hm_2(hd(z), hm_2(c(w), hd(w))) \\
& \quad + c(z) \otimes hm_2(hm_2(c(w), hd(w)), hd(z))] \\
& = \int d^d z d^d w \omega(d(x), c(z)) [\omega(d(y), hm_2(hd(z), hm_2(c(w), hd(w)))) \\
& \quad + \omega(d(y), hm_2(hm_2(c(w), hd(w)), hd(z)))] \\
& = g \int d^d w \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + m^2} \omega(d(y), hm_2(hd(x), hd(w))) \\
& = \frac{g^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{m^2(k^2 + m^2)^2(l^2 + m^2)}, \tag{5.25}
\end{aligned}$$

while the bottom two give the contribution from the one-particle-irreducible (1PI) diagrams (Fig.5.1 (b)),

$$\begin{aligned}
& \langle \omega_2(x, y) | \int d^d z d^d w [c(z) \otimes hm_2(c(w), hm_2(hd(z), hd(w))) \\
& \quad + c(z) \otimes hm_2(c(w), hm_2(hd(w), hd(z)))] \\
& = \int d^d z d^d w \omega(d(x), c(z)) [\omega(d(y), hm_2(c(w), hm_2(hd(z), hd(w)))) \\
& \quad + \omega(d(y), hm_2(c(w), hm_2(hd(w), hd(z)))] \\
& = g \int d^d z d^d w \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{e^{ik \cdot (x-z)}}{k^2 + m^2} \frac{e^{il \cdot (w-z)}}{l^2 + m^2} \omega(d(y), hm_2(c(w), hd(z))) \\
& = \frac{g^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{(k^2 + m^2)^2(l^2 + m^2)((k+l)^2 + m^2)}. \tag{5.26}
\end{aligned}$$

The remaining contributions at order $\hbar^2$ correspond to disconnected two-loop diagrams (Fig.5.1 (c)),

$$\begin{aligned}
& \langle \omega_2(x, y) | \int d^d z d^d w hm_2(c(z), hd(z)) \otimes hm_2(c(w) \otimes hd(w)) \\
& = \int d^d z d^d w \omega(d(x), hm_2(c(z), hd(z))) \omega(d(y), hm_2(c(w) \otimes hd(w))) \\
& = \frac{g^2}{4} \int d^d z d^d w \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{l^2 + m^2} \omega(d(x), hd(z)) \omega(d(y), hd(w)) \\
& = \left(\frac{g}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{m^2(k^2 + m^2)} \right)^2. \tag{5.27}
\end{aligned}$$

In this way, the formula (5.9) reproduces the correlation functions via the Feynman-diagram expansion.

5.2 Algebraic Schwinger-Dyson equation

An important ingredient in the correlation function formula is $\mathbf{f}\mathbf{1}$. This quantity satisfies the relation

$$(\mathbf{M} + i\hbar\mathbf{U})\mathbf{f}\mathbf{1} = 0. \quad (5.28)$$

We refer to this physically significant identity as the algebraic Schwinger–Dyson equation, as it generates the standard Schwinger–Dyson equations. Before turning to that derivation, we first prove this identity. Let us compute the commutator of $\mathbf{Q}$ with $\mathbf{f}$:

$$\begin{aligned} [\mathbf{Q}, \mathbf{f}] &= \left[\mathbf{Q}, \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})} \right], \\ &= -\frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})} [\mathbf{Q}, \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})] \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})}, \\ &= -\frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})} ((\mathbf{m} + i\hbar\mathbf{U}) + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})^2) \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})}, \\ &= -\frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})} (\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U}))(\mathbf{m} + i\hbar\mathbf{U}) \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar\mathbf{U})}, \\ &= -(\mathbf{m} + i\hbar\mathbf{U})\mathbf{f}, \end{aligned} \quad (5.29)$$

where we used quantum A_∞ relations at the third equality. This yields

$$(\mathbf{M} + i\hbar\mathbf{U})\mathbf{f} = (\mathbf{Q} + \mathbf{m} + i\hbar\mathbf{U})\mathbf{f} = \mathbf{f}\mathbf{Q}. \quad (5.30)$$

Since $\mathbf{Q}\mathbf{1} = 0$, right-multiplying both sides by $\mathbf{1}$ yields (5.28).

We now demonstrate that the usual Schwinger–Dyson equations follow from the algebraic Schwinger–Dyson equation. Act on the algebraic Schwinger–Dyson identity from the left with

$$(\mathbb{I}^{\otimes n} \otimes Qh)\pi_{n+1}. \quad (5.31)$$

Using the representation (5.14), this can be rewritten as

$$\left\langle \Phi^{\otimes n} \otimes Qh\pi_1 \mathbf{M} \frac{1}{1 - \Phi} \right\rangle + i\hbar \sum_{k=0}^{n-1} \sum_a \int d^d x \left\langle \Phi^{\otimes k} \otimes e_a(x) \otimes \Phi^{\otimes n-k-1} \otimes Qh\tilde{e}^a(x) \right\rangle = 0. \quad (5.32)$$

Here e_a is restricted to $V^{(Qh)}$ (by the projector Qh), while $\tilde{e}^a$ is restricted to $V^{(hQ)}$. By acting with

$$(-1)^{e_{a_{n+1}}+1} \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \otimes \langle \omega | (e_{a_{n+1}}(x_{n+1}) \otimes \pi_1), \quad (5.33)$$

where $e_{a_{n+1}} \in V^{(hQ)}$, one can eliminate the basis elements and extract the coefficients, which are the correlation functions. Let us compute the contributions of each term. First, the first term can be written as

$$(-1)^{e_{a_{n+1}}+1} \left\langle \phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n) \omega(e_{a_{n+1}}(x_{n+1}), \pi_1 \mathbf{M} \frac{1}{1 - \Phi}) \right\rangle. \quad (5.34)$$

Since

$$\omega\left(e_a(x), \pi_1 \mathbf{M} \frac{1}{1-\Phi}\right) = (-1)^{e_a+1} \frac{\delta S}{\delta \phi^a(x)}, \quad (5.35)$$

the expression (5.34) is its quantum counterpart and can be written as

$$\left\langle \phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n) \frac{\delta S}{\delta \phi^{a_{n+1}}(x_{n+1})} \right\rangle. \quad (5.36)$$

On the other hand, the second term can be calculated as

$$i\hbar \sum_{k=0}^{n-1} (-1)^{e_{a_{n+1}}(e_{a_{k+2}}+\cdots+e_{a_{n+1}})+1} \delta_{a_{n+1}}^{a_{k+1}} \delta^d(x_{k+1} - x_{n+1}) \left\langle \phi^{a_1}(x_1) \cdots \widehat{\phi^{a_{k+1}}(x_{k+1})} \cdots \phi^{a_n}(x_n) \right\rangle. \quad (5.37)$$

Here, $\widehat{\phi^{a_{k+1}}(x_{k+1})}$ denotes the exclusion of $\phi^{a_{k+1}}(x_{k+1})$. Ultimately, combining the two contributions, we obtain

$$\begin{aligned} 0 = & \left\langle \phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n) \frac{\delta S}{\delta \phi^{a_{n+1}}(x_{n+1})} \right\rangle \\ & - i\hbar \sum_{k=0}^{n-1} (-1)^{e_{a_{n+1}}(e_{a_{k+2}}+\cdots+e_{a_{n+1}})} \delta_{a_{n+1}}^{a_{k+1}} \delta^d(x_{k+1} - x_{n+1}) \left\langle \phi^{a_1}(x_1) \cdots \widehat{\phi^{a_{k+1}}(x_{k+1})} \cdots \phi^{a_n}(x_n) \right\rangle. \end{aligned} \quad (5.38)$$

This is precisely the Schwinger-Dyson equation itself. Note that the degree of e_a is the same as that of the associating field ϕ^a .

The Algebraic Schwinger-Dyson equation (5.28) was an identity that holds when $\mathbf{f}\mathbf{1}$ is given by (5.7); conversely, if an element $\mathbf{f}\mathbf{1}$ that satisfies (5.28) exists, a correlation function that satisfies the Schwinger-Dyson equation can be defined from it. This equation can also be interpreted as the quantum version of the classical equation of motion (3.22). Since $\mathbf{M} + i\hbar \mathbf{U}$ is nilpotent, its cohomology defines physical space in the quantum theory. However, the specific introduction of $\mathbf{f}\mathbf{1}$ above assumed the existence of h , which corresponds to gauge fixing; therefore, the cohomology is defined by $(\mathbf{M} + i\hbar \mathbf{U})$ -closed. If one could construct a solution to the algebraic Schwinger-Dyson equation without introducing h , it would mean that the correlation function is defined without gauge fixing; however, the author is, at least, unaware of such an algebraic construction method. Furthermore, since $\mathbf{f}$ is given merely by a formal power series expansion here, its non-perturbative existence is unknown. The non-perturbative construction of solutions to the algebraic Schwinger-Dyson equation is also an important challenge. Such an attempt has been made in the context of zero-dimensional field theory in [56].

6 Ward-Takahashi Identity from Homotopy Algebras

In quantum field theory, the Ward-Takahashi identity

$$\langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \partial_\mu j^\mu(x) \rangle = -i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \phi^{a_1}(x_1) \dots \delta \phi^{a_i}(x_i) \dots \phi^{a_n}(x_n) \rangle, \quad (6.1)$$

is recognized as one of the most crucial formulas regarding symmetries. This identity is valid only under the condition that no anomaly is present. The remainder of this section investigates how this foundational formula is expressed within the framework of homotopy algebras. The content of this section is primarily based on [38].

6.1 Infinitesimal symmetry

First, let us examine how the transformation of fields and invariance of the action are expressed in the language of homotopy algebras. To express the transformation, we introduce a degree-even multilinear map $A_n : V^{\otimes n} \rightarrow V$, and the transformation is represented by

$$\delta \Phi = \sum_{e_a \in V^{(hQ)}} \int d^d x \delta \phi^a(x) e_a(x) = A_0(\mathbf{1}) + A_1(\Phi) + A_2(\Phi, \Phi) + \dots. \quad (6.2)$$

Furthermore, we assume that the operator A_n is cyclic:

$$\omega(\Phi_1, A_n(\Phi_2, \dots, \Phi_{n+1})) = -\omega(A_n(\Phi_1, \dots, \Phi_n), \Phi_{n+1}). \quad (6.3)$$

Let us denote the coderivations induced by A_n as $\mathbf{A}_n$. We then construct the operator $\mathbf{A}$ as the infinite sum of these components:

$$\mathbf{A} = \sum_{n=0}^{\infty} \mathbf{A}_n. \quad (6.4)$$

As a result, the transformation given in the equation (6.2) admits a compact representation:

$$\delta \Phi = \pi_1 \mathbf{A} \frac{1}{1 - \Phi}. \quad (6.5)$$

Henceforth, taking the transformation to be infinitesimal, the variation of the action is expressed as

$$\delta S = - \sum_{n=0}^{\infty} \frac{1}{n+1} \omega \left(\Phi, \pi_1 [\mathbf{M}, \mathbf{A}] \pi_n \frac{1}{1 - \Phi} \right). \quad (6.6)$$

From this result, we establish that the condition sufficient for the infinitesimal transformation generated by $\mathbf{A}$ to be a symmetry transformation is that $\mathbf{A}$ satisfies the commutation relation:

$$[\mathbf{M}, \mathbf{A}] = 0. \quad (6.7)$$

In the subsequent discussion, we restrict our attention solely to symmetry transformations generated by $\mathbf{A}$ that satisfy this condition. Furthermore, we stipulate that A_n maps into $V^{(hQ)}$ if all of its inputs are elements of $V^{(hQ)}$. Specifically, this means that A_n must satisfy the condition:

$$h A_n(\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_n) = 0, \quad (6.8)$$

for all $\Phi_1, \dots, \Phi_n \in V^{(hQ)}$. This assumption ensures that the transformation is compatible with the gauge fixing condition. To properly define A_n cyclically, it is essential to note that non-trivial values must typically be assigned even when only one of the inputs belongs to $V^{(Qh)}$.

6.2 Conserved current and conserved charge

Prior to a detailed discussion of the homotopy description of the Ward-Takahashi identity, we first address the homotopy description of the conserved current and conserved charge. As it is well known, when the action remains invariant under an infinitesimal transformation, a conserved current $j^\mu(x)$ associated with that transformation emerges. This conserved current obeys the conservation law:

$$\partial_\mu j^\mu(x) \approx 0, \quad (6.9)$$

where the symbol $\approx$ indicates that the equality holds up to the equation of motion. In the standard procedure of quantum field theory, the conserved current is derived by replacing the global infinitesimal parameter ϵ with a local function $\epsilon(x)$. In this context, the infinitesimal transformation is modified as follows: For a symmetry transformation $\delta\phi(x)$, we define $\delta^\epsilon\phi(x)$ such that:

$$\delta^\epsilon\phi(x) = \epsilon(x) \delta\phi(x), \quad (6.10)$$

where $\epsilon(x)$ is an arbitrary infinitesimal function. The conserved current is subsequently obtained from the variation of the action under this localized transformation, which yields the expression:

$$\delta^\epsilon S = - \int d^d x \epsilon(x) \partial_\mu j^\mu(x). \quad (6.11)$$

We now translate the aforementioned procedure into the framework of homotopy algebras. As previously established, the symmetry transformation is described by a degree-even cyclic operator A_n that satisfies the condition (6.7). Correspondingly, we define a degree-even cyclic operator A_n^ϵ with respect to the localized transformation $\delta^\epsilon\phi(x)$:

$$\delta^\epsilon\Phi = \sum_{e_a \in V^{(hQ)}} \int d^d x \epsilon(x) \delta\phi^a(x) e_a(x) = A_0^\epsilon(\mathbf{1}) + A_1^\epsilon(\Phi) + A_2^\epsilon(\Phi, \Phi) + \dots, \quad (6.12)$$

The coderivation induced by A_n^ϵ is denoted as $\mathbf{A}_n^\epsilon$, and we define the coderivation $\mathbf{A}^\epsilon$ as the summation:

$$\mathbf{A}^\epsilon = \sum_{n=0}^{\infty} \mathbf{A}_n^\epsilon. \quad (6.13)$$

Let us introduce the operator $\mathcal{O}_\epsilon$ defined by

$$\mathcal{O}_\epsilon = \sum_a (-1)^{\tilde{e}^a} \int d^d x \epsilon(x) e_a(x) \langle \omega | (\tilde{e}^a(x) \otimes \mathbb{I}). \quad (6.14)$$

Its action on the basis elements $e_a(x)$ is

$$\mathcal{O}_\epsilon e_a(x) = \epsilon(x) e_a(x). \quad (6.15)$$

Utilizing this operator, the multilinear map A_n^ϵ , which generates the coderivation $\mathbf{A}_n^\epsilon$, can be expressed as

$$A_n^\epsilon(\Phi_1, \dots, \Phi_n) = \mathcal{O}_\epsilon A_n(\Phi_1, \dots, \Phi_n), \quad (6.16)$$

for all $\Phi_1, \dots, \Phi_n \in V^{(hQ)}$. In addition, the cyclic requirement for A_n^ϵ necessitates that it be defined as

$$A_n^\epsilon(\Phi_1, \dots, \tilde{\Phi}, \dots, \Phi_{n-1}) = A_n(\Phi_1, \dots, \mathcal{O}_\epsilon \tilde{\Phi}, \dots, \Phi_{n-1}), \quad (6.17)$$

for $\Phi_1, \dots, \Phi_{n-1} \in V^{(hQ)}$ and $\tilde{\Phi} \in V^{(Qh)}$. The operator $\mathbf{A}^\epsilon$ no longer satisfies the symmetry condition: $[\mathbf{M}, \mathbf{A}^\epsilon] \neq 0$. The variation of the action under this transformation is expressed as

$$\begin{aligned} \delta_\epsilon S &= - \sum_{n=0}^{\infty} \frac{1}{n+1} \omega \left(\Phi, \pi_1 [\mathbf{M}, \mathbf{A}^\epsilon] \pi_n \frac{1}{1-\Phi} \right), \\ &= - \sum_{n=0}^{\infty} \frac{1}{n+1} \langle \omega | (\mathbb{I} \otimes \pi_1 [\mathbf{M}, \mathbf{A}^\epsilon] \pi_n) \frac{1}{1-\Phi}. \end{aligned} \quad (6.18)$$

Given that this expression is the homotopy algebra counterpart of the equation (6.11), we obtain the conservation law in the following form:

$$\int d^d x \epsilon(x) \partial_\mu j^\mu(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \langle \omega | (\mathbb{I} \otimes \pi_1 [\mathbf{M}, \mathbf{A}^\epsilon] \pi_n) \frac{1}{1-\Phi}. \quad (6.19)$$

From the completeness of basis (2.17), the cyclicity of $\mathbf{M}$ and $\mathbf{A}^\epsilon$ and the definition of $\mathcal{O}_\epsilon$, this expression can be rewritten as

$$\int d^d x \epsilon(x) \partial_\mu j^\mu(x) = - \sum_{e_a \in V^{(hQ)}} \int d^d x \epsilon(x) \left(\langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \frac{1}{1-\Phi}. \quad (6.20)$$

Here, we note that since $\mathbf{A}$ maps to $V^{(hQ)}$ when all of its inputs belong to $V^{(hQ)}$, the element $\tilde{e}^a(x)$ is restricted to $V^{(hQ)}$, which implies that $e_a(x)$ is restricted to $V^{(hQ)}$. To be fully precise, we restore the notation for the product ∇ and the coproduct Δ , which were previously suppressed. The complete expression for the quantity under consideration is

$$- \sum_{e_a \in V^{(hQ)}} \int d^d x \, \epsilon(x) \nabla \left(\langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes' \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \Delta \frac{1}{1 - \Phi}. \quad (6.21)$$

The notation for the product ∇ and coproduct Δ is suppressed wherever the context is clear. From this expression, the divergence of the conserved current, $\partial_\mu j^\mu(x)$, is shown to be given in homotopy algebraic form by

$$\partial_\mu j^\mu(x) = - \sum_{e_a \in V^{(hQ)}} \left(\langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \frac{1}{1 - \Phi}, \quad (6.22)$$

The right-hand side is proportional to the equation of motion (3.22). This leads to the re-derivation of conservation law in the context of homotopy algebras. As a consistency check, the right-hand side must vanish upon integration. This fact can be verified as follows. By utilizing the completeness of the basis (2.17), the integral of the right-hand side can be expressed as

$$\langle \omega | (\pi_1 \mathbf{A} \otimes \pi_1 \mathbf{M}) \frac{1}{1 - \Phi}. \quad (6.23)$$

By exploiting the cyclicity property, this result is equivalent to

$$\sum_{n=0}^{\infty} \omega(\Phi, [\mathbf{M}, \mathbf{A}] \pi_n \frac{1}{1 - \Phi}). \quad (6.24)$$

Since $\mathbf{A}$ and $\mathbf{M}$ commute, the expression is zero.

To obtain the conserved current from the expression for the current divergence (6.22), one needs only to integrate after multiplying by a step function in time:

$$H(t_0) = \int_{t=t_0} d^{d-1} x \, J^0 = - \int d^d x \, \Theta(t - t_0) \partial_\mu J^\mu(x). \quad (6.25)$$

This allows us to obtain the conserved charge on the time slice at $t = t_0$. This procedure is realized in the homotopy algebra formalism by the following definitions. The operator $\mathcal{O}_{\Theta t_0}$ is defined by replacing $\epsilon(x)$ with $\Theta(t - t_0)$ in the definition of $\mathcal{O}_\epsilon$. Consequently, $A_n^{\Theta t_0}$, its coderivation $\mathbf{A}_n^{\Theta t_0}$, and the operator $\mathbf{A}^{\Theta t_0}$ are defined analogously. Utilizing this, $\mathcal{Q}$ can be expressed as

$$\begin{aligned} H(t_0) &= \sum_{e_a \in V^{(hQ)}} \int d^d x \left(\langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}^{\Theta t_0}) \right) \frac{1}{1 - \Phi}, \\ &= \langle \omega | (\pi_1 \mathbf{A}^{\Theta t_0} \otimes \pi_1 \mathbf{M}) \frac{1}{1 - \Phi}, \end{aligned} \quad (6.26)$$

and further application of cyclicity allows it to be written as

$$H(t_0) = \sum_{n=0}^{\infty} \omega(\Phi, [\mathbf{M}, \mathbf{A}^{\Theta_{t_0}}] \pi_n \frac{1}{1-\Phi}). \quad (6.27)$$

This expression is identical to the result obtained in [57]. In that work, the formula was derived from a symplectic form constructed within the context of the covariant phase space formalism [58]. Furthermore, a more general approach was adopted in [57], where a sigmoid function was utilized instead of a step function. At first glance, Expression (6.26) appears to be proportional to the equation of motion; however, this is not the case. The expression actually contains a divergence proportional to the infinite volume of spacetime, which originates from the step function. Consequently, even if Φ is chosen to be a solution to the equation of motion, the expression does not vanish. Proper treatment of this requires the introduction of a cutoff. For a detailed discussion on this matter, we refer the reader to [58]. When the time slice undergoes a small shift, $t_0 \rightarrow t_0 + \delta t_0$, the corresponding change in the conserved charge is obtained by substituting a delta function for the step function in the expression (6.26). Because this result is a local quantity, it avoids the spacetime volume divergence and is directly proportional to the equation of motion. In other words, the charge is independent of time up to the equations of motion:

$$\frac{\partial}{\partial t_0} H(t_0) \approx 0. \quad (6.28)$$

In the subsequent subsection, we quantize the classical analysis presented here, focusing particularly on the divergence of the conserved current, and obtain the Ward-Takahashi identity.

6.3 Ward-Takahashi identity

Since the formula for the divergence of the conserved current, $\partial_\mu j^\mu(x)$, has been established, the subsequent discussion focuses on its expectation value $\langle \partial_\mu j^\mu(x) \rangle$. Since $\mathbf{f}\mathbf{1}$ is identified with the quantum version of $\frac{1}{1-\Phi}$, the calculation of the expectation value $\langle \partial_\mu j^\mu(x) \rangle$ is achieved by the direct replacement of $\frac{1}{1-\Phi}$ with $\mathbf{f}\mathbf{1}$ in the expression (6.22):

$$\langle \partial_\mu j^\mu(x) \rangle = - \sum_{e_a \in V^{(hQ)}} (\langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A})) \mathbf{f}\mathbf{1}. \quad (6.29)$$

This quantity is expected to vanish in the absence of anomalies, but must be non-zero when an anomaly is present. We now investigate this result from the perspective of homotopy algebras. Utilizing the fact that $\mathbf{M}$ is a coderivation, this expression can be written as

$$\begin{aligned} \langle \partial_\mu j^\mu(x) \rangle = & - \sum_{e_a \in V^{(hQ)}} (-1)^{e_a} (\langle \omega | (e_a(x) \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A})) \mathbf{M} \mathbf{f}\mathbf{1}, \\ & + \sum_{e_a \in V^{(hQ)}} (-1)^{e_a} (\langle \omega | (e_a(x) \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} \mathbf{M})) \mathbf{f}\mathbf{1}, \end{aligned} \quad (6.30)$$

however, the second term vanishes because both arguments of the first symplectic form belong to $V^{(hQ)}$. Here, applying the algebraic Schwinger-Dyson equation allows us to replace $\mathbf{M}$ with $-i\hbar\mathbf{U}$, enabling this expression to be written as

$$\begin{aligned}
& \langle \partial_\mu j^\mu(x) \rangle \\
&= i\hbar \sum_{e_a \in V^{(hQ)}} (-1)^{e_a} \left(\langle \omega | (e_a(x) \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \mathbf{U} \mathbf{f1}, \\
&= i\hbar \sum_{e_a \in V^{(hQ)}} \sum_{\tilde{e}^b \in V^{(Qh)}} \int d^d y (-1)^{e_a + e_a \tilde{e}^b} \left(\langle \omega | (e_a(x) \otimes \tilde{e}^b(y)) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_b(y)) \right) \mathbf{f1}.
\end{aligned} \tag{6.31}$$

Furthermore, by utilizing the normalization of the basis, we finally obtain

$$\langle \partial_\mu j^\mu(x) \rangle = i\hbar \sum_{e_a \in V^{(hQ)}} \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \mathbf{f1}. \tag{6.32}$$

Generally, the right-hand side of this equation contains divergences. For example, if the symmetry transformation $\mathbf{A}$ is simply linear and corresponds to scaling by a constant, this quantity is proportional to $\delta^d(0)$. Consequently, some appropriate regularization is essential. The specifics of regularization are discussed in detail later. Based on these observations, this quantity is expected to represent anomalies. We demonstrate in a later section that this quantity is consistent with known anomalies when applied to specific cases.

Extending the equation (6.19), the expectation value between the fields and $\partial_\mu j^\mu(x)$ is given by

$$\begin{aligned}
& \langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \partial_\mu j^\mu(x) \rangle \\
&= - \sum_{e_a \in V^{(hQ)}} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \mathbf{f1}.
\end{aligned} \tag{6.33}$$

The Ward-Takahashi identity is derivable from this expression. As previously established, the coderivation $\mathbf{M}$ can be shifted to the position immediately preceding $\mathbf{f1}$. Consequently, by applying the Algebraic Schwinger-Dyson equation, we obtain the following expression:

$$\begin{aligned}
& \langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \partial_\mu j^\mu(x) \rangle \\
&= i\hbar \sum_{e_a \in V^{(hQ)}} (-1)^{e_a} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \mathbf{U} \mathbf{f1}.
\end{aligned} \tag{6.34}$$

Among the outputs of $\mathbf{U}$, the basis element belonging to $V^{(Qh)}$ must enter the form $\langle \omega | (e_a(x) \otimes \pi_1)$, otherwise they vanish; however, the basis element belonging to the other space, $V^{(hQ)}$, have

two contributions: one that enters $\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n$ and one that enters $\langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A})$. Consequently, this expression can be written as

$$\begin{aligned} & \langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \partial_\mu j^\mu(x) \rangle \\ &= i\hbar \sum_{e_a \in V(hQ)} (-1)^{e_a} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \mathbf{e}_a(x) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \mathbf{f} \mathbf{1} \\ &+ i\hbar \sum_{e_a \in V(hQ)} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} \mathbf{e}_a(x)) \right) \mathbf{f} \mathbf{1}. \end{aligned} \quad (6.35)$$

The second term on the right-hand side corresponds to the anomaly. We proceed to reformulate the first term to elucidate its physical significance. This term is decomposed into a summation over n terms, according to the definition of $\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) |$, where $e_a(x)$ is placed in one of the n positions:

$$\begin{aligned} i\hbar \sum_{e_a \in V(hQ)} \sum_{i=1}^n (-1)^{n+e_a+e_a(e_{a_{i+1}}+\dots+e_{a_n})} & \left(\langle \omega | (\tilde{e}^{a_1}(x_1) \otimes \pi_1) \otimes \dots \otimes \langle \omega | (\tilde{e}^{a_i}(x_i) \otimes e_a(x)) \otimes \dots \right. \\ & \left. \otimes \langle \omega | (\tilde{e}^{a_n} \otimes \pi_1) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \right) \mathbf{f} \mathbf{1}. \end{aligned} \quad (6.36)$$

Following a careful treatment of the sign factors, and utilizing the normalization condition (5.3), we arrive at the simplified expression:

$$\begin{aligned} -i\hbar \sum_{i=1}^n (-1)^n \delta^d(x - x_i) & \left(\langle \omega | (\tilde{e}^{a_1}(x_1) \otimes \pi_1) \otimes \dots \otimes \langle \omega | (\tilde{e}^{a_i}(x_i) \otimes \pi_1 \mathbf{A}) \otimes \dots \right. \\ & \left. \otimes \langle \omega | (\tilde{e}^{a_n}(x_n) \otimes \pi_1) \right) \mathbf{f} \mathbf{1}, \end{aligned} \quad (6.37)$$

Applying the definition of $\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) |$ once more, the expression transforms into

$$-i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f} \mathbf{1}. \quad (6.38)$$

Recognizing that $\mathbf{A}$ generates the infinitesimal transformation (6.5) and that the replacement $\frac{1}{1-\Phi} \rightarrow \mathbf{f} \mathbf{1}$ implements the expectation value, this term is identified as a contact term equivalent to

$$\begin{aligned} & -i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f} \mathbf{1} \\ &= -i\hbar \sum_{i=1}^n \int d^d x \delta^d(x - x_i) \langle \phi^{a_1}(x_1) \dots \delta \phi^{a_i}(x_i) \dots \phi^{a_n}(x_n) \rangle. \end{aligned} \quad (6.39)$$

In summary, the Ward-Takahashi identity is formulated in terms of homotopy algebras as follows:

$$\begin{aligned}
& - \sum_{e_a \in V(hQ)} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A})) \right) \mathbf{f} \mathbf{1} \\
& = -i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f} \mathbf{1} \\
& \quad + i\hbar \sum_{e_a \in V(hQ)} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \right) \mathbf{f} \mathbf{1}. \tag{6.40}
\end{aligned}$$

In this expression, the left-hand side represents the expectation value between the fields and the derivative of the current $\partial_\mu j^\mu(x)$. The first term on the right-hand side corresponds to the contact term, while the second term represents the contribution of the anomaly.

