

An uncountable number Of II_1 , II_∞ .-factors

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1. Introduction. Recently, great progress has been made in the investigation of the isomorphism classes of II_1 -factors ([1], [2],[4], [6],[7], [8]).

In particular, McDuff [4] proved the existence of a countably infinite number of II_1 -factors on a separable Hilbert space.

In this paper, by using the method of McDuff, we shall show the existence of an uncountable number of non-isomorphic II_1 -factors on a separable Hilbert space.

Moreover, by using this result and tensor products, we shall show the existence of an uncountable number of II_∞ factors on a separable Hilbert space,

Concerning III-factors, Powers [11] has shown the existence of an uncountable number.

Added in proof. After writing this paper, the author received other two papers of McDuff [9], [10] in which she proves the existence of an uncountable number of II_1 -factors. But, the construction is different from ours.

2. Construction of examples. Suppose $G_1, G_2, \dots$; $H_1, H_2, \dots$ are two sequences of groups. We denote by $(G_1, G_2, \dots ; H_1, H_2, \dots)$ the group generated by the G_i 's and the H_i 's with additional relations that H_i, H_j

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commute elementwise for $i \neq j$ and G_i, H_j commute elementwise for $i \leq j$. This situation was considered in [2].

Let $L_1 = (G_1, G_2, \dots; H_1, H_2, \dots)$ with $G_i = Z$, and $H_i = Z$ for all i , where Z is the infinite cyclic group. Define L_k inductively by $L_k = (G_1, G_2, \dots; H_1, H_2, \dots)$ with $G_i = Z$, $H_i = L_{k-1}$ for all i and $k \geq 2$.

Let I be the set of all positive integers and let I_1 be ~~a sequence of positive integers~~ ^{a subset of I} . Let $M_n(I_1) = \sum_{i=1}^n \oplus L_{p_i}$, if $I_1 = (p_1, p_2, \dots)$ is infinite, and $M_n(I_1) = \sum_{i=1}^n \oplus L_{p_i}$ for $n \leq n_0$ and $M_n(I_1) = M_{n_0}$ for $n > n_0$, if $I_1 = (p_1, p_2, \dots, p_{n_0})$ is finite.

Let $G[I_1] = (G_1, G_2, \dots; H_1, H_2, \dots)$ with $G_i = Z$ and $H_i = M_i(I_1)$ for all i .

For a discrete group G , $U(G)$ is the W^* -algebra generated by the left regular representation of G .

Then, we shall show the following theorem

^{sequences}
Theorem 1. Let $I_1 = (p_i)$ and $I_2 = (q_j)$ be two subsets of I such that I_2 contains a positive integer q such that $q \geq 2$, $q-1 \notin I_1$. Then, $U(G[I_1])$ is not $*$ -isomorphic to $U(G[I_2])$. In particular, $U(G[I_1])$ is not $*$ -isomorphic to $U(G[I_2])$, if I_1 and I_2 are two subsets of ~~even~~ positive integers and $I_1 \neq I_2$ as a set.

As a corollary, we have

Corollary 1. There exists an uncountable number of non-isomorphic II_1 -factors on a separable Hilbert space. To prove Theorem 1, we shall provide some considerations.

Definition 1 ([4]). For a W^* -algebra $U(G)$ we shall write $(U(G))_1$ to denote the unit sphere of $U(G)$. If B and C are subalgebras of $U(G)$ and $\delta > 0$, then we shall write $B \overset{\delta}{\subset} C$ to mean that given any $T \in (B)_1$, there exists some $S \in (C)_1$ with $\|T - S\|_2 < \delta$ where $\|x\|_2$ is the $L^2(G)$ -norm of x , when $U(G)$ is embedded into $L^2(G)$ canonically.