6.4 Examples

In this subsection, we provide several examples of the formula (6.40). We specifically present two cases: the U(1) symmetry in complex scalar field theories and the chiral U(1) symmetry in vector-like gauge theories. We note that the latter case serves as an example where an anomaly exists.

6.4.1 The U(1) symmetry in complex scalar field theories

We begin by considering the U(1) symmetry within a complex scalar field theory. The action under consideration is defined as

$$S = - \int d^d x \left[\varphi^\dagger(x) (-\partial^2 + m^2) \varphi(x) + \sum_{n=2}^{\infty} \frac{g_n}{2n} (\varphi^\dagger(x) \varphi(x))^n \right]. \tag{6.41}$$

This action exhibits invariance under the following U(1) transformation:

$$\begin{aligned}
\varphi(x) & \rightarrow \varphi'(x) = e^{-i\alpha} \varphi(x), \\
\varphi^\dagger(x) & \rightarrow \varphi'^\dagger(x) = e^{i\alpha} \varphi^\dagger(x).
\end{aligned} \tag{6.42}$$

The infinitesimal form of this transformation, $\varphi(x) \rightarrow \varphi(x) + \delta\varphi(x)$ and $\varphi^\dagger(x) \rightarrow \varphi^\dagger(x) + \delta\varphi^\dagger(x)$, is given by

$$\delta\varphi(x) = -i\alpha\varphi(x), \quad \delta\varphi^\dagger(x) = i\alpha\varphi^\dagger(x). \tag{6.43}$$

In terms of quantum A_∞ algebras, we consider an action of the form

$$S = -\frac{1}{2} \omega(\Phi, Q\Phi) - \sum_{n=1}^{\infty} \frac{1}{2n} \omega(\Phi, m_{2n-1}(\Phi, \dots, \Phi)), \tag{6.44}$$

where

$$\Phi = \int d^d x [\varphi(x) c^\dagger(x) + \varphi^\dagger(x) c(x)]. \quad (6.45)$$

Here, the degree-even basis elements that belong to $V^{(hQ)}$ are denoted by $c(x)$ and $c^\dagger(x)$. In addition to these, we introduce the degree-odd basis elements d and $d^\dagger$ belonging to $V^{(Qh)}$. The non-vanishing symplectic form is defined by

$$\begin{aligned} \omega(c(x_1), d^\dagger(x_2)) &= \delta^d(x_1 - x_2), \\ \omega(c^\dagger(x_1), d(x_2)) &= \delta^d(x_1 - x_2), \\ \omega(d(x_2), c^\dagger(x_1)) &= -\delta^d(x_1 - x_2), \\ \omega(d^\dagger(x_2), c(x_1)) &= -\delta^d(x_1 - x_2). \end{aligned} \quad (6.46)$$

Corresponding to the notation of the bases introduced thus far, the basis $e_a(x)$ is given by

$$\{e_a(x)\}_{a=1,\dots,4} = \{c(x), c^\dagger(x), d(x), d^\dagger(x)\}, \quad (6.47)$$

and its dual basis $\tilde{e}^a(x)$ is given by

$$\{\tilde{e}^a(x)\}_{a=1,\dots,4} = \{d^\dagger(x), d(x), c(x), c^\dagger(x)\}. \quad (6.48)$$

We define the operator Q by specifying its action on the basis elements as

$$\begin{aligned} Q c(x) &= (-\partial^2 + m^2) d(x), \quad Q d(x) = 0, \\ Q c^\dagger(x) &= (-\partial^2 + m^2) d^\dagger(x), \quad Q d^\dagger(x) = 0. \end{aligned} \quad (6.49)$$

This definition allows the kinetic and mass terms of the action to be expressed concisely via the symplectic form

$$-\frac{1}{2} \omega(\Phi, Q\Phi) = - \int d^d x \varphi^\dagger(x) (-\partial^2 + m^2) \varphi(x). \quad (6.50)$$

To define the operators m_{2n-1} , which describe the interaction terms, let the $2n-1$ inputs be denoted by $a_1(x_1), \dots, a_{2n-1}(x_{2n-1})$. The definition of m_{2n-1} depends on the composition of its inputs:

1. If $a_1, \dots, a_{2n-1}$ contains n instances of c and $n-1$ instances of $c^\dagger$, we define:

$$m_{2n-1}(a_1(x_1), \dots, a_{2n-1}(x_{2n-1})) = \frac{n!(n-1)!}{2(2n-1)!} g_n \delta^d(x_1 - x_2) \cdots \delta^d(x_1 - x_{2n-1}) d(x_1), \quad (6.51)$$

2. If $a_1, \dots, a_{2n-1}$ contains $n-1$ instances of c and n instances of $c^\dagger$, we define:

$$m_{2n-1}(a_1(x_1), \dots, a_{2n-1}(x_{2n-1})) = \frac{n!(n-1)!}{2(2n-1)!} g_n \delta^d(x_1 - x_2) \cdots \delta^d(x_1 - x_{2n-1}) d^\dagger(x_1). \quad (6.52)$$

3. Otherwise, m_{2n-1} is defined to vanish:

$$m_{2n-1}(a_1(x_1), \dots, a_{2n-1}(x_{2n-1})) = 0. \quad (6.53)$$

Associated with the operator Q , we introduce the contracting homotopy h , defined by its action on the basis elements as

$$\begin{aligned} h c(x) &= 0, & h d(x) &= \int d^d y \Delta(x-y) c(y), \\ h c^\dagger(x) &= 0, & h d^\dagger(x) &= \int d^d y \Delta(x-y) c^\dagger(y), \end{aligned} \quad (6.54)$$

where $\Delta(x-y)$ is the scalar propagator given by

$$\Delta(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 + m^2}. \quad (6.55)$$

Next, we examine the symmetry transformations within the homotopy algebra formalism. We introduce the degree-even operator A , whose action on the basis elements is defined as

$$\begin{aligned} A c(x) &= i \alpha c(x), & A d(x) &= i \alpha d(x), \\ A c^\dagger(x) &= -i \alpha c^\dagger(x), & A d^\dagger(x) &= -i \alpha d^\dagger(x). \end{aligned} \quad (6.56)$$

This operator accurately reproduces the infinitesimal symmetry transformation (6.43), as demonstrated by the following calculation:

$$A\Phi = \int d^d x [\varphi(x) A c^\dagger(x) + \varphi^\dagger(x) A c(x)] \quad (6.57)$$

$$\begin{aligned} &= \int d^d x \{ [-i \alpha \varphi(x)] c^\dagger(x) + [i \alpha \varphi^\dagger(x)] c(x) \}, \\ &= \int d^d x [\delta \varphi(x) c^\dagger(x) + \delta \varphi^\dagger(x) c(x)], \\ &= \delta \Phi. \end{aligned} \quad (6.58)$$

It is straightforward to verify that the symmetry condition, $[\mathbf{M}, \mathbf{A}] = 0$, is satisfied.

The Ward-Takahashi identity in terms of homotopy algebras (6.40) are described by

$$\begin{aligned} &- \sum_{e_a \in \{c, c^\dagger\}} (\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{M}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A})) \rangle \mathbf{f} \mathbf{1} \\ &= -i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \rangle \mathbf{f} \mathbf{1} \\ &\quad + i\hbar \sum_{e_a \in \{c, c^\dagger\}} (\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \rangle \rangle \mathbf{f} \mathbf{1}. \end{aligned} \quad (6.59)$$

We first consider the case of the free theory. In this scenario, where $\mathbf{M}$ is replaced by $\mathbf{Q}$ and the interaction terms $\mathbf{m}_n$ are zero, the identities (6.59) simplify to

$$\begin{aligned}
& - \sum_{e_a \in \{c, c^\dagger\}} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{Q}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \rangle \right) \mathbf{f} \mathbf{1}, \\
& = -i\hbar \sum_{i=1}^n \delta^d(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f} \mathbf{1}, \\
& \quad + i\hbar \sum_{e_a \in \{c, c^\dagger\}} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \rangle \right) \mathbf{f} \mathbf{1}. \tag{6.60}
\end{aligned}$$

For simplification, we choose $n = 2k$ and set the field components such that $e_{a_1} = e_{a_2} = \dots = e_{a_k} = c^\dagger$ and $e_{a_{k+1}} = e_{a_{k+2}} = \dots = e_{a_{2k}} = c$. Under these specifications, the operator $\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) |$ is written as

$$\begin{aligned}
\langle \omega_{2k}^{a_1, \dots, a_{2k}}(x_1, \dots, x_{2k}) | & = \langle \omega | (d(x_1) \otimes \pi_1) \otimes \dots \otimes \langle \omega | (d(x_k) \otimes \pi_1), \\
& \quad \otimes \langle \omega | (d^\dagger(x_{k+1}) \otimes \pi_1) \otimes \dots \otimes \langle \omega | (d^\dagger(x_{2k}) \otimes \pi_1). \tag{6.61}
\end{aligned}$$

Using the relationship

$$\pi_n \mathbf{f} \mathbf{1} = \langle \Phi^{\otimes n} \rangle = \left\langle \left[\int d^d x \left(\varphi(x) c^\dagger(x) + \varphi^\dagger(x) c(x) \right) \right]^{\otimes n} \right\rangle, \tag{6.62}$$

the left-hand side of the equation (6.60) is expanded to

$$\begin{aligned}
& - \int d^d y d^d z \langle \varphi(x_1) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{2k}) \\
& \quad \left[\varphi(y) \varphi^\dagger(z) \langle \omega | (c(x) \otimes Q c^\dagger(y)) \otimes \langle \omega | (d^\dagger(x) \otimes A c(z)) \right. \\
& \quad \left. + \varphi^\dagger(y) \varphi(z) \langle \omega | (c^\dagger(x) \otimes Q c(y)) \otimes \langle \omega | (d(x) \otimes A c^\dagger(z)) \right] \rangle. \tag{6.63}
\end{aligned}$$

The quantity inside the square brackets in the equation (6.63) is calculated by substituting the definitions (6.49) and (6.56). We first evaluate the first term:

$$\begin{aligned}
& - \int d^d y d^d z \varphi(y) \varphi^\dagger(z) \langle \omega | (c(x) \otimes Q c^\dagger(y)) \otimes \langle \omega | (d^\dagger(x) \otimes A c(z)), \\
& = -i\alpha \int d^d y d^d z (-\partial_y^2 + m^2) \varphi(y) \varphi^\dagger(z) \langle \omega | (c(x) \otimes d^\dagger(y)) \otimes \langle \omega | (d^\dagger(x) \otimes c(z)), \\
& = i\alpha \varphi^\dagger(x) (-\partial^2 + m^2) \varphi(x). \tag{6.64}
\end{aligned}$$

Similarly, the second term is calculated as

$$\begin{aligned}
& - \int d^d y d^d z \varphi^\dagger(y) \varphi(z) \langle \omega | (c^\dagger(x) \otimes Q c(y)) \otimes \langle \omega | (d(x) \otimes A c^\dagger(z)), \\
& = i\alpha \int d^d y d^d z (-\partial_y^2 + m^2) \varphi^\dagger(y) \varphi(z) \langle \omega | (c^\dagger(x) \otimes d(y)) \otimes \langle \omega | (d(x) \otimes c^\dagger(z)) \\
& = -i\alpha \varphi(x) (-\partial^2 + m^2) \varphi(x). \tag{6.65}
\end{aligned}$$

Combining these results, the quantity inside the square brackets in (6.63) simplifies to

$$i\alpha(\varphi(x)\partial^2\varphi^\dagger(x) - \varphi^\dagger(x)\partial^2\varphi(x)). \quad (6.66)$$

We recognize that this result is equal to the divergence of the conserved current, $\partial_\mu j^\mu(x)$, where $j^\mu(x)$ is the current associated with the U(1) symmetry (6.43). Therefore, the left-hand side of the equation (6.60) is exactly equivalent to

$$\langle \varphi(x_1)\varphi(x_2)\dots\varphi(x_k)\varphi^\dagger(x_{k+1})\varphi^\dagger(x_{k+2})\dots\varphi^\dagger(x_{2k})\partial_\mu j^\mu(x) \rangle. \quad (6.67)$$

Next, we evaluate the first term on the right-hand side of the equation (6.60). Utilizing the relation (6.58), the operation $(\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{2k-i}) \mathbf{f} \mathbf{1}$ is interpreted as the expectation value involving the infinitesimal field transformation $\delta\Phi$:

$$(\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{2k-i}) \mathbf{f} \mathbf{1} = \langle \Phi^{\otimes i-1} \otimes \delta\Phi \otimes \Phi^{\otimes (2k-i)} \rangle. \quad (6.68)$$

Therefore, the first term on the right-hand side of the equation (6.60) is

$$\begin{aligned} & -i\hbar \sum_{i=1}^{2k} \delta^d(x-x_i) \langle \omega_{2k}^{a_1, \dots, a_{2k}}(x_1, \dots, x_{2k}) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{2k-i}) \mathbf{f} \mathbf{1} \\ & = -i\hbar \sum_{i=1}^k \delta^d(x-x_i) \langle \varphi(x_1) \dots \varphi(x_{i-1}) \delta\varphi(x_i) \varphi(x_{i+1}) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{2k}) \rangle \\ & \quad - i\hbar \sum_{i=1}^k \delta^d(x-x_i) \langle \varphi(x_1) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{k+i-1}) \delta\varphi^\dagger(x_{k+i}) \varphi^\dagger(x_{k+i+1}) \dots \varphi^\dagger(x_{2k}) \rangle. \end{aligned} \quad (6.69)$$

Furthermore, the second term on the right-hand side of the equation (6.60) is evaluated as

$$\begin{aligned} & i\hbar \sum_{e_a \in \{c, c^\dagger\}} (\langle \omega_{2k}^{a_1, \dots, a_n}(x_1, \dots, x_{2k}) | \pi_{2k} \otimes \langle \omega | (\tilde{e}^a(x)) \otimes \pi_1 \mathbf{A} e_a(x)) \mathbf{f} \mathbf{1} \\ & = i\hbar \langle \varphi(x_1) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{2k}) [\langle \omega | (d^\dagger(x) \otimes Ac(x)) + \langle \omega | (d(x) \otimes Ac^\dagger(x))] \rangle. \end{aligned} \quad (6.70)$$

We obtain the result:

$$\langle \omega | (d^\dagger(x) \otimes Ac(x)) + \langle \omega | (d(x) \otimes Ac^\dagger(x)) = i\alpha \delta^d(0) - i\alpha \delta^d(0). \quad (6.71)$$

This expression is ill-defined and necessitates regularization. However, for convenience in the current discussion, we ignore this divergence and assume that the term vanishes. The detailed

discussion concerning regularization is deferred to a later subsection. The resulting identity derived from the equation (6.60) is

$$\begin{aligned}
& \langle \varphi(x_1) \varphi(x_2) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \varphi^\dagger(x_{k+2}) \dots \varphi^\dagger(x_{2k}) \partial_\mu j^\mu(x) \rangle, \\
& = -i\hbar \sum_{i=1}^k \delta^d(x - x_i) \langle \varphi(x_1) \dots \varphi(x_{i-1}) \delta \varphi(x_i) \varphi(x_{i+1}) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{2k}) \rangle, \\
& \quad - i\hbar \sum_{i=1}^k \delta^d(x - x_i) \langle \varphi(x_1) \dots \varphi(x_k) \varphi^\dagger(x_{k+1}) \dots \varphi^\dagger(x_{k+i-1}) \delta \varphi^\dagger(x_{k+i}) \varphi^\dagger(x_{k+i+1}) \dots \varphi^\dagger(x_{2k}) \rangle.
\end{aligned} \tag{6.72}$$

This is exactly the known form of the Ward-Takahashi identity for the free complex scalar theory.

In fact, for the action (6.41), the interaction term involving $\mathbf{m}_{2n-1}$ does not alter the identity (6.59). In the interacting theory, the term containing

$$\sum_{n=1}^{\infty} \sum_{e_a \in \{c, c^\dagger\}} \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{m}_{2n-1}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \tag{6.73}$$

is added to the identity for the free theory. We demonstrate that this term vanishes. It is sufficient to evaluate its action on the field power series

$$\Phi^{\otimes 2n} = \left[\int d^d x (\varphi(x) c^\dagger(x) + \varphi^\dagger(x) c(x)) \right]^{\otimes 2n}. \tag{6.74}$$

Terms for which the number of c and $c^\dagger$ inputs is unequal trivially vanish due to the definition of m_{2n-1} . Two types of non-trivial terms remain.

1. The first type is the term where $\mathbf{m}_{2n-1}$ takes n components of c and $n-1$ components of $c^\dagger$ as inputs, and $\mathbf{A}$ takes c as an input. There are $\binom{2n}{n}$ such terms. These are evaluated as

$$-i\alpha \frac{n!(n-1)!}{2(2n-1)!} g_n (\varphi(x) \varphi^\dagger(x))^n \times \binom{2n}{n} = -\frac{i\alpha g_n}{2} (\varphi(x) \varphi^\dagger(x))^n. \tag{6.75}$$

2. The second type is the term where $\mathbf{m}_{2n-1}$ takes $n-1$ components of c and n components of $c^\dagger$ as inputs, and $\mathbf{A}$ takes $c^\dagger$ as an input. There are also $\binom{2n}{n}$ such terms. These are evaluated as

$$i\alpha \frac{n!(n-1)!}{2(2n-1)!} g_n (\varphi(x) \varphi^\dagger(x))^n \times \binom{2n}{n} = \frac{i\alpha g_n}{2} (\varphi(x) \varphi^\dagger(x))^n. \tag{6.76}$$

Consequently, these two contributions cancel each other out. Therefore, the interaction term does not affect the current identity, and we obtain the same form of the Ward-Takahashi identity as in the equation (6.72).

6.4.2 The chiral $U(1)$ symmetry in vector-like gauge theories

Let us consider the chiral $U(1)$ symmetry in a vector-like gauge theory. Consider the action of a massless Dirac fermion, given by

$$S = \int d^d x \, \bar{\Psi}_s(x) i \not{D}_{st}(x) \Psi_t(x), \quad (6.77)$$

where s and t are spinor indices. We adopt the Einstein summation convention. We omit the spinor indices if there is no confusion; they run from 1 to 4. The operator $\not{D}$ is defined as

$$\not{D}(x) = \gamma^\mu (\partial_\mu - i A_\mu(x)), \quad (6.78)$$

and γ^μ is a gamma matrix satisfying

$$\{\gamma^\mu, \gamma^\nu\} = -2 \eta^{\mu\nu}. \quad (6.79)$$

Furthermore, we treat the gauge field $A_\mu(x)$ as a background field. This theory possesses a chiral $U(1)$ symmetry, defined by the transformation:

$$\Psi(x) \rightarrow e^{-i\alpha\gamma_5} \Psi(x), \quad \bar{\Psi}(x) \rightarrow \bar{\Psi}(x) e^{-i\alpha\gamma_5}, \quad (6.80)$$

where the chiral matrix γ_5 is given by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (6.81)$$

The infinitesimal transformations are given by

$$\delta\Psi(x) = -i\alpha\gamma_5\Psi(x), \quad \delta\bar{\Psi}(x) = -i\alpha\bar{\Psi}(x)\gamma_5. \quad (6.82)$$

The action defined earlier can be rewritten in terms of homotopy algebras as

$$S = -\frac{1}{2} \omega(\Phi, Q\Phi), \quad (6.83)$$

where the degree-even element $\Phi \in V^{(hQ)}$ is defined by

$$\Phi = \int d^4 x \, [\Psi_s(x) \bar{\theta}_s(x) + \bar{\Psi}_s(x) \theta_s(x)], \quad (6.84)$$

and $\{\theta_s(x), \bar{\theta}_s(x)\}$ are the degree-odd basis vectors of $V^{(hQ)}$. We introduce the degree-even basis vectors $\{\lambda_s(x), \bar{\lambda}_s(x)\}$ of $V^{(Qh)}$, which satisfy the following normalization conditions with respect to the symplectic form:

$$\begin{aligned} \omega(\theta_s(x), \bar{\lambda}_t(y)) &= -\delta_{st} \delta^4(x-y), \\ \omega(\bar{\theta}_s(x), \lambda_t(y)) &= -\delta_{st} \delta^4(x-y), \\ \omega(\lambda_s(x), \bar{\theta}_t(y)) &= \delta_{st} \delta^4(x-y), \\ \omega(\bar{\lambda}_s(x), \theta_t(y)) &= \delta_{st} \delta^4(x-y). \end{aligned} \quad (6.85)$$

Corresponding to the notation of the bases introduced thus far, the basis $e_a(x)$ is given by

$$\{e_a(x)\} = \{\theta_s(x), \bar{\theta}_s(x), \lambda_s(x), \bar{\lambda}_s(x)\}, \quad (6.86)$$

and its dual basis $\tilde{e}^a(x)$ is given by

$$\{\tilde{e}^a(x)\} = \{\bar{\lambda}_s(x), \lambda_s(x), \bar{\theta}_s(x), \theta_s(x)\}. \quad (6.87)$$

We do not require the introduction of a separate basis vector for the background field A_μ . We define the operator Q by specifying its action on the basis elements as

$$\begin{aligned} Q \theta_s(x) &= -i \mathcal{D}_{st}(x) \lambda_t(x), & Q \lambda_s(x) &= 0, \\ Q \bar{\theta}_s(x) &= i \bar{\lambda}_t(x) \overleftarrow{\mathcal{D}}_{ts}(x), & Q \bar{\lambda}_s(x) &= 0, \end{aligned} \quad (6.88)$$

where $\overleftarrow{\mathcal{D}}$ is defined by

$$\overleftarrow{\mathcal{D}}(x) = \gamma^\mu (-\overleftarrow{\partial}_\mu - i A_\mu(x)), \quad (6.89)$$

and $\overleftarrow{\partial}_\mu$ denotes the differential acting from the right. The action is successfully reproduced in the homotopy algebra formalism:

$$-\frac{1}{2} \omega(\Phi, Q\Phi) = \int d^4x \bar{\Psi}(x) i \mathcal{D}(x) \Psi(x). \quad (6.90)$$

Associated with the operator Q , we introduce the contracting homotopy h , whose action on the basis elements is defined as

$$\begin{aligned} h \theta_s(x) &= 0, & h \lambda_s(x) &= i \left[\mathcal{D}(x) \int_0^\infty d\tau e^{-\tau \mathcal{D}^2(x)} \right]_{st} \theta_t(x), \\ h \bar{\theta}_s(x) &= 0, & h \bar{\lambda}_s(x) &= -i \bar{\theta}_t(x) \left[\overleftarrow{\mathcal{D}}(x) \int_0^\infty d\tau e^{-\tau \overleftarrow{\mathcal{D}}^2(x)} \right]_{ts}. \end{aligned} \quad (6.91)$$

The degree-even operator A , which corresponds to the symmetry transformation (6.82), is defined by its action on the basis elements

$$\begin{aligned} A \theta_s(x) &= -i\alpha (\gamma_5)_{st} \theta_t(x), & A \lambda_s(x) &= i\alpha (\gamma_5)_{st} \lambda_t(x), \\ A \bar{\theta}_s(x) &= -i\alpha \bar{\theta}_t(x) (\gamma_5)_{ts}, & A \bar{\lambda}_s(x) &= i\alpha \bar{\lambda}_t(x) (\gamma_5)_{ts}. \end{aligned} \quad (6.92)$$

This operator successfully reproduces the infinitesimal symmetry transformation $\delta\Phi$, as verified by the following calculation:

$$\begin{aligned} A\Phi &= \int d^4x [\Psi_s(x) A \bar{\theta}_s(x) + \bar{\Psi}_s(x) A \theta_s(x)] \\ &= -i\alpha \int d^4x [\Psi_s(x) (\gamma_5)_{ts} \bar{\theta}_t(x) + \bar{\Psi}_s(x) (\gamma_5)_{st} \theta_t(x)], \\ &= \int d^4x [\delta\Psi_t(x) \bar{\theta}_t(x) + \delta\bar{\Psi}_t(x) \theta_t(x)], \\ &= \delta\Phi. \end{aligned} \quad (6.93)$$

Moreover, we can confirm that the operator A generates the symmetry, as the commutation relation $[Q, A] = 0$ is satisfied. This verification for the $\theta_s(x)$ basis element is performed as

$$\begin{aligned}
[Q, A] \theta_s(x) &= -i\alpha (\gamma_5)_{st} Q \theta_t(x) + i \not{D}_{st}(x) A \lambda_t(x) \\
&= -\alpha (\gamma_5)_{st} \not{D}_{tu}(x) \lambda_u(x) - \alpha \not{D}_{st}(x) (\gamma_5)_{tu} \lambda_u(x), \\
&= -\alpha (\gamma_5)_{st} \not{D}_{tu}(x) \lambda_u(x) + \alpha (\gamma_5)_{st} \not{D}_{tu}(x) \lambda_u(x), \\
&= 0,
\end{aligned} \tag{6.94}$$

where the anti-commutation relation $\{\gamma_5, \gamma^\mu\} = 0$ was utilized. Similarly, we find that the commutation relation holds for the $\bar{\theta}_s(x)$ element:

$$[Q, A] \bar{\theta}_s(x) = 0. \tag{6.95}$$

The Ward-Takahashi identity in terms of homotopy algebras (6.40) is described by

$$\begin{aligned}
& - \sum_{e_a \in \{\theta_s, \bar{\theta}_s\}} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (e_a(x) \otimes \pi_1 \mathbf{Q}) \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A}) \rangle \right) \mathbf{f1} \\
&= -i\hbar \sum_{i=1}^n \delta^4(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f1} \\
&\quad + i\hbar \sum_{e_a \in \{\theta_s, \bar{\theta}_s\}} \left(\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \otimes \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \rangle \right) \mathbf{f1}.
\end{aligned} \tag{6.96}$$

For simplicity in the subsequent calculations, we set $n = 2k$ and choose the basis elements e_{a_i} such that $e_{a_i} = \bar{\theta}_{s_i}$ for $i = 1, \dots, k$ and $e_{a_i} = \theta_{s_i}$ for $i = k+1, \dots, 2k$. Since

$$\pi_n \mathbf{f1} = \langle \Phi^{\otimes n} \rangle = \left\langle \left[\int d^4x (\Psi_s(x) \bar{\theta}_s(x) + \bar{\Psi}_s(x) \theta_s(x)) \right]^{\otimes n} \right\rangle, \tag{6.97}$$

the left-hand side of the equation (6.96) is written as

$$\begin{aligned}
& - \int d^4y d^4z \langle \Psi_{s_1}(x_1) \dots \Psi_{s_k}(x_k) \bar{\Psi}_{s_{k+1}}(x_{k+1}) \dots \bar{\Psi}_{s_{2k}}(x_{2k}) \\
&\quad \left[\Psi_t(y) \bar{\Psi}_u(z) \langle \omega | (\theta_s(x) \otimes Q \bar{\theta}_t(y)) \otimes \langle \omega | (\bar{\lambda}_s(x) \otimes A \theta_u(z)) \right. \\
&\quad \left. + \bar{\Psi}_t(y) \Psi_u(z) \langle \omega | (\bar{\theta}_s(x) \otimes Q \theta_t(y)) \otimes \langle \omega | (\lambda_s(x) \otimes A \bar{\theta}_u(z)) \right] \rangle.
\end{aligned} \tag{6.98}$$

Evaluation of the terms inside the square brackets yields the following two results:

$$\begin{aligned}
& - \int d^4y d^4z \Psi_t(y) \bar{\Psi}_u(z) \langle \omega | (\theta_s(x) \otimes Q \bar{\theta}_t(y)) \otimes \langle \omega | (\bar{\lambda}_s(x) \otimes A \theta_u(z)) \rangle, \\
&= \alpha \bar{\Psi}(x) \not{D}(x) \gamma_5 \Psi(x),
\end{aligned} \tag{6.99}$$

and

$$\begin{aligned}
& - \int d^4 y d^4 z \bar{\Psi}_t(y) \Psi_u(z) \langle \omega | (\bar{\theta}_s(x) \otimes Q \theta_t(y)) \otimes \langle \omega | (\lambda_s(x) \otimes A \bar{\theta}_u(z)) , \\
& = -\alpha \bar{\Psi}(x) \overleftrightarrow{\mathcal{D}}(x) \gamma_5 \Psi(x) .
\end{aligned} \tag{6.100}$$

Combining these two expressions, the quantity inside the square brackets simplifies to the divergence of the current as follows:

$$\begin{aligned}
& - \int d^4 y d^4 z \left[\Psi_t(y) \bar{\Psi}_u(z) \langle \omega | (\theta_s(x) \otimes Q \bar{\theta}_t(y)) \otimes \langle \omega | (\bar{\lambda}_s(x) \otimes A \theta_u(z)) \right. \\
& \quad \left. + \bar{\Psi}_t(y) \Psi_u(z) \langle \omega | (\bar{\theta}_s(x) \otimes Q \theta_t(y)) \otimes \langle \omega | (\lambda_s(x) \otimes A \bar{\theta}_u(z)) \right] , \\
& = \partial_\mu \left[\alpha \bar{\Psi}_s(x) \gamma^\mu \gamma_5 \Psi(x) \right] , \\
& = \partial_\mu j_5^\mu(x) ,
\end{aligned} \tag{6.101}$$

where $j_5^\mu(x)$ denotes the conserved current associated with the chiral U(1) symmetry (6.82). Thus, the left-hand side of the equation (6.96) is found to be equal to the expectation value:

$$\langle \Psi_{s_1}(x_1) \dots \Psi_{s_k}(x_k) \bar{\Psi}_{s_{k+1}}(x_{k+1}) \dots \bar{\Psi}_{s_{2k}}(x_{2k}) \partial_\mu j_5^\mu(x) \rangle . \tag{6.102}$$

The first term on the right-hand side of the equation (6.96) is expanded in a manner similar to the preceding examples as

$$\begin{aligned}
& - i\hbar \sum_{i=1}^n \delta^4(x - x_i) \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | (\pi_{i-1} \otimes \pi_1 \mathbf{A} \otimes \pi_{n-i}) \mathbf{f} \mathbf{1} \\
& = -i\hbar \sum_{i=1}^k \delta^4(x - x_i) \langle \Psi_{s_1}(x_1) \dots \Psi_{s_{i-1}}(x_{i-1}) \delta \Psi_{s_i}(x_i) \Psi_{s_{i+1}}(x_{i+1}) \dots \Psi_{s_k}(x_k) \\
& \quad \bar{\Psi}_{s_{k+1}}(x_{k+1}) \dots \bar{\Psi}_{s_{2k}}(x_{2k}) \rangle \\
& - i\hbar \sum_{i=k+1}^{2k} \delta^4(x - x_i) \langle \Psi_{s_1}(x_1) \dots \Psi_{s_k}(x_k) \\
& \quad \bar{\Psi}_{s_{k+1}}(x_{k+1}) \dots \bar{\Psi}_{s_{i-1}}(x_{i-1}) \delta \bar{\Psi}_{s_i}(x_i) \bar{\Psi}_{s_{i+1}}(x_{i+1}) \dots \bar{\Psi}_{s_{2k}}(x_{2k}) \rangle .
\end{aligned} \tag{6.103}$$

The second term on the right-hand side still contains an ill-defined factor

$$\langle \omega | (\bar{\lambda}_s(x) \otimes A \theta_s(x)) + \langle \omega | (\lambda_s(x) \otimes A \bar{\theta}_s(x)) = -2i\alpha \lim_{y \rightarrow x} \text{tr} \gamma_5 \delta^4(x - y) . \tag{6.104}$$

As is well known, the chiral symmetry is anomalous, and this factor does not vanish after regularization. We explore the method for regularizing this term from the homotopy algebraic point of view in the next subsection.

6.5 Regularization with stubs

As we mentioned in the previous section, we need to regularize the term

$$i\hbar \sum_{e_a \in V(hQ)} \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \mathbf{f} \mathbf{1}, \quad (6.105)$$

to mediate the divergence. To preserve the homotopy algebraic structure, we employ a regularization method involving stubs. The concept of stubs, widely used in string field theory (see [7]), and recently applied to quantum field theory [59], is adopted here for regularization. In this specific regularization scheme, the operators $\mathbf{M}$ and $\mathbf{U}$ are modified via the heat kernel in a manner that ensures the preservation of the underlying homotopy algebraic structure. Repeating the derivations using these regularized operators, the correct anomalous term is obtained. It is important to comment on the technical aspects regarding the heat kernel. In the Lorentzian signature, the heat kernel does not inherently regularize path integrals. The solution adopted here is to proceed with the calculation formally and then introduce a Wick rotation when performing the integral involving the heat kernel.

We now explain the detailed treatment of stubs within the context of homotopy algebras. The incorporation of stubs into the homotopy algebraic formalism is achieved by employing the Homological perturbation lemma (HPL). This approach has been extensively developed and utilized in recent literature on the subject [41, 42, 43, 44, 59]. Refer to Subsection 4.1 for details on HPL. To apply the HPL to adding a stub, we define two degree-even maps, π and ι , as follows:

$$\pi e_a(x) = \iota e_a(x) = K_{\lambda/2} e_a(x). \quad (6.106)$$

Here, $K_{\lambda/2}$ is a heat kernel labeled by the positive parameter λ . The degree-odd map η that satisfies

$$Q\eta + \eta Q + \iota\pi = \mathbb{I}, \quad (6.107)$$

for these π and ι is defined by

$$\eta = (1 - K_\lambda)h. \quad (6.108)$$

In the context of scalar field theory, we define K_λ using the heat kernel of the scalar field as

$$K_\lambda = e^{-\lambda(-\partial^2 + m^2)}. \quad (6.109)$$

Consequently, the basis vectors $c(x)$ and $d(x)$ are mapped as

$$\pi c(x) = \iota c(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} c(x), \quad \pi d(x) = \iota d(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} d(x). \quad (6.110)$$

This procedure serves as a regularization method, as demonstrated by an analysis of the contact terms in the Ward-Takahashi identities. Furthermore, note that the limit $\lambda \rightarrow 0$ corresponds to the original theory without stubs. Unfortunately, the defined maps π , ι , and η do not satisfy conditions $\pi\iota = \mathbb{I}$ and $\pi\eta = \eta\iota = 0$, although the condition $\eta^2 = 0$ holds. Consequently, the perturbed operator

$$\mathbf{M}' = \pi \mathbf{Q} \iota + \pi \mathbf{m} \frac{1}{\mathbf{I} + \eta(\mathbf{m} + i\hbar \mathbf{U})} \iota, \quad (6.111)$$

does not act as a coderivation. Here, the boldface operators are those lifted to the tensor algebra, and their detailed definitions are the same as in Subsection 4.1. According to [41], the resolution is straightforward: we can simply proceed by formally assuming these conditions are satisfied (i.e., we pretend they hold). A more precise treatment of this matter is provided in Appendix A of [42]. Alternatively, we can ensure that these conditions are met by defining the product appropriately with respect to $K_{\lambda/2}$ and K_λ . If we define the products as

$$K_{\lambda/2} \cdot K_{\lambda/2} = K_{\lambda/2} \cdot K_\lambda = K_\lambda \cdot K_{\lambda/2} = K_\lambda \cdot K_\lambda = K_\lambda, \quad (6.112)$$

then the operator $\iota\pi$ can be treated as a projection.

We define the symplectic form $\langle \omega' |$ on πV by

$$\langle \omega' | = \langle \omega | (\iota \otimes \iota). \quad (6.113)$$

The stubbed action S' is defined using the operator $\mathbf{M}'$ and the symplectic form $\langle \omega' |$ as

$$S' = - \sum_{n=0}^{\infty} \frac{1}{n+1} \langle \omega' | (\mathbb{I} \otimes \pi_1 \mathbf{M}'_n) \pi \frac{1}{1 - \Phi}. \quad (6.114)$$

This stubbed action S' is quasi-isomorphic to the original action and reproduces the original action in the limit $\lambda \rightarrow 0$. Starting from this stubbed action, we can construct the counterpart of the correlation function $\mathbf{f}\mathbf{1}$ in the stubbed theory, denoted $\mathbf{f}'\mathbf{1}$, as

$$\mathbf{f}'\mathbf{1} = \frac{1}{\mathbf{I} + \mathbf{h}'(\mathbf{m}' + i\hbar \mathbf{U}')} \mathbf{1}, \quad (6.115)$$

where $\mathbf{m}'$, $\mathbf{U}'$ and $\mathbf{h}'$ are defined by

$$\mathbf{m}' = \mathbf{M}' - \pi \mathbf{Q} \iota, \quad \mathbf{U}' = \pi \mathbf{U} \iota, \quad \mathbf{h}' = \pi \mathbf{h} \iota. \quad (6.116)$$

The operator $\mathbf{h}'$ is specifically defined to satisfy the following relation

$$\pi \mathbf{Q} \iota \mathbf{h}' + \mathbf{h}' \pi \mathbf{Q} \iota = \mathbf{I}. \quad (6.117)$$

As discussed in Subsection 5.2, the algebraic Schwinger-Dyson equation holds for $\mathbf{f}'\mathbf{1}$:

$$(\mathbf{M}' + i\hbar \mathbf{U}') \mathbf{f}'\mathbf{1} = 0. \quad (6.118)$$

Let us consider the transformation $\mathbf{A}'$, defined as the counterpart of the symmetry transformation $\mathbf{A}$:

$$\mathbf{A}' = \pi \mathbf{A} \iota. \quad (6.119)$$

This $\mathbf{A}'$ is not a symmetry transformation because it does not commute with $\mathbf{M}'$. However, in the limit $\lambda \rightarrow 0$, where the maps π and ι approach the identity operator $\mathbb{I}$, the transformation $\mathbf{A}'$ commutes with $\mathbf{M}'$ and becomes a symmetry transformation. By repeating the arguments presented in Subsection 6.3 using $\mathbf{f}'\mathbf{1}$ and $\mathbf{A}'$, and subsequently setting $\lambda \rightarrow 0$, we obtain the regularized anomalous term as

$$\begin{aligned} & i\hbar \lim_{\lambda \rightarrow 0} \sum_{e_a \in V(hQ)} \langle \omega' | (\pi \tilde{e}^a(x) \otimes \pi_1 \mathbf{A}' \pi e_a(x) \iota) \mathbf{f}'\mathbf{1} \\ &= i\hbar \lim_{\lambda \rightarrow 0} \sum_{e_a \in V(hQ)} \langle \omega | (\iota \pi \tilde{e}^a(x) \otimes \iota \pi_1 \pi \mathbf{A} \iota \pi e_a(x) \iota) \mathbf{f}'\mathbf{1}. \end{aligned} \quad (6.120)$$

Since the projection $\iota \pi$ satisfies

$$\begin{aligned} \langle \omega | (\iota \pi \otimes \mathbb{I}) &= \langle \omega | (\mathbb{I} \otimes \iota \pi) = \langle \omega | (\iota \pi \otimes \iota \pi), \\ \iota \pi e_a(x) \otimes \tilde{e}^a(x) &= e_a(x) \otimes \iota \pi \tilde{e}^a(x) = \iota \pi e_a(x) \otimes \iota \pi \tilde{e}^a(x), \end{aligned} \quad (6.121)$$

the regularized anomalous term can also be expressed as

$$i\hbar \lim_{\lambda \rightarrow 0} \sum_{e_a \in V(hQ)} \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} \iota \pi e_a(x) \iota) \mathbf{f}'\mathbf{1}. \quad (6.122)$$

In particular, if the symmetry transformation $\mathbf{A}$ is linear, as was the case in the examples in the previous subsection, it can be simplified as

$$i\hbar \lim_{\lambda \rightarrow 0} \sum_{e_a \in V(hQ)} \langle \omega | (\tilde{e}^a(x) \otimes A K_\lambda e_a(x)) \mathbf{f}'\mathbf{1}. \quad (6.123)$$

This term is regularized due to the smearing effect of the heat kernel on the delta function.

Let us now compute the anomaly for the example introduced in the previous subsection. We first consider the case of the $U(1)$ symmetry in complex scalar theories. The symmetry transformation $\mathbf{A}$ is defined by equation (6.56). The stub is chosen such that the maps π and ι are defined by the heat kernel $K_{\lambda/2}$:

$$\begin{aligned} \pi c(x) &= \iota c(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} c(x), & \pi d(x) &= \iota d(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} d(x), \\ \pi c^\dagger(x) &= \iota c^\dagger(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} c^\dagger(x), & \pi d^\dagger(x) &= \iota d^\dagger(x) = e^{-\frac{\lambda}{2}(-\partial^2 + m^2)} d^\dagger(x). \end{aligned} \quad (6.124)$$

The anomalous term can then be evaluated as follows:

$$\begin{aligned}
& i \hbar \lim_{\lambda \rightarrow 0} \sum_{a \in \{c, c^\dagger\}} \langle \omega | (\tilde{e}^a(x) \otimes A K_\lambda e_a(x)) \\
&= \hbar \alpha \lim_{\lambda \rightarrow 0} \lim_{y \rightarrow x} [e^{-\lambda(-\partial_x^2 + m^2)} \delta^d(x - y) - e^{-\lambda(-\partial_x^2 + m^2)} \delta^d(x - y)], \\
&= i \hbar \alpha \lim_{\lambda \rightarrow 0} \left[\int \frac{d^d k}{(2\pi)^d} e^{-\lambda(k^2 + m^2)} - \int \frac{d^d k}{(2\pi)^d} e^{-\lambda(k^2 + m^2)} \right], \\
&= 0,
\end{aligned} \tag{6.125}$$

where the third line was obtained by carrying out the Wick rotation. Therefore, we confirm that there is no anomaly in this specific case.

Next, let us consider the chiral $U(1)$ symmetry in vector-like gauge theories. The symmetry transformation $\mathbf{A}$ is defined by equation (6.92). The stub can be chosen as

$$\begin{aligned}
\pi \theta_s(x) &= \iota \theta_s(x) = (e^{-\lambda \not{p}^2(x)})_{st} \theta_t(x), \quad \pi \lambda_s(x) = \iota \lambda_s(x) = (e^{-\lambda \not{p}^2(x)})_{st} \lambda_t(x), \\
\pi \bar{\theta}_s(x) &= \iota \bar{\theta}_s(x) = \bar{\theta}_t(x) (e^{-\lambda \overleftarrow{\not{p}}^2(x)})_{ts}, \quad \pi \bar{\lambda}_s(x) = \iota \bar{\lambda}_s(x) = \bar{\lambda}_t(x) (e^{-\lambda \overleftarrow{\not{p}}^2(x)})_{ts}.
\end{aligned} \tag{6.126}$$

The anomalous term can then be calculated as

$$\begin{aligned}
& i \hbar \lim_{\lambda \rightarrow 0} [\langle \omega | (\bar{\lambda}_s(x) \otimes A K_\lambda \theta_s(x)) + \langle \omega | (\lambda_s(x) \otimes A K_\lambda \bar{\theta}_s(x))] \\
&= \hbar \alpha \lim_{\lambda \rightarrow 0} \lim_{y \rightarrow x} \text{tr} [\gamma_5 e^{-\lambda \not{p}^2(x)} \delta^4(x - y) + \delta^4(x - y) e^{-\lambda \overleftarrow{\not{p}}^2(x)} \gamma_5] \\
&= \hbar \alpha \lim_{\lambda \rightarrow 0} \int \frac{d^4 k}{(2\pi)^4} \text{tr} [\gamma_5 e^{-ikx} e^{-\lambda \not{p}^2(x)} e^{ikx} + \gamma_5 e^{ikx} e^{-\lambda \overleftarrow{\not{p}}^2(x)} e^{-ikx}],
\end{aligned} \tag{6.127}$$

where the last line was obtained by carrying out the Wick rotation. As is well known, this quantity can be evaluated using standard techniques. Since the covariant derivative satisfies

$$e^{-ikx} D_\mu e^{ikx} = D_\mu + i k_\mu, \tag{6.128}$$

we obtain

$$e^{-ikx} e^{-\lambda \not{p}^2(x)} e^{ikx} = e^{-\lambda(k - iD)^2 + \frac{i\lambda}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}}, \tag{6.129}$$

where $F_{\mu\nu}$ denotes the field strength of the background field A_μ , defined as

$$F_{\mu\nu} = i [D_\mu, D_\nu]. \tag{6.130}$$

Integration over the momentum k yields a factor of λ^{-2} , and using the gamma matrix trace identities:

$$\text{tr} \gamma_5 = \text{tr} \gamma_5 \gamma^\mu \gamma^\nu = 0, \quad \text{tr} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4 i \epsilon^{\mu\nu\rho\sigma}, \tag{6.131}$$

the anomalous term evaluates to

$$\begin{aligned}
& i \hbar \lim_{\lambda \rightarrow 0} \left[\langle \omega | \left(\bar{\lambda}_s(x) \otimes A K_\lambda \theta_s(x) \right) + \langle \omega | \left(\lambda_s(x) \otimes A K_\lambda \bar{\theta}_s(x) \right) \right] \\
&= \hbar \alpha \lim_{\lambda \rightarrow 0} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[\gamma_5 e^{-ikx} e^{-\lambda \not{D}^2(x)} e^{ikx} + \gamma_5 e^{ikx} e^{-\lambda \overleftarrow{\not{D}}^2(x)} e^{-ikx} \right], \\
&= - \frac{i \hbar \alpha}{64 \pi^2} \text{tr} [\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] F_{\mu\nu} F_{\rho\sigma}, \\
&= - \frac{\hbar \alpha}{16 \pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{6.132}
\end{aligned}$$

This resulting quantity is the well-known chiral anomaly. Thus, the use of stubs allows for regularization that respects the underlying homotopy algebraic structure. This validates the Ward-Takahashi identity derived in this framework, ensuring its correctness even in the presence of anomalies.

7 LSZ Reduction Formula from Homotopy Algebras

In Subsection 4.2 we saw, in the context of homotopy transfer, that the minimal model reproduces on-shell amplitudes (at least at the tree level). Section 5 then introduced a homotopy-algebraic formula for correlation functions. In quantum field theory, the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula extracts amplitudes from correlation functions by amputating external lines. This suggests that, within the homotopy-algebra framework as well, there exists a procedure to obtain amplitudes from correlators. Moreover, via the LSZ reduction formula, one can remove the diagrams associated with mass renormalization even at the loop level. The goal of this section is to formulate and understand this procedure in the language of homotopy algebras. The discussion follows [45].

In the previous two sections, we considered a kinetic operator Q and a contracting homotopy h satisfying $Qh + hQ = \mathbb{I}$. This corresponded to shifting the mass by the $i\epsilon$ prescription. In this section, we turn to on-shell amplitudes. Since on-shell states are defined with respect to the kinetic operator without the $i\epsilon$ shift, this operator must be distinguished from the shifted one and written separately. In this section, Q and h denote the unshifted operators (no $i\epsilon$ shift), and they satisfy $Qh + hQ = \mathbb{I} - P$, where P is the projector onto the Q -cohomology. In this setting, the vector space V decomposes into the subspaces $V^{(Qh)}$, $V^{(hQ)}$, and the subspace projected by P , which we denote by $V^{(P)}$. Meanwhile, denote the $i\epsilon$ -shifted operators by a family Q_ϵ, h_ϵ parametrized by ϵ . For $\epsilon > 0$, we impose

$$Q_\epsilon h_\epsilon + h_\epsilon Q_\epsilon = \mathbb{I}, \tag{7.1}$$

as well as the other SDR conditions and compatibility condition with ω , and they approach the

unsifted operators in the limit $\epsilon \rightarrow 0$ ¹:

$$\lim_{\epsilon \rightarrow 0} Q_\epsilon = Q, \quad \lim_{\epsilon \rightarrow 0} h_\epsilon = h. \quad (7.2)$$

Additionally, we assume that Φ satisfies $h\Phi = 0$ and $h_\epsilon\Phi = 0$. Although not written explicitly, the multilinear products m_n generally need to be deformed for $\epsilon > 0$ so that they satisfy the quantum A_∞ relations with respect to Q_ϵ . In a gauge-fixed theory, however, these relations are automatically satisfied, so this point can be safely ignored. We denote by $\mathbf{Q}_\epsilon$ and $\mathbf{h}_\epsilon$ the lifts of Q_ϵ and h_ϵ to operators on the tensor algebra, in the same way that $\mathbf{Q}$ and $\mathbf{h}$ are the lifts of Q and h . It is crucial to note that although h_ϵ approaches h in the limit $\epsilon \rightarrow 0$, the corresponding lifted operators $\mathbf{h}_\epsilon$ and $\mathbf{h}$ are distinct. The definition of $\mathbf{h}_\epsilon$ is

$$\mathbf{h}_\epsilon\pi_0 = 0, \quad \mathbf{h}_\epsilon\pi_n = \sum_{k=0}^{n-1} \mathbb{I}^{\otimes k} \otimes h_\epsilon \quad \text{for } n \geq 1, \quad (7.3)$$

whereas the definition of $\mathbf{h}$ is

$$\mathbf{h}\pi_0 = 0, \quad \mathbf{h}\pi_n = \sum_{k=0}^{n-1} \mathbb{I}^{\otimes k} \otimes h \otimes P^{\otimes n-k-1} \quad \text{for } n \geq 1. \quad (7.4)$$

The two operators are different because we assume that $P \neq 0$.

As in the previous section, we write the basis as $e_a(x)$, separating discrete and continuous labels. We may also use the simplified notation e_s when the distinction is unnecessary.

7.1 LSZ reduction formula

Let us extract amplitudes from the homotopy–algebraic correlator formula (5.9) via the LSZ reduction. As a preliminary step, we remove the tadpole contributions. The tadpole diagram is encoded in the one-point correlation function, which we denote by

$$\Phi_* = \pi_1 \frac{1}{\mathbf{I} + \mathbf{h}_\epsilon(\mathbf{m} + i\hbar \mathbf{U})} \mathbf{1} \quad (7.5)$$

If this quantity does not vanish, one can shift $\Phi \rightarrow \Phi_* + \Phi$ and expand the theory around Φ_* to set the one-point correlation function to zero. In what follows, the theory is implicitly assumed to have a vanishing one-point function.

Let us revisit the formula for the correlation function. We write

$$\langle \phi^{a_1}(x_1) \cdots \phi^{a_2}(x_n) \rangle = \langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | \pi_n \mathbf{f}_\epsilon \mathbf{1} \rangle, \quad (7.6)$$

¹Although h_ϵ may not connect smoothly to h , in that case the limit is understood as a formal replacement of h_ϵ by h .

where $\mathbf{f}_\epsilon$ is defined by

$$\mathbf{f}_\epsilon = \frac{1}{\mathbf{I} + \mathbf{h}_\epsilon(\mathbf{m} + i\hbar\mathbf{U})}, \quad (7.7)$$

and

$$\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) | = (-1)^n \langle \omega | (\tilde{e}^{a_1}(x_1) \otimes \pi_1) \otimes \dots \otimes \langle \omega | (\tilde{e}^{a_n}(x_n) \otimes \pi_1), \quad (7.8)$$

for all $\tilde{e}_{a_1}(x_1), \dots, \tilde{e}_{a_n}(x_n) \in V^{(Qh)}$. The subscript ϵ emphasizes that $\mathbf{f}_\epsilon$ is defined in terms of $i\epsilon$ shifted contracting homotopy h_ϵ . Here, by using the identity

$$\sum_a \int d^d x h_\epsilon \tilde{e}^a(x) \otimes e_a(x) = \sum_a \int d^d x (-1)^{e_a} \tilde{e}^a(x) \otimes h_\epsilon e_a(x), \quad (7.9)$$

which is derived from (5.5), one can check that every output of $\mathbf{f}_\epsilon \mathbf{1}$ carries an h_ϵ . Finally, contracting with $\langle \omega_n^{a_1, \dots, a_n}(x_1, \dots, x_n) |$ strips off the basis elements and extracts the coefficients, i.e., the correlation functions. Naively, one might hope to amputate the external lines—and thus obtain amplitudes—by removing the h_ϵ factors. Indeed, by acting on Q_ϵ , one can eliminate h_ϵ from the output. This works at the tree level. At the loop level, however, such an amputation fails to remove the diagrams associated with mass renormalization. To eliminate those contributions, one must amputate using the operator corresponding to the inverse of the exact two-point correlation function $\hat{Q}_\epsilon$, which satisfies

$$\hat{Q}_\epsilon \pi_2 \mathbf{f}_\epsilon \mathbf{1} = -i\hbar \pi_2 \mathbf{U} \mathbf{1}, \quad (7.10)$$

where $\hat{Q}_\epsilon$ is a coderivation promoted from $\hat{Q}_\epsilon$. For example, in the scalar-field case, if the exact two-point function is

$$-i\hbar \int d^d x_1 d^d x_2 \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik(x_1-x_2)}}{K(k^2)} c(x_1) \otimes c(x_2), \quad (7.11)$$

then $\hat{Q}_\epsilon$ is given by

$$\hat{Q}_\epsilon c(x) = \int \frac{d^d k}{(2\pi)^d} K(k^2) e^{-ik(x-y)} d(y). \quad (7.12)$$

Expressed in terms of the self-energy $\Pi(k^2)$, $K(k^2)$ is given by

$$K(k^2) = k^2 + m^2 - i\epsilon - \Pi(k^2). \quad (7.13)$$

Using this operator, the n -point on-shell amplitude is given by

$$\left(\frac{i}{\hbar} \right)^{n-1} \lim_{\epsilon \rightarrow 0} \pi_n \langle \omega_n(\Phi_1, \dots, \Phi_n) | \pi_n \mathbf{f}_\epsilon \mathbf{1}, \quad (7.14)$$

where

$$\langle \omega_n(\Phi_1, \dots, \Phi_n) | = (-1)^n \langle \omega | (\widehat{Q}_\epsilon \Phi_1 \otimes \pi_1) \otimes \dots \otimes \langle \omega | (\widehat{Q}_\epsilon \Phi_n \otimes \pi_1) . \quad (7.15)$$

Here each Φ_i lies in the cohomology of $\widehat{Q} = \lim_{\epsilon \rightarrow 0} \widehat{Q}_\epsilon$. At first glance, it might seem that $\widehat{Q}_\epsilon \Phi_i$ should vanish in the limit $\epsilon \rightarrow 0$, since $\widehat{Q}_\epsilon$ approaches $\widehat{Q}$, and Φ_i belongs to the cohomology of $\widehat{Q}$. However, this is safe because the terms $\widehat{Q}_\epsilon \Phi_i$ are used to amputate the external lines and are removed from the expression before the limit $\epsilon \rightarrow 0$ is finally taken. Furthermore, as was commented in the discussion regarding the minimal model, we assume that the external lines, $\Phi_1, \dots, \Phi_n$, carry general momenta.