Let $A = U(G)$. A bounded sequence (T_n) of elements of A is called a central sequence, if for all $X \in A$,

$$\|[X, T_n]\|_2 \rightarrow 0 \quad (n \rightarrow \infty), \text{ where } [\cdot, \cdot] \text{ is the Lie product.}$$

Central sequences (T_n) , (T'_n) in A are called equivalent, if

$$\|T_n - T'_n\|_2 \rightarrow 0 \quad (n \rightarrow \infty).$$

Let H be a subgroup of a group G . H is called strongly residual in G , if it satisfies the following conditions :
there exist a subset S of the complement $G \setminus H$ of H and elements g_1, g_2 of G such that (i) $g_1 H g_1^{-1} = H$, (ii) $S \cup g_1 S g_1^{-1} = G \setminus H$, (iii) $\{g_2^n S g_2^{-n}\}_{n=0, \pm 1, \pm 2, \dots}$ forms a family of disjoint subsets of $G \setminus H$.

By the above definition, we can easily see that only one strongly residual subgroup of a commutative group G is G itself - in this case, S is the empty set.

Lemma 1 ([4]). Let G_i ($i = 1, 2, \dots, n$) be a finite family of groups and let H_i ($i = 1, 2, \dots, n$) be a subgroup of G_i . Suppose that H_i is strongly residual in G_i for each i , then $\sum_{i=1}^n \oplus H_i$ is strongly residual in $\sum_{i=1}^n \oplus G_i$.

Let H be a strongly residual subgroup of G , then H must

contain the center of G .

Let $\{H_n\}$ be a sequence of subgroups of G . $\{H_n\}$ is called a residual sequence of G , if it satisfies the following conditions : (i) H_n is strongly residual in G ; (ii) $H_n = H_{n+1} \oplus K_n$, where K_n is a subgroup of G ; (iii) $\bigcup_{n=1}^{\infty} H_n' = G$, where H_n' is the commutant of H_n in G .

Let G_i ($i = 1, 2, \dots, m$) be a finite family of groups and let $\{H_{i,n}\}$ ($i = 1, 2, \dots, m$) be a residual sequence of G_i then $\{\sum_{i=1}^m H_{i,n}\}$ is a residual sequence of $\sum_{i=1}^m G_i$. Any central sequence in $U(G)$ is equivalent to a central sequence whose elements lie in $U(H)$, if H is canonically considered as a subalgebra of $U(G)$, and H is strongly residual in G .

Definition 2 ([4]). A sequence (T_n) in the unit sphere $(A)_1$ of $A \cong U(G)$ is an ϵ -approximate central sequence, if $\limsup \| [T_n, X] \|_2 < \epsilon$ for all $X \in (A)_1$. The set of all ϵ -approximate sequences is denoted by $C_A(\epsilon)$.

If H is strongly residual in G , then for all $(T_n) \in C_{U(G)}(\epsilon)$, there exists a sequence (T_n') in the unit sphere of $U(H)$ such that $\limsup \| T_n - T_n' \|_2 < 14\epsilon$ (cf. [3], [5], [6]).

Let $G = (G_1, G_2, \dots; H_1, H_2, \dots)$ with $G_1 = Z$ and let $Q(G, n) = \sum_{j=n}^{\infty} \oplus H_j$ and $Q(G, m, n) = \sum_{j=m}^n \oplus H_j$.

Then, $\{Q(G, n)\}$ is a residual sequence in G . Let $(\Gamma_k |$

$k = 1, 2, \dots, r)$ be a finite family with the form $\Gamma_k =$

$(G_1, G_2, \dots; H_1, H_2, \dots)$ with $G_1 = Z$.

Let $Q(\sum_{k=1}^r \oplus \Gamma_k, n) = \sum_{k=1}^r \oplus Q(\Gamma_k, n)$ is a residual sequence in G . This residual sequence is called the

canonical residual sequence.

Denote $Q(\sum_{k=1}^r \oplus P_k, n, m) = \sum_{k=1}^r \oplus Q(P_k, n, m)$.