This expression seems to differ from the minimal model

$$-\omega \left(\Phi_1, \pi_1 \mathbf{P} \mathbf{m} \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U})} \mathbf{P} \pi_{n-1} \Phi_2 \otimes \dots \otimes \Phi_n \right) , \quad (7.16)$$

which is known to yield the on-shell amplitude. First, (7.14) includes disconnected contributions and is built with the $i\epsilon$ sifted contracting homotopy h_ϵ . By contrast, (7.16) contains only connected diagrams and uses the unsifted contracting homotopy h . Although $h_\epsilon \rightarrow h$ as $\epsilon \rightarrow 0$, their lifts to the tensor algebra differ. Consequently, in the lifted operator one always has the rightmost action by h_ϵ ; i.e., $\mathbf{h}_\epsilon$ places h_ϵ on the last tensor factor. By contrast, since $P \neq 0$ for $\mathbf{h}$, there is no constraint pushing h to the last slot; unless P annihilates the tensor factors to the right of the position where h acts, h can act on any tensor factor. This mismatch survives the limit $\epsilon \rightarrow 0$ and leads to distinct contributions. For example, compare

$$\pi_1 \mathbf{m}_2 \mathbf{h}_\epsilon \mathbf{m}_2 (\Phi_1 \otimes \Phi_2 \otimes \Phi_3) , \quad (7.17)$$

and

$$\pi_1 \mathbf{m}_2 \mathbf{h} \mathbf{m}_2 (\Phi_1 \otimes \Phi_2 \otimes \Phi_3) , \quad (7.18)$$

for fields Φ_1, Φ_2, Φ_3 that are annihilated by h (and also by h_ϵ). The former evaluates to

$$m_2(\Phi_1, h_\epsilon m_2(\Phi_2, \Phi_3)) , \quad (7.19)$$

whereas the latter becomes

$$m_2(\Phi_1, h m_2(\Phi_2, \Phi_3)) + m_2(h m_2(\Phi_1, \Phi_2), P \Phi_3) . \quad (7.20)$$

When $P \Phi_3 \neq 0$, the two expressions do not coincide even in the limit $\epsilon \rightarrow 0$.

In the following subsections, we bridge this gap. In fact, if one extracts the connected part of (7.14), it agrees with (7.16).

7.2 Connected correlation functions and connected amplitudes

First, let us extract the part of $\mathbf{f}_\epsilon \mathbf{1}$ that corresponds to connected diagrams. Let us recall the example of the two-point correlation function from (5.23). Here, the correlation function can be written as:

$$\begin{aligned}
\pi_2 \mathbf{f}_\epsilon \mathbf{1} = & -i\hbar \sum_a \int d^d x e_a(x) \otimes h_\epsilon \tilde{e}^a(x) \\
& + (i\hbar)^2 \sum_{a,b} \int d^d x d^d y \left[e_a(x) \otimes h_\epsilon m_{0,2}(h\tilde{e}^a(x), h_\epsilon m_{0,2}(e_b(y), h\tilde{e}^b(y))) \right. \\
& \quad + (-1)^{e_a e_b} e_a(x) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon m_{0,2}(h\tilde{e}^a(x), h_\epsilon \tilde{e}^b(y))) \\
& \quad + e_a(x) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon m_{0,2}(h_\epsilon \tilde{e}^b(y), h_\epsilon \tilde{e}^a(x))) \\
& \quad + e_a(x) \otimes h_\epsilon m_{0,2}(h_\epsilon m_{0,2}(e_b(y), h_\epsilon \tilde{e}^b(y)), h_\epsilon \tilde{e}^a(x)) \\
& \quad \left. + h_\epsilon m_{0,2}(e_a(x), h_\epsilon \tilde{e}^a(x)) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon \tilde{e}^b(y)) \right] \\
& + \mathcal{O}(\hbar^3).
\end{aligned} \tag{7.21}$$

This expression contained the term

$$\begin{aligned}
& -i\hbar \sum_a \int d^d x e_a(x) \otimes h_\epsilon \tilde{e}^a(x) \\
& + (i\hbar)^2 \sum_{a,b} \int d^d x d^d y \left[e_a(x) \otimes h_\epsilon m_{0,2}(h\tilde{e}^a(x), h_\epsilon m_{0,2}(e_b(y), h\tilde{e}^b(y))) \right. \\
& \quad + (-1)^{e_a e_b} e_a(x) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon m_{0,2}(h_\epsilon \tilde{e}^a(x), h_\epsilon \tilde{e}^b(y))) \\
& \quad + e_a(x) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon m_{0,2}(h_\epsilon \tilde{e}^b(y), h_\epsilon \tilde{e}^a(x))) \\
& \quad \left. + e_a(x) \otimes h_\epsilon m_{0,2}(h_\epsilon m_{0,2}(e_b(y), h_\epsilon \tilde{e}^b(y)), h_\epsilon \tilde{e}^a(x)) \right],
\end{aligned} \tag{7.22}$$

which corresponds to the connected diagrams, and the term

$$(i\hbar)^2 \sum_{a,b} \int d^d x d^d y h_\epsilon m_{0,2}(e_a(x), h_\epsilon \tilde{e}^a(x)) \otimes h_\epsilon m_{0,2}(e_b(y), h_\epsilon \tilde{e}^b(y)), \tag{7.23}$$

which corresponds to the disconnected diagrams. A key feature of the connected parts is that m_2 does not appear in any position other than the rightmost one. This property holds in the general case.

To make this feature of the connected parts more explicit, we compute $\pi_n \mathbf{f}_\epsilon \mathbf{1}$. Alternatively, using relation (2.31), this can also be expressed as

$$\nabla(\pi_{n-1} \otimes' \pi_1) \Delta \mathbf{f}_\epsilon \mathbf{1}. \tag{7.24}$$

Here, we provide a useful identity to proceed with the calculation. For an arbitrary contracting homotopy h that satisfies the SDR conditions (not necessarily h_ϵ), the following relation holds:

$$\begin{aligned}
& \Delta \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U})} \mathbf{P} \\
&= \left(\frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m} + i\hbar \mathbf{h} \mathbf{U}} \mathbf{P} \otimes' \frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m} + i\hbar \mathbf{h} \mathbf{U}} \mathbf{P} \right) \Delta, \\
&+ \sum_{n=1}^{\infty} \int d^d x_1 \dots d^d x_n \left[\left(\prod_{i=1}^n e_{a_i}(x_i) \right) \frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m} + i\hbar \mathbf{h} \mathbf{U}} \mathbf{P} \right. \\
&\quad \left. \otimes' \prod_{i=1}^n \left(\frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m} + i\hbar \mathbf{h} \mathbf{U}} (-i\hbar \mathbf{h} \tilde{e}^{a_i}(x_i)) \right) \frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m} + i\hbar \mathbf{h} \mathbf{U}} \mathbf{P} \right] \Delta. \tag{7.25}
\end{aligned}$$

In the specific case where $h = h_\epsilon$, $\mathbf{P}$ corresponds to π_0 . Applying this formula, $\pi_n \mathbf{f}_\epsilon \mathbf{1}$ can be written as follows:

$$\begin{aligned}
\pi_n \mathbf{f}_\epsilon \mathbf{1} &= \nabla(\pi_{n-1} \otimes' \pi_1) \Delta \mathbf{f}_\epsilon \mathbf{1}, \\
&= \sum_{k=0}^{\infty} \sum_{a_1, \dots, a_k} \int d^d x_1 \dots d^d x_k (-1)^{e_{a_1} + \dots + e_{a_k}} (-i\hbar)^k \pi_{n-1} e_{a_1}(x_1) \dots e_{a_k}(x_k) \mathbf{f}_\epsilon \mathbf{1} \\
&\quad \otimes \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_k}(x_k) \dots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_1}(x_1) \mathbf{f}_\epsilon \mathbf{1}. \tag{7.26}
\end{aligned}$$

From this expression, it is clear that the connected part (characterized by the absence of the $\mathbf{m}$ operation except in the rightmost position) is given by

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{a_1, \dots, a_n} \int d^d x_1 \dots d^d x_n (-1)^{e_{a_1} + \dots + e_{a_n}} (-i\hbar)^n e_{a_1}(x_1) \dots e_{a_n}(x_n) \mathbf{1} \\
&\quad \otimes \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_n}(x_n) \dots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_1}(x_1) \mathbf{f}_\epsilon \mathbf{1}. \tag{7.27}
\end{aligned}$$

The connected part of the n -point correlation function, $\langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \rangle_{\text{conn}}$, can then be obtained by acting with the operator $(-1)^n \langle \omega | (\tilde{e}^{a_1}(x_1) \otimes \pi_1) \dots \langle \omega | (\tilde{e}^{a_n}(x_n) \otimes \pi_1)$ on this expression:

$$\begin{aligned}
& \langle \phi^{a_1}(x_1) \dots \phi^{a_n}(x_n) \rangle_{\text{conn}} \\
&= (-i\hbar)^{n-1} \sum_{\sigma \in S_{n-1}} (-1)^{1+e_{a_n} (e_{a_1} + \dots + e_{a_{n-1}}) + \epsilon(\sigma)} \\
&\quad \times \omega \left(\tilde{e}^{a_n}(x_n), \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_{\sigma(1)}}(x_{\sigma(1)}) \dots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_{\sigma(n-1)}}(x_{\sigma(n-1)}) \mathbf{f}_\epsilon \mathbf{1} \right). \tag{7.28}
\end{aligned}$$

Here, S_{n-1} is the symmetric group of degree $n-1$, and $(-1)^{\epsilon(\sigma)}$ is the sign factor arising from permuting $e_{a_1}, \dots, e_{a_{n-1}}$ into the order $e_{a_{\sigma(1)}}, \dots, e_{a_{\sigma(n-1)}}$. This expression includes the connected correlation functions for all cyclic permutations of the external fields $\phi^{a_1}, \dots, \phi^{a_n}$. In

particular, the component corresponding to the specific ordering $\phi^{a_1}(x_1), \dots, \phi^{a_n}(x_n)$ (such as clockwise along the disk boundary) is denoted $\langle\langle\phi^{a_1}(x_1) \cdots \phi^{a_n}(x_n)\rangle\rangle_{\text{conn}}$ and is given by

$$\begin{aligned} & \langle\langle\phi^{a_1}(x_1) \cdots \phi^{a_{n+1}}(x_n)\rangle\rangle_{\text{conn}} \\ &= (-i\hbar)^{n-1} (-1)^{1+\text{rev}(e_{a_1}, \dots, e_{a_n})} \omega\left(\tilde{e}^{a_n}(x_n), \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_{n-1}}(x_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{e}^{a_1}(x_1) \mathbf{f}_\epsilon \mathbf{1}\right), \end{aligned} \quad (7.29)$$

where $(-1)^{\text{rev}(e_{a_1}, \dots, e_{a_{n+1}})}$ is the sign factor resulting from reversing the order of $e_{a_1}, \dots, e_{a_{n+1}}$.

From this expression, to obtain the on-shell amplitude via the LSZ reduction formula, we need to replace the external lines $\tilde{e}^{a_i}$ with $\widehat{Q}_\epsilon \Phi_i$, multiply the factor $(\frac{i}{\hbar})^{n-1}$ and then take the limit $\epsilon \rightarrow 0$. However, it must be noted that $\tilde{e}^a$ is a coderivation promoted from $\tilde{e}^a$. Therefore, the substituted expression $\widehat{Q}_\epsilon \Phi_i$ must also be lifted to a coderivation. Let $\mathbf{D}(a)$ for $a \in V$ be the coderivation defined by

$$\mathbf{D}(a)\pi_0 = a, \quad \mathbf{D}(a)\pi_n = \sum_{i=0}^{n-1} \mathbb{I}^{\otimes i} \otimes a \otimes \mathbb{I}^{\otimes n-i-1} \quad \text{for } n \geq 1. \quad (7.30)$$

Ultimately, the connected amplitude can be written as follows:

$$-\lim_{\epsilon \rightarrow 0} \omega\left(\widehat{Q}_\epsilon \Phi_n, \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(\widehat{Q}_\epsilon \Phi_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(\widehat{Q}_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}\right), \quad (7.31)$$

7.3 Relation to the minimal model

Let us compare the expression (7.31) for the connected amplitude with the minimal model (7.16). In fact, the two expressions agree at the tree-level. At the loop-level, the expression (7.31) agrees with the minimal model if we replace $\widehat{Q}_\epsilon$ with Q_ϵ . This yields

$$-\lim_{\epsilon \rightarrow 0} \omega\left(Q_\epsilon \Phi_n, \pi_1 \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}\right), \quad (7.32)$$

for all $\Phi_1, \dots, \Phi_n \in V^{(P)}$. This substitution, where $\widehat{Q}_\epsilon$ is replaced by Q_ϵ , is precisely equivalent to performing a naive amputation of the external lines (i.e., neglecting mass shift factors). First, utilizing the cyclicity of Q_ϵ and the formula (5.29), this expression can be written as

$$\begin{aligned} & -\lim_{\epsilon \rightarrow 0} \omega\left(\Phi_n, \pi_1 \mathbf{m} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}\right) \\ & + \lim_{\epsilon \rightarrow 0} \omega\left(\Phi_n, \pi_1 \mathbf{f}_\epsilon \mathbf{Q}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}\right). \end{aligned} \quad (7.33)$$

Regarding the second term, since the contracting homotopies h and h_ϵ annihilate Φ_i , the leftmost factor $\mathbf{f}_\epsilon$ reduces to the identity operator $\mathbf{I}$. This yields

$$\lim_{\epsilon \rightarrow 0} \omega\left(\Phi_n, \pi_1 \mathbf{Q}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{n-1}) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}\right) \quad (7.34)$$

This term vanishes trivially when $n \geq 3$, because the projection π_1 acts on a tensor product involving at least two factors. In the case of $n = 2$, the term $\omega(\Phi_2, Q_\epsilon \Phi_1)$ survives. However, as $\epsilon \rightarrow 0$, Q_ϵ becomes Q . Since Φ_1 and Φ_2 are elements of $V^{(P)}$, the action of Q on Φ_1 causes the entire expression to vanish. Therefore, the second term in (7.33) vanishes. To proceed with the calculation of the remaining terms, let us analyze the operator $\mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i)$. We consider the action of this operator on a generic element $a_1 \otimes \cdots \otimes a_k$, where all a_i vanish under the action of h_ϵ . The operator $\mathbf{D}(Q_\epsilon \Phi_i)$ acts in the form:

$$\begin{aligned} & Q_\epsilon \Phi_i \otimes a_1 \otimes \cdots \otimes a_k \\ & + \sum_{l=1}^{k-1} (-1)^{a_1 + \cdots + a_l} a_1 \otimes \cdots \otimes a_l \otimes Q_\epsilon \Phi_i \otimes a_{l+1} \otimes \cdots \otimes a_k \\ & + (-1)^{a_1 + \cdots + a_k} a_1 \otimes \cdots \otimes a_k \otimes Q_\epsilon \Phi_i. \end{aligned} \quad (7.35)$$

However, since $\mathbf{h}_\epsilon$ acts only on the rightmost tensor factor, only the contribution where $Q_\epsilon \Phi_i$ is located at the rightmost position survives the action of $\mathbf{h}_\epsilon$. This action of $\mathbf{h}_\epsilon$ then removes Q_ϵ , simplifying the expression to

$$\mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i)(a_1 \otimes \cdots \otimes a_k) = a_1 \otimes \cdots \otimes a_k \otimes h_\epsilon Q_\epsilon \Phi_i. \quad (7.36)$$

Taking the previous result into account, and leveraging the relation $h_\epsilon Q_\epsilon \Phi_i = \Phi_i$ (since Φ_i is annihilated by h_ϵ), the remaining term in (7.33) is expressed as

$$- \lim_{\epsilon \rightarrow 0} \omega \left(\Phi_n, \pi_1 \mathbf{m} \mathbf{f}_\epsilon \left(\cdots \mathbf{f}_\epsilon (\mathbf{f}_\epsilon (\mathbf{f}_\epsilon \mathbf{1} \otimes \Phi_1) \otimes \Phi_2) \cdots \otimes \Phi_{n-1} \right) \right). \quad (7.37)$$

Here, the operator $\mathbf{f}_\epsilon$ acts solely on the argument contained within the parenthesis that directly succeeds it.

On the other hand, let us consider

$$\mathbf{f} = \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U})} = \sum_{n=0}^{\infty} (-\mathbf{h}(\mathbf{m} + i\hbar \mathbf{U}))^n. \quad (7.38)$$

The operator $\mathbf{h}$ has two patterns of action: (i) h acts on the rightmost factor, or (ii) h acts at an intermediate position, but the element immediately following it is subject to the projection P (which also covers the rightmost factor). The former pattern of action is identical to that of $\mathbf{h}_\epsilon$. Due to the identity $Ph = 0$, once pattern (i) has occurred (i.e., h has acted on the rightmost factor), pattern (ii) is precluded from appearing anywhere in the subsequent chain

of operators. Consequently, for $\Phi_1, \dots, \Phi_n \in V^{(P)}$, the action of $\mathbf{f}$ can be written as

$$\begin{aligned} & \mathbf{f}(\Phi_1 \otimes \dots \otimes \Phi_n) \\ &= \lim_{\epsilon \rightarrow 0} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(-\mathbf{h}_{\epsilon}(\mathbf{m} + i\hbar \mathbf{U}) \right)^k \left[\left(-\mathbf{h}(\mathbf{m} + i\hbar \mathbf{U}) \right)^l (\Phi_1 \otimes \dots \otimes \Phi_{n-1}) \otimes P\Phi_n \right], \\ &= \lim_{\epsilon \rightarrow 0} \mathbf{f}_{\epsilon}(\mathbf{f}(\Phi_1 \otimes \dots \otimes \Phi_{n-1}) \otimes P\Phi_n). \end{aligned} \quad (7.39)$$

Here, the action corresponding to pattern (i) is realized by $\lim_{\epsilon \rightarrow 0} \mathbf{h}_{\epsilon}$. By applying this recursively and finally utilizing the relation $\mathbf{f}\mathbf{1} = \mathbf{f}_{\epsilon}\mathbf{1}$, we obtain

$$\mathbf{f}(\Phi_1 \otimes \dots \otimes \Phi_n) = \lim_{\epsilon \rightarrow 0} \mathbf{f}_{\epsilon} \left(\dots \mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}\mathbf{1} \otimes P\Phi_1) \otimes P\Phi_2) \dots \otimes P\Phi_n \right). \quad (7.40)$$

Since $P\Phi_i = \Phi_i$, the right-hand side is precisely the term included in (7.37). Consequently, the expression (7.37) can ultimately be written as

$$-\omega \left(\Phi_n, \pi_1 \mathbf{m} \frac{1}{\mathbf{I} + \mathbf{h}(\mathbf{m} + i\hbar \mathbf{U})} \pi_n (\Phi_1 \otimes \dots \otimes \Phi_{n-1}) \right), \quad (7.41)$$

which is exactly the minimal model itself.

Furthermore, it is possible to obtain an expression corresponding to (7.37) from (7.31) without performing the substitution $\widehat{Q}_{\epsilon} \rightarrow Q_{\epsilon}$. This procedure corresponds to the correct amputation of the external lines. This is achieved simply by the substitution $\Phi_i \rightarrow h_{\epsilon} \widehat{Q}_{\epsilon} \Phi_i$:

$$-\lim_{\epsilon \rightarrow 0} \omega \left(h_{\epsilon} \widehat{Q}_{\epsilon} \Phi_n, \pi_1 \mathbf{m} \mathbf{f}_{\epsilon} \left(\dots \mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}\mathbf{1} \otimes h_{\epsilon} \widehat{Q}_{\epsilon} \Phi_1) \otimes h_{\epsilon} \widehat{Q}_{\epsilon} \Phi_2) \dots \otimes h_{\epsilon} \widehat{Q}_{\epsilon} \Phi_{n-1}) \right) \right). \quad (7.42)$$

where the transformed Φ_i belongs to the cohomology of $\widehat{Q}$. We must re-emphasize the caution: although taking the limit $\epsilon \rightarrow 0$ might seemingly cause the expression to vanish, this procedure is safe because $\widehat{Q}_{\epsilon}$ is utilized to amputate the external lines and is subsequently eliminated before the final limit is taken.

7.4 Off-shell amplitude

The minimal model (7.16) restricted its input to the space $V^{(P)}$. Correspondingly, the expression (7.37) obtained via the LSZ reduction formula also limited its input to $V^{(P)}$. Alternatively, the expression (7.42), obtained through the correct amputation of external lines, restricted the input to the cohomology of $\widehat{Q}$. However, the majority of the calculations required to obtain expression (7.37) hold true provided that the input is simply annihilated by h and h_{ϵ} . The only step where the restriction $\Phi_i \in V^{(P)}$ was utilized was in evaluating the second term in (7.33), where we specifically required $\omega(\Phi_2, Q\Phi_1) = 0$. Consequently, if we do not restrict the input to $V^{(P)}$ but instead extend it to the larger space $V^{(P)} \oplus V^{(hQ)}$, we obtain the following expression:

$$\delta_{n,2} \omega(\Phi_2, Q\Phi_1) - \lim_{\epsilon \rightarrow 0} \omega \left(\Phi_n, \pi_1 \mathbf{m} \mathbf{f}_{\epsilon} \left(\dots \mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}(\mathbf{f}_{\epsilon}\mathbf{1} \otimes \Phi_1) \otimes \Phi_2) \dots \otimes \Phi_{n-1}) \right) \right), \quad (7.43)$$

where, $\delta_{n,2}$ equals 1 if $n = 2$ and 0 otherwise. This expression is defined on a space broader than $V^{(P)}$ (the on-shell space) and can therefore be interpreted as an off-shell amplitude. However, since this expression is derived using a naive amputation of the external lines, one must use the substitution $\Phi_i \rightarrow h_\epsilon \widehat{Q} \epsilon \Phi_i$ to obtain the correct amplitude, as discussed at the end of the previous subsection. The first term can be interpreted as the tree-level two-point off-shell amplitude. This term vanishes in the on-shell regime, which corresponds to the absence of the two-point on-shell amplitude in the minimal model. In conventional field theory, the two-point on-shell amplitude is generally non-zero, but our current formalism cannot reproduce it. A method for obtaining the two-point on-shell amplitude within the context of homotopy algebra by considering boundary terms has been proposed in [60]. However, since the scope of this work does not include the consideration of boundary contributions, we shall consequently omit the two-point on-shell amplitude from our subsequent analysis.

Now, let us consider the consequences of extending the input space to include elements belonging to $V^{(Qh)}$. If an element of $V^{(Qh)}$ is directly inserted as the input for expression (7.43), it generates contributions that are distinct from the diagrams produced by the minimal model, resulting in terms that cannot be interpreted as an amplitude. As long as the input is restricted to the space $V^{(P)} \oplus V^{(hQ)}$ (where elements are annihilated by h), the contracting homotopy h only acts on m_n or U . These actions correspond to propagators connecting vertices or forming loops. However, if the input includes elements from $V^{(Qh)}$ (which are not annihilated by h), contributions arise where h acts directly on these input elements. Such contributions cannot be interpreted as an amplitude. Therefore, let us now consider an expression that can expand the input space to include $V^{(Qh)}$ while preserving the generated diagrams.

In the tree-level case, where $\mathbf{f}_\epsilon$ is replaced by $\mathbf{f}_\epsilon^{(0)} = \frac{1}{I + h_\epsilon \mathbf{m}}$, a relatively simple expression is available [46]. Here, the operator $\mathbf{m}$ is understood not to include quantum corrections and merely satisfies the A_∞ relations. Our starting point is to consider a cohomomorphism $\mathbf{F}$ satisfying the relation

$$\pi_1 \mathbf{F} = \pi_1 (\mathbb{I} - h \mathbf{m} \mathbf{F}) . \quad (7.44)$$

Here, a cohomomorphism is a map $\mathbf{F} : \mathcal{TV} \rightarrow \mathcal{TV}$ that satisfies the relation

$$\Delta \mathbf{F} = (\mathbf{F} \otimes' \mathbf{F}) \Delta , \quad (7.45)$$

with respect to the coproduct Δ . Furthermore, $\mathbf{F}$ is entirely determined by its component $\pi_1 \mathbf{F}$, with the higher components given by the tensor powers:

$$\mathbf{F} = \pi_0 + \sum_{n=1}^{\infty} (\pi_1 \mathbf{F})^{\otimes n} . \quad (7.46)$$

Formally, such a cohomomorphism $\mathbf{F}$ can be constructed by solving the defining relation recursively, yielding the expression for the first component as

$$\pi_1 \mathbf{F} = \pi_1 \mathbb{I} - \pi_1 \mathbf{h} \mathbf{m} + \pi_1 \mathbf{h} \mathbf{m} \mathcal{D}(\pi_1 \mathbf{h} \mathbf{m}) - \pi_1 \mathbf{h} \mathbf{m} \nabla (\mathcal{D}(\pi_1 \mathbf{h} \mathbf{m}) \otimes' \mathcal{D}(\pi_1 \mathbf{h} \mathbf{m})) \triangle + \cdots. \quad (7.47)$$

where $\mathcal{D}(\pi_1 \mathbf{C})$ for a map $\pi_1 \mathbf{C} : \mathcal{T}V \rightarrow V$ represents the coderivation generated by $\pi_1 \mathbf{C}$, as defined by

$$\mathcal{D}(\pi_1 \mathbf{C}) := \sum_{n=0}^{\infty} \mathcal{D}(\pi_1 \mathbf{C} \pi_n), \quad (7.48a)$$

$$\mathcal{D}(\pi_1 \mathbf{C} \pi_n) \pi_k = 0 \quad \text{for } k < n, \quad \mathcal{D}(\pi_1 \mathbf{C} \pi_n) \pi_k = \sum_{i=0}^{k-n} \mathbb{I}^{\otimes i} \otimes \pi_1 \mathbf{C} \pi_n \otimes \mathbb{I}^{\otimes k-n-i} \quad \text{for } k \geq n \quad (7.48b)$$

where $\deg(\mathbf{C})$ denotes the degree of $\pi_1 \mathbf{C}$. Here, $\mathbf{C}$ is not necessarily a coderivation, but if it is, the relation $\mathcal{D}(\pi_1 \mathbf{C}) = \mathbf{C}$ holds.