A group G is called of type 0, if it is commutative
 G is called of type i , if $G = \sum_{j=1}^n \oplus G_j$ with $G_j = L_i$; G is
called of type i_∞ , if $G = \sum_{j=1}^\infty \oplus G_j$ with $G_j = L_i$; G is called
of type $(i_1, i_2, \dots, i_n)$, if $G = \sum_{j=1}^n \oplus G_j$, where G_j is
of type i_j ; G is called of type $(i_1, i_2, \dots, i_n)_\infty$, if
 $G = \sum_{j=1}^\infty \oplus G_j$ and some of G_j are of type i_j and others are of
type i_j .

Now let $U(G[I_1]) = A$ and $U(G[I_2]) = B$. Suppose that
 A is $*$ -isomorphic to B . Then, under the identification
 $A = B$, we have two expressions $U(G[I_1])$ and $U(G[I_2])$.

Henceforward, we shall assume that $A = B$ and conclude
a contradiction.

Lemma 2 ([4]). For $\delta > 0$ and a positive integer n_1
there exists a positive integer n_2 such that
 $U(Q(G[I_2], n_2)) \overset{\delta}{\subset} U(Q(G[I_1], n_1))$

Moreover, since $U(Q(G[I_1], n, n+1))$ is a factor, we
have

Lemma 3 ([4]). For a positive integer m_2 with $m_2 > n_2$,
there exists a positive integer m_1 such that $m_1 > n_1$ and
 $U(Q(G[I_2], n_2, m_2)) \overset{\delta}{\subset} U(Q(G[I_1], n_1, m_1))$.

Now let $I_1 = (p_i)$ and $I_2 = (q_j)$. Without the loss
of generality, we can assume that $q = q_1$.

For $t = 10^{q_1}$, by applying Lemma 2 for I_1 and the symmetric
form of Lemma 2 for I_2 , we can choose positive integers n_1

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, $n_2, \dots, n_t$ such that $n_2 < n_4 < n_6 < \dots < n_t$,

and $n_1 < n_3 < \dots < n_{t-1}$ and

$$U(Q(G[I_2], n_t)) \overset{\delta}{\subset} U(Q(G[I_1], n_{t-1})) \overset{\delta}{\subset} \dots \overset{\delta}{\subset} U(Q(G[I_2], n_2)) \overset{\delta}{\subset} U(Q(G[I_1], n_1)).$$

Then, by Lemma 3, we can choose positive integers $m_1,$

$m_2, \dots, m_t$ such that $m_2 > m_4 > \dots > m_t$ and

$m_1 > m_3 > \dots > m_{t-1}$ with $m_t > n_t$ and

$$U(Q(G[I_2], n_t, m_t)) \overset{\delta}{\subset} U(Q(G[I_1], n_{t-1}, m_{t-1})) \overset{\delta}{\subset} \dots \overset{\delta}{\subset} U(Q(G[I_1], n_1, m_1)).$$

Since $Q(G[I_i], h, k)$ is a finite sum of the form $(G_1, G_2, \dots$

$H_1, H_2, \dots)$ with $G_i = Z$, it has the canonical residual sequence $\{Q(Q(G[I_i], h, k), n)\}$.

For simplicity, we shall denote $Q(G[I_i], h, k)$ (resp.

$Q(Q(G[I_i], h, k), n)$) by $Q_i(h, k)$ (resp. $Q_i^2((h, k), n)$).

Lemma 4. Suppose $U(Q_1(h, k)) \overset{\delta}{\subset} U(Q_2(i, j))$
 $\overset{\delta}{\subset} U(Q_1(p, q))$ with $h > p$ and $q > k$.

Then for arbitrary positive integers r and w , there exists a positive integer s such that $U(Q_1^2((h, k), s)) \overset{(10)^3 \delta}{\subset} U(Q_2^2(i, j), r)$ and $s > w$.

Proof. Suppose this is not true, then there exists

$T_n \in (U(Q_1^2((h, k), n)))_1$ for each n with $n > w$ such that $\|T_n - S\|_2 \geq (10)^3 \delta$ for all $S \in (U(Q_2^2((i, j), r)))_1$.

Since $\{Q_1^2((h, k), n)\}$ is a residual sequence in $U(Q_1(h, k))$, (T_n) is a central sequence in $U(Q_1(h, k))$.

On the other hand, $Q_1(p, q) = Q_1(h, k) \oplus C$, where C is a subgroup of $Q_1(p, q)$; hence (T_n) is a central

sequence in $U(Q_1(p, q))$. Now, take $T_n' \in (U(Q_2(i, j)))_1$ such that $\|T_n - T_n'\|_2 < 9\delta$ and for arbitrary $X' \in (U(Q_2(i, j)))_1$, take $X \in (U(Q_1(p, q)))_1$ such that $\|X - X'\|_2 < 9\delta$.

Then,

$$\begin{aligned} \| [X', T_n'] \|_2 &= \| [X', T_n' - T_n] \|_2 + \| [T_n, X] \|_2 \\ + \| [T_n, X - X'] \|_2 &\leq 2 \| T_n' - T_n \|_2 + \| [T_n, X] \|_2 \\ + 2 \| X - X' \|_2. \end{aligned}$$

$$\text{Hence } \limsup \| [X', T_n'] \|_2 \leq 18\delta + 18\delta < 37\delta.$$

Therefore, there exists a sequence (T_n'') in $(U(Q_2^2(i, j), r)))_1$ such that $\limsup \| T_n' - T_n'' \|_2 < 14 \cdot 37\delta$.

$$\text{Then, } \| T_n - T_n'' \|_2 \leq \| T_n - T_n' \|_2 + \| T_n' - T_n'' \|_2 < 10^3 \delta.$$

This is a contradiction and completes the proof.

Applying this lemma for I_1 and the symmetric one for I_2 , there exist positive integers $r_2, r_3, \dots, r_t$ such that $r_2 < r_4 < \dots < r_t$ and $r_3 < r_4 < \dots < r_{t-1}$ and $U(Q_2^2((n_t, m_t), r_t)) \overset{(10)^3\delta}{\subset} \dots$

$$U(Q_1^2((n_3, m_3), r_3)) \overset{(10)^3\delta}{\subset} U(Q_2^2((n_2, m_2), r_2)).$$

$$Q(G[I_1], n, m) = \sum_{j=1}^m \oplus M_j(I_1) \text{ is of type } (p_1, p_2, \dots, p_m).$$

Then, $Q_1^2((n, m), r)$ is of type $(p_1-1, \dots, p_m-1)$.

Therefore, at this time, $Q_i^2((h, k), r)$ might contain

a type 0-group as a direct summand.

Now we shall define: for $r < s$, $RQ_i^2((h, k), (r, s))$

= the center of $Q_i^2((h, k), r) \oplus (Q_i^2((h, k), r) \oplus$

$$Q_i^2((h, k), s+1)).$$

Lemma 5. For arbitrary positive integer $s_t > r_t$, there exist positive integers $s_4, s_5, \dots, s_t$ such that $s_4 > s_6 > \dots > s_t$ and $s_5 > s_7 > \dots > s_{t-1}$ and

$$\begin{aligned} & U(RQ_2^2(n_t, m_t), (r_t, s_t)) \subset \dots \dots \dots \\ & \subset U(RQ_1^2((n_5, m_5), (r_5, s_5))) \subset U(RQ_2^2((n_4, m_4), (r_4, s_4))). \end{aligned}$$

Proof. $Q_2((n_{t-2}, m_{t-2})) = Q_2((n_t, m_t) \oplus H$, where H is a subgroup of $Q_2((n_{t-2}, m_{t-2}))$. Now, consider $Q_2^2((n_t, m_t), r) \oplus H$ in $Q_2((n_{t-2}, m_{t-2}))$, then $Q_2^2((n_t, m_t), r)$ is strongly residual in $Q_2((n_t, m_t))$ and so $Q_2^2((n_t, m_t), r) \oplus H$ is strongly residual in $Q_2((n_{t-2}, m_{t-2}))$ for each r .