To express this in a closed form, we proceed as follows: First, consider V' as a copy of the vector space V and let $\{e'_s\}$ be a basis of V' . We assume that h vanishes upon acting on V' . Here, we introduce three new linear maps, $\mathfrak{T} : V \rightarrow V'$, $\mathfrak{P} : V \oplus V' \rightarrow V'$ and $\mathfrak{J} : V \oplus V' \rightarrow V$. Here are the definitions of the maps $\mathfrak{T}$, $\mathfrak{P}$, and $\mathfrak{J}$ acting on the basis elements. The map $\mathfrak{T}$ acts as a transference, mapping the basis of V directly to the basis of V' :

$$\mathfrak{T} e_s = e'_s. \quad (7.49)$$

Conversely, $\mathfrak{P}$ acts as a projection onto V' :

$$\mathfrak{P} e_s = 0, \quad \mathfrak{P} e'_s = e'_s. \quad (7.50)$$

Finally, $\mathfrak{J}$ integrates the elements of V and V' into V :

$$\mathfrak{J} e_s = e_s, \quad \mathfrak{J} e'_s = e_s. \quad (7.51)$$

From these definitions, it immediately follows that

$$\mathfrak{J} \mathfrak{T} = \mathbb{I}_V, \quad (7.52)$$

$$\mathfrak{P} \mathfrak{T} = \mathfrak{T}. \quad (7.53)$$

We promote these to operators on the tensor algebra as follows:

$$\mathfrak{T} = \pi_0, \quad \mathfrak{T} \pi_n = \mathfrak{T}^{\otimes n} \quad \text{for } n \geq 1, \quad (7.54)$$

$$\mathfrak{P} = \pi_0, \quad \mathfrak{P} \pi_n = \mathfrak{T}^{\otimes n} \quad \text{for } n \geq 1, \quad (7.55)$$

$$\mathfrak{J} = \pi_0, \quad \mathfrak{J} \pi_n = \mathfrak{T}^{\otimes n} \quad \text{for } n \geq 1, \quad (7.56)$$

Furthermore, we define $\mathfrak{h}$ and $\mathfrak{m}$ as follows:

$$\mathfrak{h}\pi_0 = 0, \quad \mathfrak{h}\pi_n = \sum_{k=0}^{n-1} (\mathbb{I}_{V \oplus V'})^{\otimes k} \otimes h \otimes \mathfrak{P}^{\otimes n-k-1} \quad \text{for } n \geq 1, \quad (7.57)$$

$$\mathfrak{m} = \mathcal{D}(\pi_1 \mathfrak{m} \mathfrak{J}). \quad (7.58)$$

Using these operators, the cohomomorphism $\mathbf{F}$ satisfying condition (7.44) can be constructed as

$$\mathbf{F} = \mathfrak{J} \frac{1}{I + \mathfrak{h}\mathfrak{m}} \mathfrak{T}. \quad (7.59)$$

However, for the subsequent discussion, the specific form of this expression is not crucial; what is important is that it satisfies condition (7.44).

This cohomomorphism $\mathbf{F}$ possesses a left inverse, which is also a cohomomorphism and is given by

$$\pi_1 \mathbf{F}^{-1} = \pi_1 (I + \mathfrak{h}\mathfrak{m}). \quad (7.60)$$

Indeed, it is straightforward to verify that this is the left inverse. Given that the composition of cohomomorphisms is again a cohomomorphism, it is sufficient to check only the π_1 component:

$$\begin{aligned} \pi_1 \mathbf{F}^{-1} \mathbf{F} &= \pi_1 (I + \mathfrak{h}\mathfrak{m}) \mathbf{F}, \\ &= \pi_1 \mathbf{F} + \pi_1 \mathfrak{h}\mathfrak{m} \mathbf{F}, \\ &= \pi_1 I. \end{aligned} \quad (7.61)$$

Showing that $\mathbf{F}^{-1}$ is also a right inverse is less trivial. In general, we compute the following composition:

$$\begin{aligned} \pi_1 \mathbf{F} \mathbf{F}^{-1} &= \pi_1 (I - \mathfrak{h}\mathfrak{m} \mathbf{F}) \mathbf{F}^{-1}, \\ &= \pi_1 \mathbf{F}^{-1} - \pi_1 \mathfrak{h}\mathfrak{m} \mathbf{F} \mathbf{F}^{-1}, \\ &= \pi_1 I + \pi_1 \mathfrak{h}\mathfrak{m} - \pi_1 \mathfrak{h}\mathfrak{m} \mathbf{F} \mathbf{F}^{-1}. \end{aligned} \quad (7.62)$$

The resulting recursive expression makes it non-trivial whether the second and third terms on the right-hand side cancel out. However, within the framework of perturbation theory, solving this expression recursively reveals that the second and third terms cancel at every order. Therefore, within perturbation theory, $\mathbf{F}^{-1}$ acts as both the left and right inverse of $\mathbf{F}$. In the following discussion, we assume this perturbative behavior and its inverse $\mathbf{F}^{-1}$ satisfies the full relations

$$\mathbf{F}^{-1} \mathbf{F} = \mathbf{F} \mathbf{F}^{-1} = I. \quad (7.63)$$

This cohomomorphism $\mathbf{F}$ satisfies

$$\begin{aligned}\mathbf{F}(\Phi_1 \otimes \cdots \otimes \Phi_n) &= \lim_{\epsilon \rightarrow 0} \pi_1 \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1}, \\ &= \lim_{\epsilon \rightarrow 0} \pi_1 \mathbf{f}_\epsilon^{(0)} \left(\cdots \mathbf{f}_\epsilon^{(0)} (\mathbf{f}_\epsilon^{(0)} (\mathbf{f}_\epsilon^{(0)} \mathbf{1} \otimes \Phi_1) \otimes \Phi_2) \cdots \otimes \Phi_n \right),\end{aligned}\quad (7.64)$$

for all inputs $\Phi_1, \dots, \Phi_n \in V^{(P)} \oplus V^{(hQ)}$. Note that $\mathbf{f}_\epsilon^{(0)} \mathbf{1} = \mathbf{1}$. We intentionally write $\mathbf{f}_\epsilon^{(0)} \mathbf{1}$ here to facilitate comparison with the loop-level case. Moreover, the set of generated diagrams remains invariant even if the input includes elements from $V^{(Qh)}$, because the contracting homotopy h always appears in the form $\pi_1 \mathbf{h}$. Consequently, this expression allows us to write down the off-shell amplitude that is valid even for inputs in $V^{(Qh)}$. The proof of (7.64) is as follows: The proof utilizes the identities

$$\Delta \mathbf{f}_\epsilon^{(0)} = [\mathbf{f}_\epsilon^{(0)} \otimes' \pi_0 + \mathbf{I} \otimes' \mathbf{f}_\epsilon^{(0)}] \Delta, \quad (7.65)$$

$$\Delta \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi) = [\mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi) \otimes' \pi_0 + \mathbf{I} \otimes' \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi)] \Delta. \quad (7.66)$$

The first identity holds true specifically on the subspace that is annihilated by the contracting homotopy h . Utilizing these identities, we proceed to demonstrate the relation

$$\lim_{\epsilon \rightarrow 0} \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1} = \Phi_1 \otimes \cdots \otimes \Phi_n. \quad (7.67)$$

Let us first examine the case where $n = 1$. The π_k component can be calculated as follows:

$$\begin{aligned}\pi_k \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1} &= \nabla (\pi_{k-1} \otimes' \pi_1) \Delta \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1}, \\ &= \pi_{k-1} \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1} \otimes \pi_1 \mathbf{F}^{-1} \mathbf{1} \\ &\quad + \pi_{k-1} \mathbf{F}^{-1} \mathbf{1} \otimes \pi_1 \mathbf{F}^{-1} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1}.\end{aligned}\quad (7.68)$$

The first term on the right-hand side vanishes because the identity $\mathbf{F}^{-1} \mathbf{1} = \mathbf{1}$ holds. (This relies on the standard tree-level assumption that $\mathbf{m} \mathbf{1} = 0$.) The second term remains only when $k = 1$. In this case, the expression can be calculated as

$$\lim_{\epsilon \rightarrow 0} \pi_1 \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1} = \lim_{\epsilon \rightarrow 0} \pi_1 (\mathbf{I} + \mathbf{h} \mathbf{m}) \frac{1}{\mathbf{I} + \mathbf{h}_\epsilon \mathbf{m}} \Phi_1. \quad (7.69)$$

As previously mentioned, $\mathbf{h}$ and $\mathbf{h}_\epsilon$ are distinct in the limit $\epsilon \rightarrow 0$. However, $\pi_1 \mathbf{h} = h$ and $\pi_1 \mathbf{h}_\epsilon = h_\epsilon$ coincide. Therefore, the expression $\pi_1 (\mathbf{I} + \mathbf{h} \mathbf{m}) \frac{1}{\mathbf{I} + \mathbf{h}_\epsilon \mathbf{m}}$ simplifies to π_1 . Consequently, we can first state the result for the $n = 1$ case as

$$\lim_{\epsilon \rightarrow 0} \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1} = \Phi_1. \quad (7.70)$$

Next, we employ mathematical induction: assuming the validity of relation (7.67) for all degrees up to $n = m$, we demonstrate its validity for $n = m + 1$. Utilizing identities (7.65) and (7.66), we obtain

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \Delta \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1} \\
&= \lim_{\epsilon \rightarrow 0} \left[\mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1} \otimes' \mathbf{F}^{-1} \mathbf{1} \right. \\
&\quad + \mathbf{F}^{-1} \mathbf{1} \otimes' \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1} \\
&\quad + \sum_{k=1}^m \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m-k+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{1} \\
&\quad \left. \otimes' \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m-k+2}) \mathbf{1} \right]. \tag{7.71}
\end{aligned}$$

From the inductive hypothesis, the third term on the right-hand side is equal to

$$\sum_{k=1}^m \Phi_1 \otimes \cdots \otimes \Phi_{m-k+1} \otimes' \Phi_{m-k+2} \otimes \cdots \otimes \Phi_{m+1}. \tag{7.72}$$

We now calculate the π_k component using the identity $\pi_k = \nabla(\pi_{k-1} \otimes' \pi_1) \Delta$. The first term vanishes for the same reason as in the $n = 1$ case. In the second term, only the $k = 1$ component of the left side of the $\otimes'$ product survives; however, in this instance, the right side always involves two or more tensor factors, causing the entire term to vanish. The third term survives only when $k = m + 1$ and only when the right-hand side of the $\otimes'$ product has exactly one tensor factor. We ultimately obtain

$$\lim_{\epsilon \rightarrow 0} \mathbf{F}^{-1} \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{m+1}) \cdots \mathbf{f}_\epsilon^{(0)} \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon^{(0)} \mathbf{1} = \Phi_1 \otimes \cdots \otimes \Phi_{m+1}. \tag{7.73}$$

This completes the proof of identity (7.64). Consequently, the tree-level off-shell amplitude can be written as

$$\delta_{n,2} \omega(\Phi_2, Q\Phi_1) - \omega\left(\Phi_n, \pi_1 \mathbf{m} \mathbf{F}(\Phi_1 \otimes \cdots \otimes \Phi_{n-1})\right). \tag{7.74}$$

This expression is applicable to the entire vector space, $\mathbf{V} = \mathbf{V}^{(P)} \oplus \mathbf{V}^{(Qh)} \oplus \mathbf{V}^{(hQ)}$. Furthermore, the component $\pi_1 \mathbf{m} \mathbf{F}$ is cyclic. This property can be shown as follows. It suffices to show that

$$\langle \omega | (\pi_1 \mathbf{m} \mathbf{F} \otimes \pi_1 + \pi_1 \otimes \pi_1 \mathbf{m} \mathbf{F}) = 0. \tag{7.75}$$

Using the definitions of $\mathbf{F}$ and $\mathbf{F}^{-1}$, we compute the following expression:

$$\begin{aligned}
\langle \omega | (\pi_1 \mathbf{m} \mathbf{F} \otimes \pi_1 + \pi_1 \otimes \pi_1 \mathbf{m} \mathbf{F}) &= \langle \omega | (\pi_1 \mathbf{m} \otimes \pi_1 \mathbf{F}^{-1} + \pi_1 \mathbf{F}^{-1} \otimes \pi_1 \mathbf{m}) \mathbf{F}, \\
&= \langle \omega | (\pi_1 \mathbf{m} \otimes \pi_1 + \pi_1 \otimes \pi_1 \mathbf{m}) \mathbf{F} \\
&\quad + \langle \omega | (\pi_1 \mathbf{m} \otimes h\pi_1 \mathbf{m} + h\pi_1 \mathbf{m} \otimes \pi_1 \mathbf{m}) \mathbf{F}. \tag{7.76}
\end{aligned}$$

The first term on the right-hand side vanishes due to the cyclicity of $\mathbf{m}$. Furthermore, the second term vanishes because of the assumed property of the contracting homotopy h given in equation (5.5). This concludes the proof that $\pi_1 \mathbf{mF}$ is cyclic, which subsequently allows us to state the following identity for the amplitude:

$$\omega\left(\Phi_n, \pi_1 \mathbf{mF}(\Phi_1 \otimes \cdots \otimes \Phi_{n-1})\right) = \omega\left(\Phi_1, \pi_1 \mathbf{mF}(\Phi_2 \otimes \cdots \otimes \Phi_n)\right). \quad (7.77)$$

This relation is precisely the statement of crossing symmetry. Note that the ability to write cyclicity in the form of (7.75) stems from the fact that the A_∞ algebra is defined on $\mathcal{TV}$. Conversely, L_∞ algebra, which is defined on $\mathcal{SV}$, cannot be written in this straightforward manner and thus requires a different treatment.

The loop-level case is more complicated and cannot be expressed using a cohomomorphism. However, the amplitude can be written in terms of an operator $\mathcal{F}$ satisfying

$$\mathcal{F} = \pi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{F}^{\wedge n}. \quad (7.78)$$

Here, $\mathbf{F}$ is an extension of the tree-level operator, satisfying

$$\pi_{k+1} \mathbf{F} = \sum_{s_1, \dots, s_k} (-1)^{e_{s_1} + \cdots + e_{s_k}} \frac{\hbar^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1 (\mathbf{I} - \hbar \mathbf{mF}) \mathbf{D}(\hbar \tilde{e}^{s_k}) \cdots \mathbf{D}(\hbar \tilde{e}^{s_1}). \quad (7.79)$$

The operator $\wedge$ denotes the symmetrized tensor product (2.48). It is important to note the appearance of the symmetrized tensor product, even though we are working within the framework of quantum A_∞ algebra at the loop level. In the classical limit ($\hbar \rightarrow 0$), only the component $\pi_1 \mathbf{F}$ remains non-trivial, and the operator $\mathcal{F}$ coincides with the tree-level cohomomorphism $\mathbf{F}$. This operator $\mathcal{F}$ is not a cohomomorphism² because it does not satisfy the required property, $\Delta \mathcal{F} \neq (\mathcal{F} \otimes' \mathcal{F}) \Delta$. In fact, the loop-level off-shell amplitude obtained via naive amputation can be written using the operator $\mathcal{F}$ as

$$\delta_{n,2} \omega(\Phi_2, Q\Phi_1) - \omega\left(\Phi_n, \pi_1 \mathbf{mF}(\Phi_1 \otimes \cdots \otimes \Phi_{n-1})\right). \quad (7.80)$$

As the derivation is rather technical, we defer the full proof to Appendix A. Furthermore, the component $\pi_1 \mathbf{mF}$ is also cyclic, leading to the crossing symmetry of the amplitude. We defer the detailed proof to Appendix B.

²In the context of quantum L_∞ algebra, a similarly constructed object is referred to as an IBL_∞ algebra. However, the author is currently unaware of the general terminology for this structure within the present quantum A_∞ algebra framework.

8 BSFT-like Action

The cohomomorphism $\mathbf{F}$, defined by equation (7.44) and derived from the tree-level off-shell amplitude in the preceding section, yields an interesting observation. This invertible cohomomorphism allows for a field redefinition in the theory. This field redefinition can be utilized to rewrite the classical action that is originally constructed based on the A_∞ algebra structure. Specifically, if we consider the action of (bosonic) open string field theory based on A_∞ algebras as the original action, the redefined action exhibits properties analogous to the action of Boundary String Field Theory (BSFT) [47, 48, 49, 50, 51]. Consequently, this redefined action is referred to as the BSFT-like action. The relationship between the A_∞ algebra and BSFT remains an open question. The existence of this BSFT-like action strongly suggests that the BSFT action can be rewritten as an A_∞ algebra-based action through a field redefinition, hinting at a deep and unclarified equivalence between the two formalisms. This section is based on [46].

In this section, we denote the basis simply as e_s , as there is no need to distinguish between discrete and continuous labels.

8.1 BSFT-like action from field redefinition

Let us consider the classical action S based on the A_∞ algebra

$$S = - \sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Phi, \pi_1 \mathbf{M} \pi_n \frac{1}{1-\Phi}\right). \quad (8.1)$$

We consider the action corresponding to open string field theory later, but for the present, we continue the discussion of general field theory based on A_∞ algebra. Here, we redefine the field Φ using the cohomomorphism $\mathbf{F}$:

$$\mathbf{F} = \pi_0 + \sum_{n=1}^{\infty} (\pi_1 \mathbf{F})^{\otimes n}, \quad (8.2)$$

$$\pi_1 \mathbf{F} = \pi_1 (\mathbf{I} - h m \mathbf{F}), \quad (8.3)$$

obtained in the previous section, introducing a new field Ψ such that

$$\Phi = \pi_1 \mathbf{F} \frac{1}{1-\Psi}. \quad (8.4)$$

Since $\mathbf{F}$ is a degree-even map, the new field Ψ must also be degree-even. The new field Ψ is gauge-fixed in the same manner as Φ . This implies that Ψ belongs to the space $V^{(P)} \oplus V^{(hQ)}$. Therefore, the field Ψ is annihilated by the contracting homotopy h . When the field Ψ is

expanded in the basis of the space $V^{(P)} \oplus V^{(hQ)}$ as

$$\Psi = \sum_{e_s \in V^{(P)} \oplus V^{(hQ)}} \psi^s e_s, \quad (8.5)$$

the coefficients ψ^s are the elements that correspond to the fields in conventional field theory. Furthermore, since $\mathbf{F}$ is a cohomomorphism and is entirely determined by its π_1 component, the following identity holds:

$$\frac{1}{1 - \Phi} = \mathbf{F} \frac{1}{1 - \Psi}. \quad (8.6)$$

Conversely, this identity implies that the field Ψ can be expressed in terms of Φ as

$$\Psi = \pi_1 \mathbf{F}^{-1} \frac{1}{1 - \Phi}. \quad (8.7)$$

By rewriting the field redefinition equation, the field Ψ can be expressed as

$$\Psi = P\Phi + \pi_1 \mathbf{h} \mathbf{M} \frac{1}{1 - \Phi}. \quad (8.8)$$

Here, we used (4.4) and gauge fixing condition $h\Phi = 0$. This equation clearly establishes the relationship between the original field Φ and the redefined field Ψ . For a solution Φ of the equation of motion $\mathbf{M} \frac{1}{1 - \Phi} = 0$, the corresponding Ψ belongs to the Q -cohomology. When the Q -cohomology is trivial, such as in bosonic open string field theory (OSFT) on the tachyon vacuum, the corresponding Ψ for any solution of the equation of motion for Φ is always zero. This outcome might seem strange because, in this case, Ψ cannot distinguish between trivial and non-trivial solutions. However, this issue stems from the subtlety of the existence of the operator $\mathbf{F}$ and the fact that $\mathbf{F}^{-1}$ being a right inverse of $\mathbf{F}$ is non-trivial in a non-perturbative setting. Since we are considering only the perturbation theory in this work, this issue does not pose a problem. In general, even if Ψ belongs to the Q -cohomology, the corresponding Φ , defined by equation (8.4), does not necessarily solve the equation of motion $\mathbf{M} \frac{1}{1 - \Phi} = 0$. The equation of motion for Φ is equivalent to the following equation, which is expressed entirely in terms of the redefined field Ψ

$$\mathbf{F}^{-1} \mathbf{M} \mathbf{F} \frac{1}{1 - \Psi} = 0. \quad (8.9)$$

Let us now evaluate the composite operator $\mathbf{F}^{-1} \mathbf{M} \mathbf{F}$. First, since $\mathbf{M}$ is a coderivation and both $\mathbf{F}^{-1}$ and $\mathbf{F}$ are cohomomorphism, it immediately follows that the composite operator $\mathbf{F}^{-1} \mathbf{M} \mathbf{F}$ is also a coderivation. Therefore, it is determined only by the π_1 component. Using

the A_∞ relations and $\mathbf{Qh} + \mathbf{hQ} + \mathbf{P} = \mathbf{I}$, we can calculate as

$$\begin{aligned}
\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} &= \pi_1 (\mathbf{I} + \mathbf{h} \mathbf{m}) (\mathbf{Q} + \mathbf{m}) \mathbf{F} , , \\
&= \mathbf{Q} \pi_1 (\mathbf{I} - \mathbf{h} \mathbf{m} \mathbf{F}) + \pi_1 \mathbf{m} \mathbf{F} + \pi_1 \mathbf{h} \mathbf{m} (\mathbf{Q} + \mathbf{m}) \mathbf{F} , \\
&= \pi_1 \mathbf{Q} + \pi_1 (\mathbf{I} - \mathbf{Qh} - \mathbf{hQ}) \mathbf{m} \mathbf{F} , \\
&= \pi_1 \mathbf{Q} + \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} ,
\end{aligned} \tag{8.10}$$

This equality holds as a coderivation:

$$\mathbf{F}^{-1} \mathbf{M} \mathbf{F} = \mathbf{Q} + \mathcal{D}(\pi_1 \mathbf{P} \mathbf{m} \mathbf{F}) , \tag{8.11}$$

Since $\mathbf{M}$ is nilpotent, the left-hand side of (8.11) also squares to zero. Ultimately, the equation of motion (specifically, π_1 component of it) is given by

$$\mathbf{Q} \Psi + \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi} = 0 . \tag{8.12}$$

By acting with the projections P and $\mathbb{I} - P$ onto Q -cohomology from the left, this equation of motion requires that the first and second terms, respectively, are zero:

$$\mathbf{Q} \Psi = 0, \quad \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi} = 0 . \tag{8.13}$$

Given that $\Psi \in V^{(P)} \oplus V^{(hQ)}$, the first equation implies that Ψ belongs to the Q -cohomology. The second equation imposes an additional condition. The operator $\pi_1 \mathbf{m} \mathbf{F}$, as seen in the previous section, gives the amplitude. Therefore, since the input Ψ belongs to the Q -cohomology (by the first equation) and the output is also restricted to Q -cohomology by P , this condition requires that the sum of the on-shell amplitudes vanishes.

Let us rewrite the action (8.1) using this field redefinition as

$$S = - \sum_{n=1}^{\infty} \frac{1}{n+1} \omega \left(\pi_1 \mathbf{F} \frac{1}{1 - \Psi}, \pi_1 \mathbf{M} \pi_n \mathbf{F} \frac{1}{1 - \Psi} \right) . \tag{8.14}$$

For computational convenience, we employ the following equivalent expression:

$$S = - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \pi_1 \mathbf{F} \frac{1}{1 - \Psi(t)}, \pi_1 \mathbf{M} \mathbf{F} \frac{1}{1 - \Psi(t)} \right) , \tag{8.15}$$

where $\Psi(t)$ is a smooth function satisfying:

$$\Psi(1) = \Psi, \quad \Psi(0) = 0 . \tag{8.16}$$

In order to rewrite it this way, we used the cyclicity of $\mathbf{M}$. Since the left input of the symplectic form ω in equation (8.15) is annihilated by the contracting homotopy h , we can insert the operator $\mathbf{F}^{-1}$ into the right input and apply the identity (8.10) to rewrite the expression as

$$S = - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \pi_1 \mathbf{F} \frac{1}{1 - \Psi(t)}, Q\Psi(t) \right) - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \pi_1 \mathbf{F} \frac{1}{1 - \Psi(t)}, \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi(t)} \right). \quad (8.17)$$

Let $\mathbf{D}(\frac{\partial}{\partial t} \Psi(t))$ denote the coderivation generated by the t -derivative $\frac{\partial}{\partial t} \Psi(t)$. The t -derivative of the component $\pi_1 \mathbf{F} \frac{1}{1 - \Psi(t)}$ is then expressed as

$$\frac{\partial}{\partial t} \pi_1 \mathbf{F} \frac{1}{1 - \Psi(t)} = \pi_1 \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)}. \quad (8.18)$$

Substituting this result into the action, we obtain the following expression:

$$S = - \int_0^1 dt \omega \left(\pi_1 \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)}, Q\Psi(t) \right), \quad (8.19)$$

$$- \int_0^1 dt \omega \left(\pi_1 \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)}, \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi(t)} \right). \quad (8.20)$$

For the second term of this expression, utilizing the identity $P\pi_1 \mathbf{F} = P$, we rewrite the expression as

$$- \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi(t)} \right). \quad (8.21)$$

We now compute the first term using the definition of $\mathbf{F}$:

$$\begin{aligned} & - \int_0^1 dt \omega \left(\pi_1 \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)}, Q\Psi(t) \right) \\ &= - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), Q\Psi(t) \right) + \int_0^1 dt \omega \left(\pi_1 h \mathbf{m} \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)}, Q\Psi(t) \right) \\ &= - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), Q\Psi(t) \right) + \int_0^1 dt \omega \left(h Q\Psi(t), \pi_1 \mathbf{m} \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)} \right). \end{aligned} \quad (8.22)$$

Applying the identity $Qh + hQ = \mathbb{I} - P$ to the second term of this expression results in

$$\int_0^1 dt \omega \left((\mathbb{I} - P)\Psi(t), \pi_1 \mathbf{m} \mathbf{F} \mathbf{D} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)} \right). \quad (8.23)$$

Next, we insert the following trivial identity (zero):

$$0 = - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi(t)} \right) + \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi(t)} \right). \quad (8.24)$$

Combining all terms, we ultimately obtain the action S in the following form:

$$\begin{aligned}
S = & - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), Q\Psi(t) \right) - \int_0^1 dt \omega \left(\frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{mF} \frac{1}{1 - \Psi(t)} \right) \\
& + \int_0^1 dt \omega \left((\mathbb{I} - P) \frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{mF} \frac{1}{1 - \Psi(t)} \right) \\
& + \int_0^1 dt \omega \left((\mathbb{I} - P) \Psi(t), \pi_1 \mathbf{mFD} \left(\frac{\partial}{\partial t} \Psi(t) \right) \frac{1}{1 - \Psi(t)} \right). \tag{8.25}
\end{aligned}$$

Thanks to the cyclicity of $\pi_1 \mathbf{mF}$ as established in the previous section, the action's integrand can be written as a total derivative with respect to t . This allows us to explicitly perform the t -integration and obtain the following expression for the action S :

$$S = -\frac{1}{2} \omega(\Psi, Q\Psi) - \sum_{n=2}^{\infty} \frac{1}{n+1} \omega \left(\Psi, \pi_1 \mathbf{mF} \pi_n \frac{1}{1 - \Psi} \right) + \omega \left((\mathbb{I} - P) \Psi, \pi_1 \mathbf{mF} \frac{1}{1 - \Psi} \right). \tag{8.26}$$

For clarity, we define the following operator:

$$\pi_1 \mathbf{W} := -Q + \pi_1 \mathbf{mF}. \tag{8.27}$$

As previously shown, the first term Q corresponds to the two-point amplitude, while the second term $\pi_1 \mathbf{mF}$ encodes the higher-point amplitudes. Note that the relative sign between these two contributions is opposite. The tree-level action S can now be compactly written as

$$S = - \sum_{n=1}^{\infty} \frac{1}{n+1} \omega \left(\Psi, \pi_1 \mathbf{W} \pi_n \frac{1}{1 - \Psi} \right) + \omega \left((\mathbb{I} - P) \Psi, \pi_1 \mathbf{W} \frac{1}{1 - \Psi} \right). \tag{8.28}$$

We refer to this final expression as the BSFT-like action. Furthermore, the variation of this BSFT-like action can be written as

$$dS = -\omega \left(\pi_1 \mathbf{FD} (d\Psi) \frac{1}{1 - \Psi}, \pi_1 \mathbf{mF} \frac{1}{1 - \Psi} \right), \tag{8.29}$$

where $\mathbf{D}(d\Psi)$ denotes the coderivation generated by $d\Psi$. The name BSFT-like is motivated by the fact that if the original action is taken to be the open string field theory action, then this action (8.28) and its variation (8.29) exhibit properties similar to those of Boundary String Field Theory (BSFT). A brief review of BSFT appears in the following section, and a comparison between BSFT and the BSFT-like action is conducted in the section immediately succeeding it.