On the other hand,

$$U(Q_1((n_{t-1}, m_{t-1}), k)) \subset U(Q((n_{t-2}, m_{t-2}))) \subset U(Q((n_{t-3}, m_{t-3}))).$$

Therefore, by the similar method with the proof of Lemma 4,

for each r there exists k with $k-1 > r_{t-1}$ such that $U(Q_1^2((n_{t-1}, m_{t-1}), k)) \subset U(Q_2^2((n_t, m_t), r) \oplus H)$. Take $r = s_t + 1$, then for $T \in (U(RQ_2^2((n_t, m_t), (r_t, s_t))))_1$

$\subset (U(Q_2^2((n_t, m_t), r_t)))_1$, there exists $T' \in (U(Q_1^2((n_{t-1}, m_{t-1}), r_{t-1})))_1$ such that $\|T - T'\|_2 < (10)^3 \delta$.

For $X' \in (U(Q_1^2((n_{t-1}, m_{t-1}), k)))_1$, take $X \in (U(Q_2^2((n_t, m_t), r) \oplus H))_1$ such that $\|X - X'\|_2 < (10)^3 \delta$, then

$$\begin{aligned} \|[T', X']\|_2 &\leq \|[T' - T, X']\|_2 + \|[X' - X, T]\|_2 \\ &+ \|[T, X]\|_2 < 2(10)^3 \delta + 2(10)^3 \delta \end{aligned}$$

, because $[T, X] = 0$.

Hence, there exists $T'' \in (U(Q_1^2((n_{t-1}, m_{t-1}), k)))'_1 \cap U(Q_1^2((n_{t-1}, m_{t-1}), r_{t-1}))$, where $(\quad)'$ is the commutant of

the W^* -algebra (), such that $\|T' - T''\|_2 \leq 4 \cdot 4(10)^3 \delta$ (cf. Lemma 4 in [4]). Hence, $\|T - T''\|_2 \leq \|T - T'\|_2 +$

$$\|T' - T''\|_2 \leq (10)^5 \delta$$

clearly, $U(Q_1^2((n_{t-1}, m_{t-1}), k)) \cap U(Q_1^2((n_{t-1}, m_{t-1}), r_{t-1})) = RQ_1^2((n_{t-1}, m_{t-1}), (r_{t-1}, k-1))$.

Take $k-1$ as s_{t-1} . The remained part is quite similar.

This completes the proof.

Remark. The proof of Lemma 5 is due to B. Vowden.

$RQ_1^2((h, k), (i, j))$ is of type $(p_1-1, p_2-1, \dots, p_k-1)$.

and $RQ_2^2((h, k), (i, j))$ is of type $(q_1-1, q_2-1, \dots, q_k-1)$.

They might contain a type 0 group as a direct summand.

$RQ_i^2((h, k), (i, j)) = D \oplus W$, where D is the center of $RQ_i^2((h, k), (i, j))$ and W is of type $(i_1, i_2, \dots, i_n)$ with

$i_u \geq 1$ for $u = 1, 2, \dots, n$.

Define the canonical residual sequence of $RQ_1^2((h, k), (i, j))$ as follows :

$$Q_1 RQ_1^2((h, k), (i, j), n) = D$$

$\oplus Q(W, n)$. Quite similarly, we define the canonical residual sequence of RQ_2^2 .

Now we shall continue this process by q_1 times.

Then we have the following situation.

$$U(\Omega_t) \overset{K\delta}{\subset} U(\Omega_{t-1}) \overset{K\delta}{\subset} U(\Omega_{t-2}) \overset{K\delta}{\subset} U(\Omega_{t-3}).$$

, where Ω_t contains a type 1-group as a direct summand

and Ω_{t-1} does not contain a type 1-group as a direct summand

: moreover $\Omega_{t-2} = \Omega_{t-1} \oplus \mathcal{R}$, where $\mathcal{R}$ is a subgroup of Ω_{t-2}

: K is a constant, which does not depend on δ .

and by the $q_1 + 1$ th process, we have

$$U(\Delta_t) \stackrel{K_1\delta}{\subset} U(\Delta_{t-1}) \stackrel{K_1\delta}{\subset} U(\Delta_{t-2})$$

, where K_1 does not depend on ϵ .

Moreover, let $\Omega_t = E \oplus H$, where E is the center of Ω_t , then

$\Delta_t = E \oplus E_1 \oplus W$, where E_1 is contained in the center of Δ_t and $E_1 = Q(L_1, n)$ for some n .

On the other hand, the center of Δ_{t-1} is same with the center C of Ω_{t-1} , because Ω_{t-1} does not contain a type 1-group as a direct summand.

Lemma 6. For $X \in (U(E_1))_1$, there exists an element $X' \in (U(C))_1$ such that $\|X - X'\|_2 < 10^2 K_1 \delta$.

Proof. Put $X_n = X$, then (x_n) is a central sequence in $U(E_1)$; it is a central sequence in $U(\Delta_{t-2})$, because

$$\Delta_{t-2} = \Delta_t \oplus \Gamma \quad \text{for some subgroup } \Gamma.$$

Let $Y' \in (U(\Delta_{t-1}))_1$ such that $\|X - Y'\|_2 < K_1 \delta$. then by the discussions in the proof of Lemma 4, $\|[Y', W']\|_2 <$

$5K_1\delta$ for all $W' \in (U(\Delta_{t-1}))_1$; hence there exists a central element X' of $(U(\Delta_{t-1}))_1$ such that $\|X' - Y'\|_2 \leq 2 \cdot 5K_1\delta$; hence $\|X - X'\|_2 \leq \|X - Y'\|_2 + \|X' - Y'\|_2 < 10^2 K_1 \delta$. This completes the proof.

Now we shall prove Theorem 1.

Proof of Theorem 1. $(U(E_1))_1 \stackrel{K_1\delta}{\subset} (U(L_1))_1 \subset (U(\Omega_t))_1 \subset (U(\Omega_{t-1}))_1$. By lemma 6, for $X \in (U(E_1))_1$, there exists an $X' \in (U(C))_1$ such that $\|X - X'\|_2 < 10^2 K_1 \delta$. For arbitrary $Y \in (U(L_1))_1$, take $Y' \in (U(\Omega_{t-1}))_1$ such that $\|Y' - Y\|_2 < K \delta$. Then,

Then,

$$\| [Y, X] \|_2 \leq \| [Y - Y', X] \|_2 + \| [Y', X - X'] \|_2 \\ + \| [Y', X'] \|_2 \leq 2K\delta + 2 \cdot 10^2 K_1 \delta.$$

Hence there exists an element $X'' \in U(L_1) \cap U(L_1)' = (\lambda 1)$, where λ are complex numbers, such that $\|X - X''\|_2 < 4(K + 10^2 K_1) \delta$.

We can choose δ as arbitrary small number; hence $U(E_1)$ must be the center of $U(L_1)$.

On the other hand, $U(E_1)$ is not the center of $U(L_1)$, because $U(E_1) = U(\sum_{j=1}^{\infty} \oplus H_j)$ with $H_j = Z$.

This is a contradiction and completes the proof,

Next, we shall show the existence of an uncountable number of II_{∞} -factors.

Let F_2 be the free group of two generators g_1, g_2 .