8.2 Overview of Boundary String Field Theory

Boundary String Field Theory (BSFT) is one of the formulations of open string field theory. Crucially, BSFT is currently formulated only at the tree-level. While conventional SFT defines

its fields as elements of the string Hilbert space, BSFT adopts a different approach. In BSFT, the field degrees of freedom correspond to the boundary perturbations introduced into the worldsheet theory. Specifically, we consider the action I_{CFT} corresponding to the closed string background and a set of local boundary operators $V(\theta)$ defined on that background. The perturbed action is then given by

$$I_{\text{QFT}} = I_{\text{CFT}} - \oint d\theta V(\theta), \quad (8.30)$$

where θ is the coordinate along the boundary. The resulting perturbed action I_{QFT} is generally not conformally invariant. This is the reason we denote the perturbed action as I_{QFT} . The degrees of freedom in BSFT correspond precisely to the freedom in choosing the local boundary operator $V(\theta)$, which is equivalent to the freedom in choosing the perturbed action I_{QFT} .

The variation of the BSFT action was originally given in [47] as

$$dS_{\text{BSFT}} = \oint d\theta_1 \oint d\theta_2 \langle d(cV(\theta_1)) \{Q_{\text{B}}, cV(\theta_2)\} \rangle_{I_{\text{QFT}}}, \quad (8.31)$$

where $\langle \cdot \rangle_{I_{\text{QFT}}}$ denotes the correlation function on the disk, which is calculated as the expectation value with respect to the perturbed worldsheet action I_{QFT} , and Q_{B} denotes the BRST operator associated with I_{CFT} . Naively, the expression appears to vanish when $\{Q_{\text{B}}, cV(\theta)\} = 0$, but this expectation is incorrect. The reason is that the expression in 8.31 involves contact divergences between the boundary insertions, which necessitate the introduction of an appropriate cut-off and subsequent renormalization. Although a universal regularization and renormalization scheme applicable to all cases is, to the best of the author's knowledge, unavailable, the appearance of nonlinear corrections in $\{Q_{\text{B}}, cV(\theta)\}$ has been demonstrated in several explicitly computable settings. As a result of this necessary regularization procedure, the anti-commutator acquires nonlinear corrections in the renormalized expression. Incorporating these corrections and with appropriate regularization, the variation of the BSFT action can be written as

$$dS_{\text{BSFT}} = \oint d\theta_1 \oint d\theta_2 \langle d(cV(\theta_1)) \{\mathcal{Q}, cV(\theta_2)\} \rangle_{I_{\text{QFT}}}, \quad (8.32)$$

where $\mathcal{Q}$ is a nilpotent operator that is no longer linear in $V(\theta)$:

$$\{\mathcal{Q}, cV(\theta)\} = \{Q_{\text{B}}, cV(\theta)\} + \mathcal{O}(V^2). \quad (8.33)$$

This variation of the BSFT action is known to be proportional to the worldsheet β -function. This relationship is given by

$$\frac{\partial S}{\partial \psi^s} = \sum_t G_{st} \beta^t. \quad (8.34)$$

Here, ψ^s denotes the coupling of the local boundary operator $V_s(\theta)$, β^s is the worldsheet β -function corresponding to that coupling and G_{st} denotes the Zamolodchikov metric defined on the worldsheet action I_{QFT} . In particular, when $V(\theta)$ is a marginal operator such that $\{Q_B, cV(\theta)\} = 0$, the higher-order terms of the non-linear operator $\mathcal{Q}$ are determined through string amplitudes [61]:

$$\langle d(cV(\pi)) \{ \mathcal{Q}, cV(0) \} \rangle_{I_{\text{QFT}}} = \sum_{n=2} \mathcal{V}^{n+1}(dV, \underbrace{V, \dots, V}_n), \quad (8.35)$$

where $\mathcal{V}^n$ denotes the n -point string amplitude ³.

The (bosonic) BSFT action derived through this variation is given by

$$S_{\text{BSFT}} = Z_{D_2} - \sum_s \beta^s \frac{\partial}{\partial \psi^s} Z_{D_2}, \quad (8.36)$$

where, Z_{D_2} is the disk partition function of I_{QFT} . In particular, the disk partition function Z_{D_2} , when expanded around I_{CFT} , can be written as

$$Z_{D_2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \oint d\theta_1 V(\theta_1) \oint d\theta_2 V(\theta_2) \cdots \oint d\theta_n V(\theta_n) \right\rangle_{I_{\text{CFT}}}. \quad (8.37)$$

However, this formal expression involves contact divergences and therefore requires appropriate regularization and renormalization. In the case of superstrings, it is expected that the action is given by the first term alone [63, 64]. However, since this topic is beyond the scope of the current discussion, we do not consider it further here.

BSFT is particularly well-suited for studying tachyon condensation [65, 66] and the structure of D-branes [67, 68]. Unlike conventional SFT, BSFT constructs the action directly from worldsheet path integrals, resulting in a manifestly background-independent formulation.

8.3 Comparison between BSFT-like and BSFT

First, let's compare the variation of the BSFT-like action (8.29) with the variation of the standard BSFT action (8.32). We use the identity

$$D\left(\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \frac{1}{1-\Psi}\right) \frac{1}{1-\Psi} = \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \frac{1}{1-\Psi}, \quad (8.38)$$

to rewrite the expression for the variation of the BSFT-like action as follows:

$$dS = -\omega\left(\pi_1 \mathbf{F} D(d\Psi) \frac{1}{1-\Psi}, \pi_1 \mathbf{F} D\left(\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \frac{1}{1-\Psi}\right) \frac{1}{1-\Psi}\right). \quad (8.39)$$

³Although it is now known that the on-shell two-point amplitude in string theory is non-zero [62], Reference [61] was published prior to this understanding and therefore does not explicitly address it. Consequently, we disregard the on-shell two-point amplitude for the purposes of the current discussion.

We now define a new symplectic form ω_Ψ as

$$\omega_\Psi(A, B) := \omega\left(\pi_1 \mathbf{F} \mathbf{D}(A) \frac{1}{1 - \Psi}, \pi_1 \mathbf{F} \mathbf{D}(B) \frac{1}{1 - \Psi}\right), \quad (8.40)$$

for $A, B \in V$ and $D(A)$ denotes a coderivation generated from $A \in V$. This new symplectic form satisfies the graded anti-symmetric property. Using this definition, the variation of the BSFT-like action can be rewritten as

$$dS = -\omega_\Psi\left(d\Psi, \pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \frac{1}{1 - \Psi}\right). \quad (8.41)$$

This form facilitates a direct comparison with the variation of the BSFT action (8.32). Considering that the initial action based on A_∞ algebra corresponds to the (bosonic) open string field theory action, the string field Ψ can generally be written using a local boundary operator $V(0)$ as

$$\Psi = cV(0) |0\rangle. \quad (8.42)$$

Furthermore, the symplectic form ω is defined from the two-point correlation function on the worldsheet, typically constructed using the BPZ inner product. In this context, the new symplectic form ω_Ψ is analogous to

$$\oint d\theta_1 \oint d\theta_2 \left\langle A(\theta_1) B(\theta_2) \right\rangle_{I_{\text{QFT}}}, \quad (8.43)$$

in the sense that both are symplectic forms depending non-linearly on the local boundary operator $V(\theta)$. Since ω_Ψ depends on the choice of the interaction terms m_n and symplectic form ω , there is a possibility that both symplectic forms could be made exactly identical, depending on how m_n and ω are chosen. Since $\mathbf{M}$ is nilpotent, the operator $\mathbf{F}^{-1} \mathbf{M} \mathbf{F}$ is also nilpotent. Furthermore, this operator can also be written in the form of (8.10). From this form, it is clear that the operator is nonlinear, and its linear part corresponds to the kinetic operator (which, in this context, is the BRST operator Q_B). In this regard, $\mathbf{F}^{-1} \mathbf{M} \mathbf{F}$ is analogous to the nilpotent operator $\mathcal{Q}$ found in renormalized BSFT. Let us specifically consider the case where the input field Ψ belongs to $V^{(P)}$ (i.e., the corresponding local boundary operator $V(\theta)$ is marginal). In this scenario, the linear part of the operator $\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F}$ vanishes, and the operator simplifies to $\pi_1 \mathbf{P} \mathbf{m} \mathbf{F}$. The variation of the BSFT-like action in this case can then be written as

$$dS|_{\text{marginal}} = -\omega\left(d\Psi, \pi_1 \mathbf{P} \mathbf{m} \mathbf{F} \frac{1}{1 - \Psi}\right). \quad (8.44)$$

This expression is precisely consistent with (8.35), as it corresponds to the sum of on-shell amplitudes.

Next, let's focus on and compare the forms of the BSFT-like action (8.28) and the BSFT action (8.36). The first term of the BSFT-like action, given by

$$-\sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Psi, \pi_1 \mathbf{W} \pi_n \frac{1}{1-\Psi}\right), \quad (8.45)$$

can be interpreted as the sum of off-shell amplitudes. Note that the n -point amplitude term is naturally associated with the factor of $1/n$. In open string theory, the tree-level amplitude is equivalent to the disk correlation function calculated on the worldsheet. Therefore, the action's first term can also be interpreted as the sum of disk correlation functions. On the other hand, the first term of the BSFT action itself is the disk partition function Z_{D_2} . While Z_{D_2} can be formally expanded around I_{CFT} as shown in (8.37), if we rewrite the expression such that the boundary insertion coordinates $\theta_1, \dots, \theta_n$ are ordered sequentially ($\theta_1 \leq \theta_2 \leq \dots \leq \theta_n$), the partition function is given by

$$Z_{D_2} = \sum_{n=0}^{\infty} \frac{1}{n} \left\langle \oint_{\theta_1 \leq \theta_2 \leq \dots \leq \theta_n} d\theta_1 d\theta_2 \dots d\theta_n V(\theta_1) V(\theta_2) \dots V(\theta_n) \right\rangle_{I_{\text{CFT}}}. \quad (8.46)$$

This expression, including the factor $1/n$, corresponds to the first term of the BSFT-like action (8.45). There is a crucial difference, however: while expression (8.46) requires appropriate regularization due to the presence of contact divergences, expression (8.45) is a formulation that inherently does not give rise to divergences. Excluding this difference in the divergence structure, both actions could potentially be made identical, depending on the specific choice of the interaction terms m_n and the symplectic form ω . Let us examine the second term in the BSFT-like action:

$$\omega\left((\mathbb{I} - P)\Psi, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right). \quad (8.47)$$

This term can be rewritten as

$$\omega\left((\mathbb{I} - P)\Psi, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right) = \sum_s (-1)^{e_s} \omega\left((\mathbb{I} - P)\Psi, \tilde{e}^s\right) \omega\left(e_s, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right), \quad (8.48)$$

When the field Ψ satisfies the equation of motion (8.13), it follows that $\Psi \in V^{(P)}$. Consequently, the factor $\omega((\mathbb{I} - P)\Psi, \tilde{e}^a)$ vanishes in this case. Furthermore, since Ψ is annihilated by h , the $\tilde{e}^s$ elements belonging to the subspace $V^{(hQ)}$ do not contribute to the factor $\omega((\mathbb{I} - P)\Psi, \tilde{e}^s)$. Consequently, regarding the second factor, $\omega\left(e_s, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right)$, the basis elements e_s belonging to the subspace $V^{(Qh)}$ also do not contribute. The remaining basis can thus be regarded as the one spanning the field Ψ , consistent with (8.5). Given that $\mathbf{W}$ is cyclic, the factor $\omega\left(e_s, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right)$ can be written as

$$\frac{\partial}{\partial \psi^s} \sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Psi, \pi_1 \mathbf{W} \pi_n \frac{1}{1-\Psi}\right). \quad (8.49)$$

Taking this into account, the second term of the BSFT-like action is structured as

$$-(\text{Terms that vanishes when } \Psi \text{ satisfies the EOM}) \times (\text{Derivative of the First Term}). \quad (8.50)$$

This structure is analogous to the second term of the BSFT action:

$$-\sum_s \beta^s \frac{\partial}{\partial \psi^s} Z_{D_2}. \quad (8.51)$$

Recall that the equation of motion for BSFT is proportional to the worldsheet β -functions. However, there is a distinction: the second term of the BSFT-like action vanishes under the weaker condition $\Psi \in V^{(P)}$, whereas the second term of the BSFT action generally does not vanish under the corresponding condition alone (i.e., when $V(\theta)$ is marginal). In summary, while both the first and second terms of the two actions exhibit broad structural similarities, there are also subtle differences. However, if we restrict the field to $\Psi \in V^{(P)}$, the BSFT-like action simplifies to

$$S = -\sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Psi, \pi_1 \mathbf{P} \mathbf{m} \frac{1}{\mathbf{I} + \mathbf{h} \mathbf{m}} \mathbf{P} \pi_n \frac{1}{1 - \Psi}\right), \quad (8.52)$$

which is a sum of on-shell amplitudes. Similarly, from (8.35), it is immediately apparent that the BSFT action also reduces to a sum of on-shell amplitudes:

$$S_{\text{BSFT}} = \sum_{n=3}^{\infty} \frac{1}{n!} \mathcal{V}^n(\underbrace{V, \dots, V}_n). \quad (8.53)$$

Therefore, the two actions coincide in this regime. In other words, while the first and second terms of the BSFT-like action and the BSFT action differ slightly from each other individually, these differences cancel out—at least within the range of $\Psi \in V^{(P)}$ —resulting in identical expressions. Considering that each term in the BSFT action was an ill-defined quantity requiring regularization, it is reasonable to interpret the BSFT-like action as a regularized formulation where each individual term is rendered well-defined.

It is worth commenting on the algebraic structure of the BSFT-like action. The variation of the BSFT-like action (8.29) is described in terms of the symplectic form ω_Ψ and a nilpotent coderivation—specifically, the products $\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \pi_n$, which satisfy the A_∞ relations. In this respect, the structure is identical to that of a standard action based on A_∞ algebras (8.1). However, a crucial distinction in the present case is that the products $\pi_1 \mathbf{F}^{-1} \mathbf{M} \mathbf{F} \pi_n$ are not cyclic with respect to the symplectic form ω_Ψ . This lack of cyclicity is the precise reason why the form of the action deviates significantly from that of a standard case. Nevertheless, when restricted to the subspace $\Psi \in V^{(P)}$, cyclicity is restored. Consequently, the action recovers the standard form based on A_∞ algebras, and the minimal model is obtained.

8.4 Extension of BSFT-like action

Thus far, we have constructed the BSFT-like action based on A_∞ algebra, bearing in mind its correspondence with BSFT, which serves as the tree-level theory of open string field theory. By virtue of this algebraic construction, the method is readily extensible to L_∞ algebra. Furthermore, let us consider the extension from tree-level to loop-level.

8.4.1 BSFT-like action for L_∞ algebras

Let us extend the construction of the BSFT-like action to the framework of L_∞ algebra. Since tree-level closed string field theory is described by an L_∞ algebra [7], this suggests the existence of a formulation for closed string field theory where the string field degrees of freedom are defined by a worldsheet QFT, analogous to BSFT.

Let us start with the action based on the L_∞ algebra:

$$S = - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \omega(\Phi, \pi_1 \mathbf{L} \pi_n(\underbrace{\Phi \wedge \cdots \wedge \Phi}_n)) . \quad (8.54)$$

By defining the exponential e^Φ as

$$e^\Phi = \mathbf{1} + \Phi + \frac{1}{2!} \Phi \wedge \Phi + \frac{1}{3!} \Phi \wedge \Phi \wedge \Phi + \cdots , \quad (8.55)$$

the action can be written as

$$S = - \sum_{n=1}^{\infty} \frac{1}{n+1} \omega(\Phi, \pi_1 \mathbf{L} \pi_n e^\Phi) . \quad (8.56)$$

In this form, the structure is nearly identical to the A_∞ algebra case, requiring only the simple substitution $\frac{1}{1-\Phi} \rightarrow e^\Phi$. The cohomomorphism $\mathbf{F}$ is constructed using the exact same procedure as in the A_∞ algebra case:

$$\pi_1 \mathbf{F} = \pi_1(\mathbf{I} - \mathbf{h} \mathbf{L} \mathbf{F}) , \quad (8.57)$$

$$\mathbf{F} = \pi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} (\pi_1 \mathbf{F})^{\wedge n} . \quad (8.58)$$

Here, we express the operator using symmetrization to reflect its action on the symmetric tensor algebra. Using this cohomomorphism $\mathbf{F}$ to redefine the field as $\Phi = \pi_1 \mathbf{F} e^\Psi$, and following a procedure parallel to the A_∞ algebra case, the action can be rewritten as

$$S = -\frac{1}{2} \omega(\Psi, Q\Psi) + \omega((\mathbb{I} - P)\Psi, \pi_1 \mathbf{L} \mathbf{F} e^\Psi) - \int_0^1 dt \omega\left(\frac{\partial}{\partial t} \Psi(t), \pi_1 \mathbf{L} \mathbf{F} e^{\Psi(t)}\right) . \quad (8.59)$$

Here, $\Psi(t)$ is defined by (8.16). Assuming the cyclicity of $\pi_1 \mathbf{lF}$, the calculation proceeds parallel to the A_∞ algebra case, leading to the result

$$S = - \sum_{n=1}^{\infty} \frac{1}{n+1} \omega \left(\Psi, \pi_1 \mathbf{W} \pi_n e^\Psi \right) + \omega \left((\mathbb{I} - P) \Psi, \pi_1 \mathbf{W} e^\Psi \right), \quad (8.60)$$

$$\pi_1 \mathbf{W} = -Q + \pi_1 \mathbf{lF}. \quad (8.61)$$

We address the proof for the cyclicity of $\pi_1 \mathbf{lF}$ subsequently; for now, we first turn our attention to commenting on this result. This expression has the exact same form as the BSFT-like action obtained in the A_∞ algebra case. This structural identity leads to a powerful interpretation: specifically, if we consider the action based on L_∞ algebra to be that of closed string field theory, the redefined field Ψ corresponds to a local bulk operator on the worldsheet CFT. The first term of the action is then interpreted as the sphere partition function of the QFT, shifted by that local bulk operator. Furthermore, a formulation that relates the worldsheet QFT degrees of freedom to spacetime fields has also been proposed in [69]. However, the expression we derived appears different from this previous formulation at first glance. Specifically, the formula in [69] gives the spacetime action as a product of the C -function on the plane and the partition function on the sphere. Although the two formalisms are expected to be equivalent, the precise relationship between them is currently unknown.

Let us demonstrate the cyclicity of $\pi_1 \mathbf{lF}$. The cyclicity condition in A_∞ algebra could be expressed as (7.75); however, since L_∞ algebra is described on the symmetric tensor algebra $\mathcal{SV}$, cyclicity cannot be written in this form. Instead, the condition:

$$\sum_s \pi_1 \mathbf{lF} \tilde{e}^s \wedge e_s = \sum_s \left(\pi_1 \mathbf{lF} \tilde{e}^s \otimes e_s + (-1)^{e_s} e_s \otimes \pi_1 \mathbf{lF} \tilde{e}^s \right) = 0, \quad (8.62)$$

is equivalent to the cyclicity condition, and we now proceed to prove this condition. First, $\pi_1 \mathbf{lF} \tilde{e}^s$ can be written as

$$\begin{aligned} \pi_1 \mathbf{lF} \tilde{e}^s &= \pi_1 \mathbf{l}(\mathbf{F} \wedge \pi_1 \mathbf{F} \tilde{e}^s \wedge \mathbf{F}), \\ &= \pi_1 \mathbf{l}(\mathbf{F} \wedge \pi_1(\mathbf{I} - \mathbf{h} \mathbf{lF}) \tilde{e}^s \wedge \mathbf{F}). \end{aligned} \quad (8.63)$$

Using this, we obtain

$$\sum_s \pi_1 \mathbf{lF} \tilde{e}^s \otimes e_s = \sum_s \left[\pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{e}^s \wedge \mathbf{F}) \otimes e_s - \pi_1 \mathbf{l}(\mathbf{F} \wedge \pi_1 \mathbf{h} \mathbf{lF} \tilde{e} \wedge \mathbf{F}) \otimes e_s \right]. \quad (8.64)$$

Here, applying the cyclicity of $\mathbf{l}$ to the first term allows us to exchange the order of $\mathbf{l}(\mathbf{F} \wedge \tilde{e}^s \wedge \mathbf{F})$ and e_s (with the appropriate sign factor). In the second term, we would also ideally use cyclicity to move the factor e_s to the left-hand tensor factor. However, performing this manipulation requires assuming the cyclicity of $\pi_1 \mathbf{lF}$. Since this cyclicity is precisely the property we are

currently trying to demonstrate, using it as an assumption at this stage is not permissible. To circumvent the problem of circular reasoning, we separate $\pi_1 \mathbf{lF}$ by the degree of the interaction $\mathbf{l}$. Let $[\pi_1 \mathbf{lF}]^{(n)}$ denote the components up to degree n . We proceed with the proof using mathematical induction on this degree n . The base case, $[\pi_1 \mathbf{lF}]^{(1)} = \pi_1 \mathbf{l}$, is cyclic. For the inductive step, we assume cyclicity holds up to degree n and then continue the calculation for degree $n+1$. Considering the component of (8.64) at degree $n+1$, the factor $\pi_1 \mathbf{lF}$ in the second term on the right-hand side is of degree n or less. Therefore, by the inductive hypothesis, it is cyclic, which allows us to calculate as

$$\begin{aligned}
\sum_s [\pi_1 \mathbf{lF}]^{(n+1)} \tilde{\mathbf{e}}^s \otimes e_s &= \sum_s (-1)^{1+e_s} \left[e_s \otimes \pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F}) \right. \\
&\quad \left. - \pi_1 \mathbf{l}(\mathbf{F} \wedge h e_s \wedge \mathbf{F}) \otimes \pi_1 \mathbf{lF} \tilde{\mathbf{e}} \right]^{(n+1)}, \\
&= \sum_s (-1)^{1+e_s} \left[e_s \otimes \pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F}) \right. \\
&\quad \left. - (-1)^{\tilde{e}^s} \pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F}) \otimes \pi_1 \mathbf{lF} \mathcal{D}(h e_s) \right]^{(n+1)}, \\
&= \sum_s (-1)^{1+e_s} \left[e_s \otimes \pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F}) \right. \\
&\quad \left. - e_s \otimes \pi_1 \mathbf{lF} \mathcal{D}(\pi_1 h \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F})) \right]^{(n+1)}. \tag{8.65}
\end{aligned}$$

In the first term of the final expression, $\pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F})$ can also be written as

$$\begin{aligned}
\pi_1 \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F}) &= \pi_1 \mathbf{lF} \mathbf{F}^{-1} \tilde{\mathbf{e}}^s \mathbf{F}, \\
&= \pi_1 \mathbf{lF} \mathcal{D}(\pi_1 \mathbf{F}^{-1} \tilde{\mathbf{e}}^s \mathbf{F}), \\
&= \pi_1 \mathbf{lF} \mathcal{D}(\pi_1 (\mathbf{I} + h \mathbf{l}) \tilde{\mathbf{e}}^s \mathbf{F}), \\
&= \pi_1 \mathbf{lF} \tilde{\mathbf{e}}^s + \pi_1 \mathbf{lF} \mathcal{D}(\pi_1 h \mathbf{l}(\mathbf{F} \wedge \tilde{\mathbf{e}}^s \wedge \mathbf{F})). \tag{8.66}
\end{aligned}$$

Combining this result with the second term, we finally obtain

$$\sum_s [\pi_1 \mathbf{lF}]^{(n+1)} \tilde{\mathbf{e}}^s \otimes e_s = \sum_s (-1)^{1+e_s} e_s \otimes [\pi_1 \mathbf{lF}]^{(n+1)} \tilde{\mathbf{e}}^s. \tag{8.67}$$

This establishes the desired cyclicity. This result guarantees that the discussion of the BSFT-like action can proceed in the L_∞ algebra framework with complete parallelism to the A_∞ case.

8.4.2 BSFT-like action in loop-level

We can consider a naive extension of the BSFT-like action, which is written in terms of tree-level amplitudes by replacing those amplitudes with loop-level amplitudes:

$$-\sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Psi, \pi_1 \mathbf{W} \pi_n \frac{1}{1-\Psi}\right) + \omega\left((\mathbb{I} - P)\Psi, \pi_1 \mathbf{W} \frac{1}{1-\Psi}\right), \quad (8.68)$$

$$\pi_1 \mathbf{W} = -Q + \pi_1 \mathbf{m} \mathcal{F}. \quad (8.69)$$

Here, the operator $\pi_1 \mathbf{W}$, as noted at the end of the preceding chapter, yields the loop-level off-shell amplitude. In the string context, considering the quantum A_∞ algebra for loop-level analysis is technically inappropriate because open string field theory is not closed at the loop level and requires the inclusion of closed strings. However, since our discussion here is strictly confined to the algebraic structure, we ignore this physical inconsistency and proceed with the A_∞ framework. We note that the same arguments hold for quantum L_∞ algebra; therefore, the reader should mentally translate the results to the L_∞ framework when considering the closed string context.

Does this naively extended loop-level expression have any physical meaning? One might naively suspect that this expression is obtained by performing the field redefinition $\Phi = \pi_1 \mathcal{F} \frac{1}{1-\Psi}$ on the quantum action based on quantum A_∞ algebra. However, this is incorrect. The core problem is that, unlike the tree-level operator $\mathbf{F}$, the loop-level operator $\mathcal{F}$ is not a cohomomorphism (meaning that the relation $\Delta \mathcal{F} = (\mathcal{F} \otimes' \mathcal{F}) \Delta$ fails to hold). Therefore, the identity $\frac{1}{1-\Phi} = \mathcal{F} \frac{1}{1-\Psi}$ does not hold, and the arguments used for the tree-level BSFT-like action cannot be applied.

To obtain this loop-level expression, while maintaining the tree-level derivation procedure, we must identify the initial action before field redefinition as the 1PI effective action Γ :

$$\Gamma = -\sum_{n=1}^{\infty} \frac{1}{n+1} \omega\left(\Phi, \pi_1 \mathbf{M}^{(\Gamma)} \pi_n \frac{1}{1-\Psi}\right). \quad (8.70)$$

Here, the operator $\mathbf{M}^{(\Gamma)}$ can be decomposed as $\mathbf{M}^{(\Gamma)} = \mathbf{Q} + \mathbf{m}^{(\Gamma)}$, and this operator satisfies the A_∞ relations, $(\mathbf{M}^{(\Gamma)})^2 = 0$. A crucial property of the 1PI effective interaction is that the minimal model of this operator $\mathbf{M}^{(\Gamma)}$ reproduces the minimal model of the quantum A_∞ algebra:

$$\mathbf{P} \mathbf{m}^{(\Gamma)} \frac{1}{\mathbf{I} + \hbar \mathbf{m}^{(\Gamma)}} \mathbf{P} = \mathbf{P} \mathbf{m} \frac{1}{\mathbf{I} + \hbar (\mathbf{m} + i \hbar \mathbf{U})} \mathbf{P}. \quad (8.71)$$

We assume that the tadpole term $\mathbf{m}^{(\Gamma)} \pi_0$ is absent. Since the operator $\mathbf{M}^{(\Gamma)}$ satisfies the A_∞ relations, we can directly apply the discussion from the previous section. By defining the

cohomomorphism $\mathbf{F}^{(\Gamma)}$ as

$$\pi_1 \mathbf{F}^{(\Gamma)} = \pi_1 (\mathbb{I} - h \mathbf{m}^{(\Gamma)} \mathbf{F}^{(\Gamma)}) , \quad (8.72)$$

we find that the operator $\pi_1 \mathbf{m}^{(\Gamma)} \mathbf{F}^{(\Gamma)}$ yields the tree-level diagram described by the 1PI effective interaction. Crucially, this expression is then equivalent to the off-shell loop amplitude $\pi_1 \mathbf{m} \mathcal{F}$. By performing the field redefinition $\Phi = \pi_1 \mathbf{F}^{(\Gamma)} \frac{1}{1-\Psi}$ on the 1PI effective action (8.70), we can derive the loop-level expression (8.68) using the exact same argument as in the tree-level case. This result suggests a profound physical interpretation: any action defined using loop-level correlation functions—such as the action defined in BSFT using the annulus partition function instead of the disk, or the action defined in closed string field theory using the partition function on a Riemann surface with handles—should be interpreted as the 1PI effective action.

9 Conclusion and Discussion

In this thesis, we have reviewed the fundamentals of homotopy algebras and their basic applications to physics in Section 2 through 4. In the first subsection of Section 5, we reviewed correlation functions, focusing on recent developments in the field. Building on these foundations, the second part of Section 5 and the subsequent sections discuss more advanced applications, specifically the Schwinger-Dyson equation, the Ward-Takahashi identity, the LSZ reduction formula, and the BSFT-like action. In what follows, we present conclusions and discussions on these respective subjects.

Schwinger-Dyson Equation and Ward-Takahashi Identity

In the second subsection of Section 5, we introduced the identity

$$(\mathbf{M} + i\hbar \mathbf{U}) \mathbf{f} \mathbf{1} = 0 , \quad (9.1)$$

and demonstrated that it reproduces the standard Schwinger-Dyson equation of conventional quantum field theory. Furthermore, this identity can be interpreted as an equation governing $\mathbf{f} \mathbf{1}$. From this perspective, it implies that the correlation functions are determined by the cohomology of the operator $\mathbf{M} + i\hbar \mathbf{U}$.

In Section 6, by applying this identity to the expectation value of the divergence of the conserved current, we derived the Ward-Takahashi identity in terms of homotopy algebras:

$$\langle \partial_\mu j^\mu(x) \rangle = i\hbar \sum_{e_a \in V^{(hQ)}} \langle \omega | (\tilde{e}^a(x) \otimes \pi_1 \mathbf{A} e_a(x)) \mathbf{f} \mathbf{1} . \quad (9.2)$$

In this formula, the right-hand side contains divergences. These divergences stem from contact terms and anomalies, necessitating appropriate regularization. In our approach, we render the theory finite by attaching stubs, with the original theory recovered in the limit of the

stub parameter $\lambda \rightarrow 0$. Furthermore, these divergences are linked to the renormalization of composite operators. The replacement procedure

$$\frac{1}{1 - \Phi} \longrightarrow \mathbf{f} \mathbf{1} \quad (9.3)$$

corresponds to taking the expectation values of the fields. We have provided a formal treatment of this process for the general case. For the examples discussed in Subsection 6.4, our results are consistent with those obtained via standard prescriptions. We anticipate that our formulation remains consistent with standard quantum field theory methods in general. The handling of these divergences is rooted in the definition of composite operators within the framework of homotopy algebras, a subject that warrants further in-depth investigation.

Finally, we discuss directions for future research. A primary motivation of this study is the application of our method to string field theory. Given the universality of the homotopy algebra description, we expect our framework to contribute significantly to the discovery and analysis of symmetries and anomalies within string field theory. While the present work is based on perturbative expansion, recent progress has been made toward a nonperturbative description of correlation functions in terms of homotopy algebras [56]. By combining our results with these developments, we hope to elucidate nonperturbative anomalies from a homotopy-algebraic perspective. Another promising avenue is the extension of this work to higher-form symmetries [70]. Since understanding the Ward-Takahashi identity is crucial for characterizing higher-form symmetries, our algebraic formulation offers a powerful vantage point. We anticipate that this algebraic approach will provide effective tools for analyzing such symmetries.

LSZ Reduction Formula

In Section 7, we proposed a formula based on homotopy algebra that provides the well-defined loop-amplitude:

$$\left(\frac{i}{\hbar} \right)^{n-1} \lim_{\epsilon \rightarrow 0} \pi_n \langle \omega_n(\Phi_1, \dots, \Phi_n) | \pi_n \mathbf{f} \epsilon \mathbf{1}, \quad (9.4)$$

$$\langle \omega_n(\Phi_1, \dots, \Phi_n) | = (-1)^n \langle \omega | (\widehat{Q}_\epsilon \Phi_1 \otimes \pi_1) \otimes \dots \otimes \langle \omega | (\widehat{Q}_\epsilon \Phi_n \otimes \pi_1). \quad (9.5)$$

This expression corresponds to the LSZ reduction formula. Here, $\widehat{Q}_\epsilon$ is the inverse operator of the exact propagator. It serves to appropriately amputate the external lines, thereby yielding a consistent loop on-shell amplitude. We also showed that the expression obtained by replacing $\widehat{Q}_\epsilon$ with Q_ϵ and extracting the connected component agrees with the conventionally known amplitude formula provided by the minimal model. Furthermore, we obtained the expression for the off-shell amplitude by generalizing the input space of this amplitude formula. At the tree level, this expression can be written using the cohomomorphism $\mathbf{F}$ as

$$\delta_{n,2} \omega(\Phi_2, Q\Phi_1) - \omega\left(\Phi_n, \pi_1 \mathbf{m} \mathbf{F}(\Phi_1 \otimes \dots \otimes \Phi_{n-1})\right). \quad (9.6)$$

At the loop level, the amplitude cannot be expressed using a standard cohomomorphism. However, the amplitude obtained via naive amputation can be represented by the following formulation:

$$\delta_{n,2} \omega(\Phi_2, Q\Phi_1) - \omega\left(\Phi_n, \pi_1 \mathbf{m}\mathcal{F}(\Phi_1 \otimes \cdots \otimes \Phi_{n-1})\right). \quad (9.7)$$

For the details regarding this expression, refer to Appendix A.

The loop-amplitude formula we proposed necessitates the use of the inverse operator of the exact propagator. This operator is precisely the kinetic operator of the 1PI effective action. This crucial observation provides strong motivation to explore methods for constructing the 1PI effective action within the language of homotopy algebra. Furthermore, since this construction involves deriving a classical homotopy algebra structure from quantum homotopy algebra data, it presents an intriguing mathematical problem in its own right. As its name implies, the 1PI effective action generates 1PI diagrams in perturbation theory. To extend the definition beyond perturbation theory, one possibility involves characterizing the 1PI effective action via the Legendre transformation and implementing this process within the homotopy algebra framework. Non-perturbatively, the Legendre transformation is only applicable for convex functions. This restriction may be related to the definition of quantum theory in terms of Lefschetz thimbles. Although the connection between the formula for correlation functions based on quantum A_∞ algebras and Lefschetz thimbles was uncovered in [56], a crucial open problem remains: identifying a principle for how contributions from multiple Lefschetz thimbles should be combined within the quantum A_∞ framework. We anticipate that research focused on a non-perturbative definition of the 1PI effective action will provide vital clues toward solving this important open problem as well.

BSFT-like Action

In Section 8, we proposed a method for obtaining the formulation we termed the BSFT-like action by redefining the field of the A_∞ action. This redefinition was specifically achieved by employing a cohomomorphism designed to generate off-shell amplitudes. This resulting BSFT-like action exhibits important similarities to a formulation known as boundary string field theory. This strongly suggests that BSFT and the well-defined A_∞ formulation of open string field theory are fundamentally connected via a field redefinition. In particular, a strict correspondence was observed when restricting the field of the BSFT-like action to the BRST cohomology, excluding the contribution of the two-point on-shell amplitude. Furthermore, we demonstrated a direction for extending the BSFT formulation to the closed string and loop-level contexts. This was achieved by applying the methodology developed in the A_∞ algebra framework to the L_∞ algebra, which governs closed string field theory and the 1PI effective action.

An important direction for future work is to investigate whether this approach indeed provides a proper regularization of BSFT. Beyond our perturbative analysis, exploring the non-perturbative aspects of this framework is crucial. Understanding whether the field redefinition remains well-defined outside of perturbation theory could provide further insights into the fundamental structure of string field theory. We have consistently neglected the on-shell two-point amplitude throughout this thesis; addressing its proper inclusion remains an open problem. A promising direction to resolve this issue is suggested in [60], where the inclusion of boundary terms plays a key role. It remains important to understand how our current framework is modified when such boundary contributions are taken into account.

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A Proof of (7.80)

The calculations here are complex, so to simplify them as much as possible, we employ the Einstein summation convention for the basis indices. We also assume that e_s is degree-even and $\tilde{e}^s$ is degree-odd. While this assumption might appear to sacrifice generality, it can always be achieved formally by introducing a degree-odd number and redefining the basis.

This appendix demonstrates the following identity required to obtain the loop-level off-shell amplitude expression (7.80):

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}^{-1} f_\epsilon h_\epsilon D(Q_\epsilon \Phi_n) \cdots f_\epsilon h_\epsilon D(Q_\epsilon \Phi_1) f_\epsilon \mathbf{1} = \Phi_1 \otimes \cdots \otimes \Phi_n, \quad (\text{A.1})$$

for all $\Phi_1, \dots, \Phi_n \in V^{(P)} \oplus V^{(hQ)}$. Here, the operator $\mathcal{F}^{-1}$ is defined by

$$\mathcal{F}^{-1} = \pi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} (\mathbf{F}^{-1})^{\wedge n}, \quad (\text{A.2})$$

$$\mathbf{F}^{-1} = \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1 (\mathbf{I} + \hbar \mathbf{m}) \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}). \quad (\text{A.3})$$

This $\mathbf{F}^{-1}$ is an extension of the operator (7.60) introduced in the tree-level discussion. We show later that this definition indeed corresponds to the inverse of $\mathcal{F}$.

First, we establish the following formula:

$$\Delta \mathcal{F}^{-1} = \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} \left(e_{s_1} \cdots e_{s_k} \mathcal{F}^{-1} \otimes' \mathcal{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \right) \Delta. \quad (\text{A.4})$$

Because of the cyclicity of $\mathbf{m}$, each component of the output of $\mathbf{F}^{-1}$ is invariant under cyclic permutation up to a sign factor. For example,

$$e_s \otimes \pi_1(\mathbf{I} + \mathbf{h}\mathbf{m}) \mathbf{D}(h\tilde{e}^s) = \pi_1(\mathbf{I} + \mathbf{h}\mathbf{m}) \mathbf{D}(h\tilde{e}^s) \otimes e_s. \quad (\text{A.5})$$

It means that all elements of the output of $\mathbf{F}^{-1}$ are equal. Let's first use the expression $\frac{1}{4!} A^{\wedge 4}$ as a simple model for the action of $\mathbf{F}^{-1}$. The coproduct Δ acts on this model as follows:

$$\begin{aligned} \Delta \frac{1}{4!} A^{\wedge 4} &= \Delta A^{\otimes 4} \\ &= \mathbf{1} \otimes' A^{\otimes 4} + A \otimes' A^{\otimes 3} + A^{\otimes 2} \otimes' A^{\otimes 2} + A^{\otimes 3} \otimes' A + A^{\otimes 4} \otimes' \mathbf{1} \\ &= \mathbf{1} \otimes' \frac{1}{4!} A^{\wedge 4} + A \otimes' \frac{1}{3!} A^{\wedge 3} + \frac{1}{2!} A^{\wedge 2} \otimes' \frac{1}{2!} A^{\wedge 2} + \frac{1}{3!} A^{\wedge 3} \otimes' A + \frac{1}{4!} A^{\wedge 4} \otimes' \mathbf{1}. \end{aligned} \quad (\text{A.6})$$

This result demonstrates that the prefactor of each term is determined by the multiplicity of elements on each side of the $\otimes'$ product. By using this result, the action of Δ on $\mathbf{F}^{-1}$ can be calculated as follows:

$$\begin{aligned} \Delta \mathbf{F}^{-1} &= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(-\hbar)^k}{l!(k-l+1)!} e_{s_1} \wedge \cdots \wedge e_{s_l} \otimes' e_{s_{l+1}} \wedge \cdots \wedge e_{s_k} \\ &\quad \wedge \pi_1(\mathbb{I} + \mathbf{h}\mathbf{m}) \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) + \mathbf{F}^{-1} \otimes' \mathbf{1} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(-\hbar)^l}{l!} e_{s_1} \wedge \cdots \wedge e_{s_l} \otimes' \pi_{k-l+1} \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_l}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) + \mathbf{F}^{-1} \otimes' \mathbf{1} \\ &= \sum_{l=0}^{\infty} \frac{(-\hbar)^l}{l!} e_{s_1} \wedge \cdots \wedge e_{s_l} \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_l}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) + \mathbf{F}^{-1} \otimes' \mathbf{1}. \end{aligned} \quad (\text{A.7})$$

Next, let's consider the expression $\frac{1}{2!} A^{\wedge 2} \wedge \frac{1}{2!} B^{\wedge 2}$ as a model for the action of $\mathbf{F}^{-1} \wedge \mathbf{F}^{-1}$. The

coproduct Δ acts on this expression as follows:

$$\begin{aligned}
& \Delta \frac{1}{2!} A^{\wedge 2} \wedge \frac{1}{2!} B^{\wedge 2} \\
&= \Delta [A \otimes A \otimes B \otimes B + A \otimes B \otimes A \otimes B + A \otimes B \otimes B \otimes A \\
&\quad + B \otimes A \otimes A \otimes B + B \otimes A \otimes B \otimes A + B \otimes B \otimes A \otimes A] \\
&= \mathbf{1} \otimes' \frac{1}{2!} A^{\wedge 2} \wedge \frac{1}{2!} B^{\wedge 2} + A \otimes' A \wedge \frac{1}{2!} B^{\wedge 2} + B \otimes' \frac{1}{2!} A^{\wedge 2} \wedge B + A \wedge B \otimes' A \wedge B \\
&\quad + \frac{1}{2!} A^{\wedge 2} \otimes' \frac{1}{2!} B^{\wedge 2} + \frac{1}{2!} A^{\wedge 2} \wedge B \otimes' B \\
&\quad + \frac{1}{2!} B^{\wedge 2} \otimes' \frac{1}{2!} A^{\wedge 2} + A \wedge \frac{1}{2!} B^{\wedge 2} \otimes' A \\
&\quad + \frac{1}{2!} A^{\wedge 2} \wedge \frac{1}{2!} B^{\wedge 2} \otimes' \mathbf{1}.
\end{aligned} \tag{A.8}$$

In this case as well, the prefactor of each term is determined by the multiplicity of elements on each side of the $\otimes'$ product. Furthermore, we can classify the terms on the right-hand side into four lines based on the presence of the elements A or B to the right of the $\otimes'$ product. The first line contains terms with at least one A and at least one B to the right of $\otimes'$. The second line contains terms with at least one B but no A to the right of $\otimes'$. The third line contains terms with at least one A but no B to the right of $\otimes'$. Finally, the fourth line contains terms with neither A nor B to the right of $\otimes'$.

$$\begin{aligned}
& \Delta \frac{1}{2!} \mathbf{F}^{-1} \wedge \mathbf{F}^{-1} \\
&= \frac{1}{2!} \Delta \left[\sum_{k=0}^{\infty} \frac{(-\hbar)^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1(\mathbf{I} + \mathbf{h}\mathbf{m}) \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \right. \\
&\quad \left. \wedge \sum_{l=0}^{\infty} \frac{(-\hbar)^l}{(l+1)!} e_{t_1} \wedge \cdots \wedge e_{t_l} \wedge \pi_1(\mathbf{I} + \mathbf{h}\mathbf{m}) \mathbf{D}(h\tilde{e}^{t_l}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) \right], \\
&= \frac{1}{2!} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{i=0}^k \sum_{j=0}^l \frac{(-\hbar)^{k+l}}{i!(k-i+1)!j!(l-j+1)!} e_{t_1} \wedge \cdots \wedge e_{t_j} \wedge e_{s_1} \wedge \cdots \wedge e_{s_i} \\
&\quad \otimes' e_{s_{i+1}} \wedge \cdots \wedge e_{s_k} \wedge \pi_1 \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&\quad \wedge e_{t_{j+1}} \wedge \cdots \wedge e_{t_l} \wedge \pi_1 \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{t_l}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) \\
&\quad + \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-\hbar)^k}{i!(k-i+1)!} \mathbf{F}^{-1} \wedge e_{s_1} \wedge \cdots \wedge e_{s_i} \\
&\quad \otimes' e_{s_{i+1}} \wedge \cdots \wedge e_{s_k} \wedge \pi_1 \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&\quad + \frac{1}{2!} \mathbf{F}^{-1} \wedge \mathbf{F}^{-1} \otimes' \mathbf{1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2!} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\hbar)^{k+l}}{k!l!} e_{t_1} \wedge \cdots \wedge e_{t_l} \wedge e_{s_1} \wedge \cdots \wedge e_{s_k} \\
&\quad \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \wedge \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{t_l}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) \\
&\quad + \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \mathbf{F}^{-1} \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&\quad + \frac{1}{2!} \mathbf{F}^{-1} \wedge \mathbf{F}^{-1} \otimes' \mathbf{1}.
\end{aligned} \tag{A.9}$$

Here, the first term of the right-hand side can be written as

$$\begin{aligned}
&\frac{1}{2!} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\hbar)^{k+l}}{k!l!} e_{t_1} \wedge \cdots \wedge e_{t_l} \wedge e_{s_1} \wedge \cdots \wedge e_{s_k} \\
&\quad \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \wedge \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{t_l}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) \\
&= \frac{1}{2!} \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(-\hbar)^k}{l!(k-l)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \\
&\quad \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_{l+1}}) \wedge \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_l}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&= \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} e_{s_1} \wedge \cdots \wedge e_{s_k} \otimes' \frac{1}{2!} (\mathbf{F}^{-1} \wedge \mathbf{F}^{-1}) \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}).
\end{aligned} \tag{A.10}$$

Consequently, we obtain

$$\begin{aligned}
&\Delta \frac{1}{2!} \mathbf{F}^{-1} \wedge \mathbf{F}^{-1} \\
&= \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} e_{s_1} \wedge \cdots \wedge e_{s_k} \otimes' \frac{1}{2!} (\mathbf{F}^{-1} \wedge \mathbf{F}^{-1}) \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&\quad + \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \mathbf{F}^{-1} \otimes' \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \\
&\quad + \frac{1}{2!} \mathbf{F}^{-1} \wedge \mathbf{F}^{-1} \otimes' \mathbf{1}.
\end{aligned} \tag{A.11}$$

Similarly, by extending this result, we obtain the general action of the coproduct Δ on the tensor power of $\mathbf{F}^{-1}$:

$$\begin{aligned}
\Delta \frac{1}{n!} (\mathbf{F}^{-1})^{\wedge n} &= \sum_{l=0}^n \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} e_{s_1} \cdots e_{s_k} \frac{1}{l!} (\mathbf{F}^{-1})^{\wedge l} \\
&\quad \otimes' \frac{1}{(n-l)!} (\mathbf{F}^{-1})^{\wedge n-l} \mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}),
\end{aligned} \tag{A.12}$$

where we define the zeroth power as $(\mathbf{F}^{-1})^{\wedge 0} = \pi_0$. By summing this equation over all n , we

finally arrive at the overall formula (A.4). Furthermore, by utilizing cyclicity, we can show that

$$\Delta \mathcal{F}^{-1} = \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} \left(\mathcal{F}^{-1} D(h\tilde{e}^{s_k}) D(h\tilde{e}^{s_{n-1}}) \cdots D(h\tilde{e}^{s_1}) \otimes' e_{s_1} e_{s_2} \cdots e_{s_k} \mathcal{F}^{-1} \right) \Delta, \quad (\text{A.13})$$

holds.

Let us use this identity to demonstrate (A.1). To begin, the π_1 component can be calculated as

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \pi_1 \mathcal{F}^{-1} f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1}, \\ &= \lim_{\epsilon \rightarrow 0} \pi_1 (\mathbb{I} + h m) \frac{1}{I + h_{\epsilon} (m + i \hbar U)} h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1}, \\ &= \lim_{\epsilon \rightarrow 0} \pi_1 (I + h(m + i \hbar U)) \frac{1}{I + h_{\epsilon} (m + i \hbar U)} h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1}, \\ &= \lim_{\epsilon \rightarrow 0} \pi_1 h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1}, \\ &= \begin{cases} \Phi_1 & \text{for } n = 1, \\ 0 & \text{for } n \neq 1. \end{cases} \end{aligned} \quad (\text{A.14})$$

We proceed by mathematical induction. We take the following relation as our inductive hypothesis:

$$\lim_{\epsilon \rightarrow 0} \pi_m \mathcal{F}^{-1} f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1} = \begin{cases} \Phi_1 \otimes \cdots \otimes \Phi_n & \text{for } n = m, \\ 0 & \text{for } n \neq m, \end{cases} \quad (\text{A.15})$$

holds true as our inductive hypothesis, we now aim to show that the π_{m+1} component also satisfies the analogous identity. First, utilizing identity (A.4), $\pi_{m+1} \mathcal{F}^{-1}$ can be written as

$$\begin{aligned} \pi_{m+1} \mathcal{F}^{-1} &= \nabla (\pi_1 \otimes' \pi_m) \Delta \mathcal{F}^{-1}, \\ &= \nabla \left(\pi_1 \mathcal{F} \otimes' \pi_m \mathcal{F} - \hbar e_s \otimes' \pi_m \mathcal{F}^{-1} D(h\tilde{e}^s) \right) \Delta. \end{aligned} \quad (\text{A.16})$$

Next, let us consider the action of the coproduct Δ on $f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_n) \cdots f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi_1) f_{\epsilon} \mathbf{1}$. To calculate this, we use the following formulas:

$$\begin{aligned} \Delta f &= \sum_{k=0}^{\infty} \hbar^k e_{s_1} \cdots e_{s_k} f_{\epsilon} \otimes' f_{\epsilon} h_{\epsilon} \tilde{e}^{s_k} \cdots f_{\epsilon} h_{\epsilon} \tilde{e}^{s_1} f_{\epsilon} \pi_0 \\ &\quad + \sum_{k=0}^{\infty} \hbar^k e_{s_1} \cdots e_{s_k} \otimes' f_{\epsilon} h_{\epsilon} \tilde{e}^{s_k} \cdots f_{\epsilon} h_{\epsilon} \tilde{e}^{s_1} f_{\epsilon} (I - \pi_0), \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} \Delta f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi) &= \sum_{k=0}^{\infty} \hbar^k e_{s_1} \cdots e_{s_k} f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi) \otimes' f_{\epsilon} h_{\epsilon} \tilde{e}^{s_k} \cdots f_{\epsilon} h_{\epsilon} \tilde{e}^{s_1} f_{\epsilon} \pi_0 \\ &\quad + \sum_{k=0}^{\infty} \hbar^k e_{s_1} \cdots e_{s_k} \otimes' f_{\epsilon} h_{\epsilon} \tilde{e}^{s_k} \cdots f_{\epsilon} h_{\epsilon} \tilde{e}^{s_1} f_{\epsilon} h_{\epsilon} D(Q_{\epsilon} \Phi), \end{aligned} \quad (\text{A.18})$$

where $\mathbf{e}_{s_1} \cdots \mathbf{e}_{s_k}$ for $k = 0$ is $\mathbf{I}$. For notational simplicity, we introduce the following abbreviations for products of operators:

$$\mathbf{E}(s_1, s_k) = \mathbf{e}_{s_1} \cdots \mathbf{e}_{s_k}, \quad (\text{A.19})$$

$$\tilde{\mathbf{E}}(s_1, s_k) = \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{\mathbf{e}}^{s_k} \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \tilde{\mathbf{e}}^{s_1}. \quad (\text{A.20})$$

Utilizing these formulas and notations, we obtain

$$\begin{aligned} & \triangle \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ &= \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \hbar^{k_i + \cdots + k_n} \mathbf{E}(s_1^{(i)}, s_{k_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & \quad \otimes' \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}. \end{aligned} \quad (\text{A.21})$$

We note that the term corresponding to $i = 0$ is given by

$$\begin{aligned} & \sum_{k_0, \dots, k_n=0}^{\infty} \hbar^{k_0 + \cdots + k_n} \mathbf{E}(s_1^{(0)}, s_{k_0}^{(0)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{1} \\ & \quad \otimes' \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(1)}, s_{k_1}^{(1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \tilde{\mathbf{E}}(s_1^{(0)}, s_{k_0}^{(0)}) \mathbf{f}_\epsilon \mathbf{1}, \end{aligned} \quad (\text{A.22})$$

and the term corresponding to $i = n$ is expressed as

$$\sum_{k_i, \dots, k_n=0}^{\infty} \hbar^{k_n} \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{1}. \quad (\text{A.23})$$

Utilizing this result, we obtain

$$\begin{aligned} & \nabla (\pi_1 \mathcal{F} \otimes' \pi_m \mathcal{F}) \triangle \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ &= \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \hbar^{k_i + \cdots + k_n} \\ & \quad \pi_1 \mathcal{F}^{-1} \mathbf{E}(s_1^{(i)}, s_{k_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & \quad \otimes \pi_m \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}. \end{aligned} \quad (\text{A.24})$$

Next, let us consider the second term on the right-hand side of (A.16). Utilizing the identity for $\mathcal{F}$,

$$\mathcal{F}^{-1} \mathbf{D}(h \tilde{\mathbf{e}}^a) = \mathbf{F}^{-1} \mathbf{D}(h \tilde{\mathbf{e}}^a) \wedge \mathcal{F}^{-1}, \quad (\text{A.25})$$

which follows immediately from the definition (A.2), it can be written as

$$\begin{aligned}
& -\hbar \nabla \left(\pi_1 \mathbf{e}_s \otimes' \pi_m \mathcal{F}^{-1} \mathbf{D}(h\tilde{e}^s) \right) \Delta \\
& = -\hbar \nabla \left(\pi_1 \mathbf{e}_s \otimes' \pi_m (\mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^s) \wedge \mathcal{F}^{-1}) \right) \Delta, \\
& = \sum_{l=0}^{\infty} \frac{(-\hbar)^{l+1}}{(l+1)!} \nabla \left(\pi_1 \mathbf{e}_s \otimes' \pi_m (e_{t_1} \wedge \cdots \wedge e_{t_l} \wedge \pi_1 \mathbf{F}^{-1} \mathbf{D}(h\tilde{e}^{t_1}) \cdots \mathbf{D}(h\tilde{e}^{t_l}) \mathbf{D}(h\tilde{e}^s) \wedge \mathcal{F}^{-1}) \right) \Delta.
\end{aligned} \tag{A.26}$$

Furthermore, by using the identity $\pi_1 \mathbf{F}^{-1} = \pi_1 \mathcal{F}^{-1}$ together with cyclicity, we obtain

$$\begin{aligned}
& -\hbar \nabla \left(\pi_1 \mathbf{e}_s \otimes' \pi_m \mathcal{F}^{-1} \mathbf{D}(h\tilde{e}^s) \right) \Delta \\
& = \sum_{l=1}^{\infty} \frac{(-\hbar)^l}{l!} \nabla \left(\pi_1 \mathcal{F}^{-1} e_{t_1} \cdots e_{t_l} \otimes' h\tilde{e}^{t_1} \wedge \cdots \wedge h\tilde{e}^{t_l} \wedge \pi_{m-l} \mathcal{F}^{-1} \right) \Delta.
\end{aligned} \tag{A.27}$$

Applying this to $\mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}$ yields

$$\begin{aligned}
& -\hbar \nabla \left(\pi_1 \mathbf{e}_s \otimes' \pi_m \mathcal{F}^{-1} \mathbf{D}(h\tilde{e}^s) \right) \Delta \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\
& = \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \sum_{l=1}^{\infty} (-1)^l \frac{\hbar^{k_i + \cdots + k_n + l}}{l!} \\
& \quad \pi_1 \mathcal{F}^{-1} e_{t_1} \cdots e_{t_l} \mathbf{E}(s_1^{(i)}, s_{k_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\
& \quad \otimes h\tilde{e}^{t_1} \wedge \cdots \wedge h\tilde{e}^{t_l} \\
& \quad \wedge \pi_{m-l} \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}.
\end{aligned} \tag{A.28}$$

From the inductive hypothesis (A.15), $\pi_{m-l} \mathcal{F}^{-1}$ can be written as

$$h\tilde{e}^{s_1^{(i)}} \otimes \cdots \otimes h\tilde{e}^{s_{k_i}^{(i)}} \otimes \Phi_{i+1} \otimes h\tilde{e}^{s_1^{(i+1)}} \otimes \cdots \otimes h\tilde{e}^{s_{k_{i+1}}^{(i+1)}} \otimes \Phi_{i+2} \otimes \cdots \otimes \Phi_n \otimes h\tilde{e}^{s_1^{(n)}} \otimes \cdots \otimes h\tilde{e}^{s_{k_n}^{(n)}}, \tag{A.29}$$

and the factor on the right side of $\otimes$ in (A.28) is then interpreted as the result of inserting the elements $h\tilde{e}^{t_1}, \dots, h\tilde{e}^{t_l}$ into this structure, while preserving the original relative order of the terms given in (A.29). Let l_i be the number of elements from the set $h\tilde{e}^{t_1}, \dots, h\tilde{e}^{t_l}$ that are inserted to the left of Φ_i . Similarly, let l_{i+1} be the number of elements inserted between Φ_{i+1} and Φ_{i+2} , and so on, until l_n , which is the number of elements inserted to the right of Φ_n . The contribution from the factor on the right side of $\otimes$ in (A.28) resulting from this insertion procedure can be concisely written as

$$\pi_m \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n + l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1} + l_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i + l_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}. \tag{A.30}$$

Correspondingly, by moving $\mathbf{e}_{t_1}, \dots, \mathbf{e}_{t_l}$ on the left side of $\otimes$, it can be written as

$$\pi_1 \mathcal{F}^{-1} \mathbf{E}(s_1^{(i)}, s_{k_i+l_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n+l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}. \quad (\text{A.31})$$

The multiplicity for each insertion pattern is given by

$${}_{k_i+1} H_{l_i} \cdots {}_{k_n+1} H_{l_n}. \quad (\text{A.32})$$

Here, ${}_k H_l$ is defined as

$${}_k H_l = \frac{(k+l-1)!}{l!(k-1)!}. \quad (\text{A.33})$$

Ultimately, we obtained the following expression:

$$\begin{aligned} & \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \sum_{l_i, \dots, l_n=0}^{\infty} (-1)^{l_i + \dots + l_n} \hbar^{(k_i+l_i) + \dots + (k_n+l_n)} {}_{k_i+1} H_{l_i} \cdots {}_{k_n+1} H_{l_n} \\ & \pi_1 \mathcal{F}^{-1} \mathbf{E}(s_1^{(i)}, s_{k_i+l_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n+l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & \otimes \pi_m \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n+l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}+l_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i+l_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1} \\ & - \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \hbar^{k_i + \dots + k_n} \pi_1 \mathcal{F}^{-1} \mathbf{E}(s_1^{(i)}, s_{k_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & \otimes \pi_m \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}. \end{aligned} \quad (\text{A.34})$$

Here, the second term serves to remove the contribution corresponding to the case where $l_i = \dots = l_n = 0$.