Let S be the set of $g \in F_2$ which, when written as a power of g_1, g_2 of minimum length, end with a g_1^n , $n = \pm 1, \pm 2, \dots$, then it is clear that $S \cap g_1 S g_1^{-1} = \emptyset$, $g_1 e g_1^{-1} = e$, and $\{g_2^n S g_2^{-n} \mid n = 0, \pm 1, \dots\}$ are disjoint subsets of $F_2 \setminus \{e\}$, where e is the unit of F_2 ; therefore $\{e\}$ is strongly residual.

Now let $R_j = F_2$ for $j = 1, 2, \dots$ and $\Gamma' = \sum_{j=1}^{\infty} \oplus R_j$.

Put $\Gamma_n = \sum_{j=1}^n \oplus R_j$, then Γ_n is strongly residual in Γ' , because Γ_n is strongly residual in Γ'_n and $\sum_{j=1}^n \oplus \{e_j\}$ is strongly residual in $\sum_{j=1}^n \oplus R_j$, where e_j is the unit of R_j ; hence $\sum_{j=1}^{\infty} \oplus R_j$ is strongly residual in Γ' .

Moreover $\Gamma_n \oplus \Gamma_{n+1}' = R_n$ and $\bigcup_{n=1}^{\infty} \Gamma_n' = \sum_{j=1}^{\infty} \oplus R_j = \Gamma'$.

; hence $\{ \bar{I}'_n \}$ is a residual sequence in $\bar{I}'$.

Put $\bar{\Phi}_n(I_1) = \bar{I}'_n \oplus Q(G[I_1], n)$ for $n = 1, 2, \dots$, then $\{ \bar{\Phi}_n(I_1) \}$ is a residual sequence in $\bar{I}' \oplus G[I_1]$.

Now, we shall show

Theorem 2. Let I_1 and I_2 be two subsets of I satisfying the conditions of Theorem 1, then $U(\bar{I}' \oplus G[I_1])$ is not *-isomorphic to $U(\bar{I}' \oplus G[I_2])$.

Proof. Since $\bar{\Phi}_n(I_1) \ominus \bar{\Phi}_n(I_1) = (\bar{I}'_n \ominus \bar{I}'_{n+1}) \oplus (Q(G[I_1], n) \ominus Q(G[I_1], n+1))$, $U(\bar{\Phi}_n(I_1) \ominus \bar{\Phi}_{n+1}(I_1))$ is a factor for $n = 1, 2, \dots$ and $i = 1, 2$.

Therefore, we can apply the lemmas of McDuff [4].

Now, suppose that $U(\bar{I}' \oplus G[I_1]) = U(\bar{I}' \oplus G[I_2])$, then we have the similar situations with Lemmas 2 and 3 for two residual sequences $\{ \bar{\Phi}_n(I_i) \}$ ($i = 1, 2$).

On the other hand, $\bar{I}'_{m,n} = \bar{I}'_m \oplus \bar{I}'_n$ $m < n$ has the strong residual subgroup $\{ e \}$; hence we have the same relation with the previous case such that $U(Q_2^2((n_t, m_t), r_t)) \stackrel{(U_0)^3 \delta}{\subset} \dots \stackrel{(U_0)^3 \delta}{\subset} U(Q_1^2((n_3, m_3), r_3)) \stackrel{(U_0)^3 \delta}{\subset} U(Q_2^2((n_2, m_2), r_2))$.

This is a contradiction and completes the proof.

Theorem 3. Suppose that I_1, I_2 satisfy the conditions of Theorem 1 and let B be a type I -factor, then $B \otimes U(\bar{I}' \oplus G[I_1])$ is not *-isomorphic to $B \otimes U(\bar{I}' \oplus G[I_2])$.

Proof. $B \otimes U(\bar{I}' \oplus G[I_1]) = B \otimes U(\bar{I}') \otimes U(G[I_1]) = B \otimes U(G[I_1]) \otimes \bigotimes_{n=1}^{\infty} U(R_n)$, where $\bigotimes_{n=1}^{\infty} U(R_n)$ is the canonical infinite tensor product of $\{ U(R_n) \}$ (cf. [6]); hence $B \otimes U(\bar{I}' \oplus G[I_1]) \otimes A$ is *-isomorphic to

$B \otimes U(\tau \oplus G[I_1])$ ($i = 1, 2$), where A is the hyperfinite II_1 -factor (cf. [6]).

We shall denote $U(\tau \oplus G[I_1])$ by N . Let φ be a normal, faithful semi-finite trace on B , and let τ_1 (resp. τ_2) be the normalized trace on N (resp. A), then $\varphi \otimes \tau_1 \otimes \tau_2$ will define a normal, faithful semi-finite trace on $B \otimes N \otimes A$. Now, let E be a minimal projection of B , then $E \otimes l_N$ is a finite projection in $B \otimes N$, where l_N is the unit of N ; moreover $(E \otimes l_N) B \otimes N (E \otimes l_N) = E \otimes N$; hence it is $*$ -isomorphic to N .

For arbitrary positive α with $\alpha \leq \varphi \otimes \tau_1 \otimes \tau_2 (E \otimes l_N \otimes l_A)$, we have a projection P in A such that $\varphi \otimes \tau_1 \otimes \tau_2 (E \otimes l_N \otimes P) = \alpha$, where l_A is the unit of A .

Now suppose that $B \otimes N \otimes A$ is $*$ -isomorphic to $B \otimes U(\tau \oplus G[I_2]) \otimes A$. then there exists a finite projection E_1 in $B \otimes N \otimes A$ such that $E_1 (B \otimes N \otimes A) E_1$ is $*$ -isomorphic to $U(\tau \oplus G[I_2]) \otimes A$.

Take $P_0 \in A$ such that $n_0 \varphi \otimes \tau_1 \otimes \tau_2 (E \otimes l_N \otimes P_0) = \varphi \otimes \tau_1 \otimes \tau_2 (E_1)$ for some positive integer n_0 .

Then, there exists a family $(E_{1,i} | i = 1, 2, \dots, n_0)$ of mutually orthogonal, equivalent projections in $B \otimes N \otimes A$ such that $E_{1,i} \sim E \otimes l_N \otimes P_0$, $E_{1,i} \leq E_1$ and

$$\sum_{i=1}^{n_0} E_{1,i} = E_1.$$

Since $E_{1,i} \sim E \otimes l_N \otimes P_0$, $E_{1,i} (B \otimes N \otimes A) E_{1,i}$ is $*$ -isomorphic to $(E \otimes l_N \otimes P_0) (B \otimes N \otimes A) (E \otimes l_N \otimes P_0)$.

On the other hand, $(E \otimes l_N \otimes P_0) (B \otimes N \otimes A) (E \otimes l_N \otimes P_0) = E \otimes N \otimes P_0 A P_0$; since $P_0 A P_0$ is $*$ -isomorphic to A ,

$E \otimes N \otimes P_0 A P_0$ is $*$ -isomorphic to $N \otimes A$.

Since $E_1(B \otimes N \otimes A)E_1$ is $*$ -isomorphic to $E_{1,i}(B \otimes N \otimes A)E_{1,i} \otimes B_{n_0}$ and so it is $*$ -isomorphic to $N \otimes A$, where B_{n_0} is the type I_{n_0} -factor.

Hence, $U(\mathcal{F} \oplus G[I_2]) \otimes A$ is $*$ -isomorphic to $N \otimes A$.

Since $U(\mathcal{F} \oplus G[I_i]) \otimes A$ is $*$ -isomorphic to $U(\mathcal{F} \oplus G[I_1])$, we have a contradiction, where $i = 1, 2$.

This completes the proof.

As a corollary, we have

Corollary 2. There exists an uncountable number of II_∞ -factors on a separable Hilbert space.

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