Evaluating the full expression $\pi_{m+1} \mathcal{F}^{-1} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1}$ using these results shows that the contribution from (A.24) and the second term in equation (A.34) cancel each other. This leads

$$\begin{aligned} & \pi_{m+1} \mathcal{F}^{-1} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & = \sum_{i=0}^n \sum_{k_i, \dots, k_n=0}^{\infty} \sum_{l_i, \dots, l_n=0}^{\infty} (-1)^{l_i + \dots + l_n} \hbar^{(k_i+l_i) + \dots + (k_n+l_n)} {}_{k_i+1} H_{l_i} \cdots {}_{k_n+1} H_{l_n} \\ & \pi_1 \mathcal{F}^{-1} \mathbf{E}(s_1^{(i)}, s_{k_i+l_i}^{(i)}) \cdots \mathbf{E}(s_1^{(n)}, s_{k_n+l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \\ & \otimes \pi_m \mathcal{F}^{-1} \tilde{\mathbf{E}}(s_1^{(n)}, s_{k_n+l_n}^{(n)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \tilde{\mathbf{E}}(s_1^{(i+1)}, s_{k_{i+1}+l_{i+1}}^{(i+1)}) \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \tilde{\mathbf{E}}(s_1^{(i)}, s_{k_i+l_i}^{(i)}) \mathbf{f}_\epsilon \mathbf{1}. \end{aligned} \quad (\text{A.35})$$

Let us concentrate on the factor depending on the index l_i :

$$\sum_{k_i=0}^{\infty} \sum_{l_i=0}^{\infty} (-1)^{l_i} \frac{(k_i + l_i)!}{l_i! k_i!} \times (\text{factor depending on } k_i + l_i). \quad (\text{A.36})$$

Redefining the summation index $k_i + l_i$ as k_i , the expression becomes

$$\sum_{k_i=0}^{\infty} \sum_{l_i=0}^{k_i} (-1)^{l_i} \frac{(k_i!)}{l_i!(k_i - l_i)!} \times (\text{factor depending on } k_i). \quad (\text{A.37})$$

The term corresponding to the indices where $k_i \geq 1$ can be recognized as the binomial expansion of $(1 - 1)^{k_i}$:

$$\sum_{l_i=0}^{k_i} (-1)^{l_i} \frac{(k_i!)}{l_i!(k_i - l_i)!} = (1 - 1)^{k_i}, \quad (\text{A.38})$$

which therefore vanishes. Consequently, only the term with $k_i = 0$ survives, yielding the following result:

$$\sum_{i=0}^n \pi_1 \mathcal{F}^{-1} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_i) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} \otimes \pi_m \mathcal{F}^{-1} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_{i+1}) \mathbf{f}_\epsilon \mathbf{1}. \quad (\text{A.39})$$

From the inductive hypothesis (A.15), the factor to the right of $\otimes$ is non-zero only when $n - i = m$, and the factor to the left of $\otimes$ is non-zero only when $i = 1$. This establishes the desired result:

$$\lim_{\epsilon \rightarrow 0} \pi_{m+1} \mathcal{F}^{-1} \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_n) \cdots \mathbf{f}_\epsilon \mathbf{h}_\epsilon \mathbf{D}(Q_\epsilon \Phi_1) \mathbf{f}_\epsilon \mathbf{1} = \begin{cases} \Phi_1 \otimes \cdots \otimes \Phi_n & \text{for } n = m + 1, \\ 0 & \text{for } n \neq m + 1. \end{cases} \quad (\text{A.40})$$

It now remains only to demonstrate that the inverse of the operator $\mathcal{F}^{-1}$ defined in equation (A.2) is indeed the operator $\mathcal{F}$ defined in equation (7.78), in order to complete the proof. Here, we treat $\mathcal{F}$ as an unknown operator and require that it satisfies the inverse relations $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathbf{I}$. Then we verify that this requirement forces $\mathcal{F}$ to take the form given in equation (7.79). It is important to note that $\mathbf{F}$ is not the inverse of $\mathbf{F}^{-1}$; however, $\mathcal{F}$ is imposed to be the inverse of $\mathcal{F}^{-1}$.

Starting from the identity $\pi_1 \mathbf{I} = \pi_1 \mathcal{F}^{-1} \mathcal{F}$, we use the definition of $\mathcal{F}^{-1}$ to obtain

$$\begin{aligned} \pi_1 \mathbf{I} &= \pi_1 (\mathbf{I} + \mathbf{h} \mathbf{m}) \mathcal{F} \\ &= \pi_1 \mathbf{F} + \pi_1 \mathbf{h} \mathbf{m} \mathcal{F}. \end{aligned} \quad (\text{A.41})$$

Thus, we can rearrange the terms to find

$$\pi_1 \mathbf{F} = \pi_1 (\mathbf{I} - \mathbf{h} \mathbf{m} \mathcal{F}). \quad (\text{A.42})$$

We perform the proof by mathematical induction. We begin by assuming the form of $\pi_k \mathcal{F}$:

$$\pi_k \mathbf{F} = \frac{\hbar^{k-1}}{k!} e_{s_1} \wedge \cdots \wedge e_{s_{k-1}} \wedge \pi_1 (\mathbf{I} - \mathbf{h} \mathbf{m} \mathcal{F}) \mathbf{D}(\hbar \tilde{e}^{s_{k-1}}) \cdots \mathbf{D}(\hbar \tilde{e}^{s_1}). \quad (\text{A.43})$$

We then proceed to demonstrate that the corresponding identity holds for $\pi_{k+1} \mathcal{F}$. To proceed with the mathematical induction, we first establish the following formula:

$$\pi_n \mathcal{F} = \sum_{k=1}^n \frac{k}{n} \pi_k \mathbf{F} \wedge \pi_{n-k} \mathcal{F}. \quad (\text{A.44})$$

This identity holds for all $n \geq 1$. The proof proceeds as follows. A morphism of the form (7.78) can be rewritten as

$$\pi_n \mathcal{F} = \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} \delta \left(\sum_{m=1}^n m l_m - n \right) \frac{1}{l_1! \cdots l_n!} (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_n \mathbf{F})^{\wedge l_n}, \quad (\text{A.45})$$

where $\delta(n)$ is defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \quad (\text{A.46})$$

Substituting (A.45) into the left-hand side of (A.44), we obtain

$$\begin{aligned} & \sum_{k=1}^n \sum_{l_1=0}^{\infty} \cdots \sum_{l_{n-k}=0}^{\infty} \delta \left(\sum_{m=1}^{n-k} m l_m - n + k \right) \frac{k}{n} \frac{1}{l_1! \cdots l_{n-k}!} \\ & (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_k \mathbf{F})^{l_k+1} \wedge \cdots \wedge (\pi_{n-k} \mathbf{F})^{\wedge l_{n-k}}. \end{aligned} \quad (\text{A.47})$$

Redefining the index $l_k + 1$ as l_k , the expression transforms into

$$\begin{aligned} & \sum_{k=1}^n \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=1}^{\infty} \cdots \sum_{l_{n-k}=0}^{\infty} \delta \left(\sum_{m=1}^{n-k} m l_m - n \right) \frac{k}{n} \frac{l_k}{l_1! \cdots l_{n-k}!} \\ & (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_k \mathbf{F})^{l_k} \wedge \cdots \wedge (\pi_{n-k} \mathbf{F})^{\wedge l_{n-k}}. \end{aligned} \quad (\text{A.48})$$

This result can also be expressed as

$$\begin{aligned} & \sum_{k=1}^n \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=1}^{\infty} \cdots \sum_{l_n=0}^{\infty} \delta \left(\sum_{m=1}^n m l_m - n \right) \frac{k}{n} \frac{l_k}{l_1! \cdots l_n!} \\ & (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_k \mathbf{F})^{l_k} \wedge \cdots \wedge (\pi_n \mathbf{F})^{\wedge l_n}, \end{aligned} \quad (\text{A.49})$$

because when $l_k \geq 1$, all l_i for $i \geq n - k + 1$ must be zero in order to hold $\sum_{m=1}^n m l_m = n$. Furthermore, since the contribution vanishes trivially when $l_k = 0$, the summation over l_k can

indeed begin at $l_k = 0$:

$$\sum_{k=1}^n \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} \delta\left(\sum_{m=1}^n m l_m - n\right) \frac{k}{n} \frac{l_k}{l_1! \cdots l_n!} (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_k \mathbf{F})^{l_k} \wedge \cdots \wedge (\pi_n \mathbf{F})^{\wedge l_n}. \quad (\text{A.50})$$

Summing over the index k and using the identity $\sum_{k=1}^n k l_k = n$, the expression simplifies to

$$\frac{k}{n} \pi_k \mathbf{F} \wedge \pi_{n-k} \mathcal{F} = \sum_{k=1}^n \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} \delta\left(\sum_{m=1}^n m l_m - n\right) \frac{1}{l_1! \cdots l_n!} (\pi_1 \mathbf{F})^{\wedge l_1} \wedge \cdots \wedge (\pi_k \mathbf{F})^{l_k} \wedge \cdots \wedge (\pi_n \mathbf{F})^{\wedge l_n}. \quad (\text{A.51})$$

Since the right-hand side is equal to $\pi_n \mathcal{F}$, this completes the proof of formula (A.44).

We now proceed with the mathematical induction using this formula. From the inverse identity $\mathcal{F} \mathcal{F}^{-1} = \mathbf{I}$, we analyze the π_n component:

$$\pi_n \mathbf{I} = \pi_n \mathcal{F} \mathcal{F}^{-1} = \sum_{k=1}^n \frac{k}{n} (\pi_k \mathbf{F} \wedge \pi_{n-k} \mathcal{F}) \mathcal{F}^{-1}. \quad (\text{A.52})$$

In the right-hand side, the output components of $\mathcal{F}^{-1}$ are distributed: some outputs feed into the first factor $\pi_k \mathcal{F}$, while the remaining outputs feed into the second factor $\pi_{n-k} \mathcal{F}$. The way the outputs of the operator $\mathcal{F}^{-1}$ split is governed by the same principle as detailed in equation (A.4) or (A.13). Using this principle, it can be written as

$$\begin{aligned} \pi_n \mathbf{I} &= \sum_{k=1}^n \sum_{l=0}^{\infty} \frac{k}{n} \frac{(-\hbar)^l}{l!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_l} \mathcal{F}^{-1} \wedge \pi_{n-k} \mathcal{F} \mathcal{F}^{-1} \mathbf{D}(h \tilde{e}^{s_l}) \cdots \mathbf{D}(h \tilde{e}^{s_1}), \\ &= \sum_{k=1}^n \sum_{l=0}^{n-k} \frac{k}{n} \frac{(-\hbar)^l}{l!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_l} \mathcal{F}^{-1} \wedge \pi_{n-k} \mathbf{D}(h \tilde{e}^{s_l}) \cdots \mathbf{D}(h \tilde{e}^{s_1}), \\ &= \sum_{k=1}^n \frac{k}{n} \frac{(-\hbar)^{n-k}}{(n-k)!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_{n-k}} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_{n-k}} \wedge \cdots \wedge h \tilde{e}^{s_1} \\ &\quad + \sum_{k=1}^{n-1} \sum_{l=0}^{n-k-1} \frac{k}{n} \frac{(-\hbar)^l}{l!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_l} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_l} \wedge \cdots \wedge h \tilde{e}^{s_1} \wedge \pi_{n-k-l} \mathbf{I}. \end{aligned} \quad (\text{A.53})$$

Here, we define

$$W_m = \sum_{k=1}^m \frac{k}{m} \frac{(-\hbar)^{m-k}}{(m-k)!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_{m-k}} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_{m-k}} \wedge \cdots \wedge h \tilde{e}^{s_1}. \quad (\text{A.54})$$

It is easy to see that

$$W_1 = \pi_1 \mathbf{F} \mathcal{F}^{-1} = \pi_1 \mathbf{I}. \quad (\text{A.55})$$

In the following, we demonstrate that $W_m = 0$ for $m \geq 2$ by employing mathematical induction. By using (A.53) into the definition of W_2 , we obtain

$$W_2 = \pi_2 \mathbf{I} - \frac{1}{2} \pi_1 \mathbf{F} \mathcal{F}^{-1} \wedge \pi_1 \mathbf{I} = 0. \quad (\text{A.56})$$

Assuming the inductive hypothesis holds, i.e., $W_m = 0$ for all $m \geq 2$, the next component W_{m+1} can be expressed as

$$W_{m+1} = \pi_{m+1} \mathbf{I} - \sum_{k=1}^m \sum_{l=0}^{m-k} \frac{k}{m+1} \frac{(-\hbar)^l}{l!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_l} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_l} \wedge \cdots \wedge h \tilde{e}^{s_1} \wedge \pi_{m-k-l+1} \mathbf{I}. \quad (\text{A.57})$$

Redefining the summation index $l+k$ as l , the second term can be written as

$$\begin{aligned} & - \sum_{l=1}^m \sum_{k=1}^l \frac{k}{m+1} \frac{(-\hbar)^{l-k}}{(l-k)!} \pi_k \mathbf{F} e_{s_1} \cdots e_{s_{l-k}} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_{l-k}} \wedge \cdots \wedge h \tilde{e}^{s_1} \wedge \pi_{m-l+1} \mathbf{I} \\ & = - \sum_{l=1}^m \frac{l}{m+1} W_l \wedge \pi_{m-l+1} \mathbf{I}. \end{aligned} \quad (\text{A.58})$$

Since $W_1 = \pi_1 \mathbf{I}$ and $W_m = 0$ for all $m \geq 2$, only the term corresponding to the index $l = 1$ survives in the summation. This remaining term simplifies to

$$-\frac{1}{m+1} \pi_1 \wedge \pi_m = -\pi_{m+1} \mathbf{I}. \quad (\text{A.59})$$

Therefore, we find that the expression for W_{m+1} vanishes:

$$W_{m+1} = 0. \quad (\text{A.60})$$

Combining the base case ($m = 1$) and the result of the inductive step ($m \geq 2$), we conclude that the component W_m is universally given by

$$W_m = \begin{cases} \pi_1 \mathbf{I} & \text{for } m = 1, \\ 0 & \text{for } m \geq 2. \end{cases} \quad (\text{A.61})$$

Using the definition (A.54) and the established result that $W_k = 0$ for $k \geq 2$, we obtain

$$\pi_1 \mathbf{F} \mathcal{F}^{-1} = - \sum_{l=1}^{k-1} \frac{l}{k} \frac{(-\hbar)^{k-l}}{(k-l)!} \pi_l \mathbf{F} e_{s_1} \cdots e_{s_{k-l}} \mathcal{F}^{-1} \wedge h \tilde{e}^{s_{k-l}} \wedge \cdots \wedge h \tilde{e}^{s_1}. \quad (\text{A.62})$$

Here, the left-hand side corresponds to the $k = m$ term in expression (A.54). By using the inductive assumption (A.43), $\pi_{k+1}\mathbf{F}\mathcal{F}$ can be calculated as

$$\begin{aligned}
\pi_{k+1}\mathbf{F}\mathcal{F}^{-1} &= - \sum_{l=1}^k \frac{l}{k+1} \frac{(-\hbar)^{k-l+1}}{(k-l+1)!} \frac{\hbar^{l-1}}{l!} e_{t_1} \wedge \cdots \wedge e_{t_{l-1}} \\
&\quad \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{t_{l-1}}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) e_{s_1} \cdots e_{s_{k-l+1}} \mathcal{F}^{-1} \wedge h\tilde{e}^{s_{k-l+1}} \wedge \cdots \wedge h\tilde{e}^{s_1}, \\
&= - \sum_{l=1}^k \frac{(-1)^{k-l+1} (\hbar)^k}{(k+1)(k-l+1)!(l-1)!} e_{s_1} \wedge \cdots \wedge e_{s_{k-l+1}} \wedge e_{t_1} \wedge \cdots \wedge e_{t_{l-1}} \\
&\quad \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{t_{l-1}}) \cdots \mathbf{D}(h\tilde{e}^{t_1}) \mathbf{D}(h\tilde{e}^{s_{k-l+1}}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \mathcal{F}^{-1}, \\
&= - \sum_{l=1}^k \frac{\hbar^k}{(k+1)!} \frac{(-1)^{k-l+1} k!}{(k-l+1)!(l-1)!} \\
&\quad e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \mathcal{F}^{-1}. \tag{A.63}
\end{aligned}$$

The summation over l can be evaluated as follows:

$$\sum_{l=1}^k \frac{(-1)^{k-l+1} k!}{(k-l+1)!(l-1)!} = \sum_{l=0}^{k-1} (-1)^{k-l} \frac{k!}{(k-l)!l!} = -1 + \sum_{l=0}^k (-1)^{k-l} \frac{k!}{(k-l)!l!}. \tag{A.64}$$

Since the summation over l in the second term of the right-hand side can be interpreted as the binomial expansion of $(1-1)^k$, which vanishes for $k \geq 1$, that term cancels out. Therefore, the expression for the π_k component simplifies to

$$\pi_{k+1}\mathbf{F}\mathcal{F}^{-1} = \frac{\hbar^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}) \mathcal{F}^{-1}. \tag{A.65}$$

By applying $\mathcal{F}$ from the right to both sides, we obtain the desired result:

$$\pi_{k+1}\mathbf{F} = \frac{\hbar^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}). \tag{A.66}$$

In conclusion, $\mathcal{F}$ satisfying $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathbf{I}$ is given by

$$\begin{aligned}
\mathcal{F} &= \pi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{F}^{\wedge n} \\
\mathbf{F} &= \sum_{k=0}^{\infty} \frac{\hbar^k}{(k+1)!} e_{s_1} \wedge \cdots \wedge e_{s_k} \wedge \pi_1(\mathbf{I} - \mathbf{h}\mathbf{m}\mathcal{F})\mathbf{D}(h\tilde{e}^{s_k}) \cdots \mathbf{D}(h\tilde{e}^{s_1}). \tag{A.67}
\end{aligned}$$

This successfully completes the proof.

B Cyclicity of $\pi_1 m\mathcal{F}$

In this appendix, we demonstrate that $\pi_1 m\mathcal{F}$ is cyclic, utilizing the operator $\mathcal{F}$ defined in (A.67). Before proceeding, we must first calculate $\Delta\mathcal{F}$. Using the inverse relation $\mathcal{F}^{-1}\mathcal{F} = I$ and the expression of $\Delta\mathcal{F}$ (A.13), we obtain

$$(I \otimes' I)\Delta = \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} \left(\mathcal{F}^{-1} D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \otimes' e_{s_1} \cdots e_{s_k} \mathcal{F}^{-1} \right) \Delta\mathcal{F}. \quad (\text{B.1})$$

We now let the operator

$$\sum_{l=0}^{\infty} \frac{h^l}{l!} \left(e_{t_1} \cdots e_{t_k} \mathcal{F} \otimes' \mathcal{F} D(h\tilde{e}^{t_k}) \cdots D(h\tilde{e}^{t_1}) \right), \quad (\text{B.2})$$

act on this expression from the left. Here, the composition

$$\sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} \sum_{l=0}^{\infty} \frac{h^l}{l!} \left(e_{t_1} \cdots e_{t_k} \mathcal{F} \otimes' \mathcal{F} D(h\tilde{e}^{t_k}) \cdots D(h\tilde{e}^{t_1}) \right) \left(\mathcal{F}^{-1} D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \otimes' e_{s_1} \cdots e_{s_k} \mathcal{F}^{-1} \right) \quad (\text{B.3})$$

can be calculated as follows:

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\hbar)^k}{k!} \frac{h^l}{l!} \\ & \left(e_{t_1} \cdots e_{t_k} \mathcal{F} \otimes' \mathcal{F} D(h\tilde{e}^{t_k}) \cdots D(h\tilde{e}^{t_1}) \right) \left(\mathcal{F}^{-1} D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \otimes' e_{s_1} \cdots e_{s_k} \mathcal{F}^{-1} \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-)^k \hbar^{k+l}}{k!l!} e_{t_1} \cdots e_{t_k} D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \otimes' \mathcal{F} D(h\tilde{e}^{t_k}) \cdots D(h\tilde{e}^{t_1}) e_{s_1} \cdots e_{s_k} \mathcal{F}^{-1}, \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-)^k \hbar^{k+l}}{k!l!} e_{t_1} \cdots e_{t_k} e_{s_1} \cdots e_{s_k} \otimes' \mathcal{F} D(h\tilde{e}^{t_k}) \cdots D(h\tilde{e}^{t_1}) D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \mathcal{F}^{-1}, \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k m!}{k!(m-k)!} \frac{\hbar^m}{m!} e_{s_1} \cdots e_{s_m} \otimes' \mathcal{F} D(h\tilde{e}^{s_m}) \cdots D(h\tilde{e}^{s_1}) \mathcal{F}^{-1}. \end{aligned} \quad (\text{B.4})$$

In the last line, the summation over k represents the binomial expansion $(1-1)^m$, which vanishes unless $m = 0$. Consequently, this expression is calculated to be

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k m!}{k!(m-k)!} \frac{\hbar^m}{m!} e_{s_1} \cdots e_{s_m} \otimes' \mathcal{F} D(h\tilde{e}^{s_m}) \cdots D(h\tilde{e}^{s_1}) \mathcal{F}^{-1} \\ &= I \otimes' \mathcal{F} \mathcal{F}^{-1}, \\ &= I \otimes' I. \end{aligned} \quad (\text{B.5})$$

Therefore, $\Delta\mathcal{F}$ can be written as

$$\Delta\mathcal{F} = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left(e_{s_1} \cdots e_{s_k} \mathcal{F} \otimes' \mathcal{F} D(h\tilde{e}^{s_k}) \cdots D(h\tilde{e}^{s_1}) \right) \Delta. \quad (\text{B.6})$$

Furthermore, by utilizing the expression in (A.4) for $\Delta\mathcal{F}^{-1}$, we obtain

$$\Delta\mathcal{F} = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left(\mathcal{F} D(h\tilde{e}^{s_k}) D(h\tilde{e}^{s_{n-1}}) \cdots D(h\tilde{e}^{s_1}) \otimes' e_{s_1} e_{s_2} \cdots e_{s_k} \mathcal{F} \right) \Delta. \quad (\text{B.7})$$

This demonstration requires showing that

$$\langle \omega | (\pi_1 \mathbf{m} \mathcal{F} \otimes \mathbb{I} + \mathbb{I} \otimes \pi_1 \mathbf{m} \mathcal{F}) = 0, \quad (\text{B.8})$$

is satisfied. To this end, we compute

$$\langle \omega | \nabla (\pi_1 \mathbf{m} \otimes' \mathbb{I} + \mathbb{I} \otimes' \pi_1 \mathbf{m}) \Delta \mathcal{F}. \quad (\text{B.9})$$

While this term is zero due to the cyclicity of $\mathbf{m}$, we can also express it in an alternative form by using (B.6) or (B.7):

$$\begin{aligned} & \langle \omega | \nabla (\pi_1 \mathbf{m} \otimes' \mathbb{I} + \mathbb{I} \otimes' \pi_1 \mathbf{m}) \Delta \mathcal{F} \\ &= \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} D \otimes \pi_1 \mathcal{F} + \pi_1 \mathcal{F} \otimes \pi_1 \mathbf{m} \mathcal{F}) + \hbar \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s) \otimes e_s + e_s \otimes \pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s)), \\ &= \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} \otimes \mathbb{I} + \mathbb{I} \otimes \pi_1 \mathbf{m} \mathcal{F}) - \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} D \otimes \pi_1 \mathbf{h} \mathbf{m} \mathcal{F} + \pi_1 \mathbf{h} \mathbf{m} \mathcal{F} \otimes \pi_1 \mathbf{m} \mathcal{F}) \\ & \quad + \hbar \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s) \otimes e_s + e_s \otimes \pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s)). \end{aligned} \quad (\text{B.10})$$

Here, the second term on the right-hand side vanishes due to the property of h (5.5). Therefore, we find

$$\langle \omega | (\pi_1 \mathbf{m} \mathcal{F} \otimes \mathbb{I} + \mathbb{I} \otimes \pi_1 \mathbf{m} \mathcal{F}) = -\hbar \langle \omega | (\pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s) \otimes e_s + e_s \otimes \pi_1 \mathbf{m} \mathcal{F} D(h\tilde{e}^s)). \quad (\text{B.11})$$

This identity allows us to inductively demonstrate the cyclicity of $\pi_1 \mathbf{m} \mathcal{F}$. First, in the zeroth order of $\hbar$, the right-hand side of the expression vanishes. Consequently, in this order, we find that $\pi_1 \mathbf{m} \mathcal{F}$ is cyclic. This result precisely corresponds to the fact that the tree-level component $\pi_1 \mathbf{m} \mathcal{F}$ is also cyclic. Next, we assume that $\pi_1 \mathbf{m} \mathcal{F}$ is cyclic up to the k -th order in $\hbar$. Focusing on the $(k+1)$ -th order in $\hbar$ in (B.11), we can verify that the right-hand side vanishes by utilizing the cyclicity of $\mathbf{m} \mathcal{F}$ at the k -th order in $\hbar$. Consequently, this demonstrates that $\pi_1 \mathbf{m} \mathcal{F}$ is also cyclic at the $(k+1)$ -th order in $\hbar$. This completes the proof that the cyclicity property holds for $\pi_1 \mathbf{m} \mathcal{F}$ to all orders in $\hbar$.

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