

Particle systems corresponding to Fleming-Viot processes with selection

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1 Introduction

Let us denote the operator L of the infinitesimal generator in $C(R^K)$ by the following:

$$L = \frac{1}{2} \sum_{i,j=1}^K x_i(\delta_{ij} - x_j) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^K b_i(x) \frac{\partial}{\partial x_i}$$

where $b_i(x) = \sum_{j=1}^K q_{ji}x_j + x_i(\sum_{j=1}^K \sigma_{ij}x_j - \sum_{k,l=1}^K \sigma_{kl}x_kx_l)$, $q_{ij} \geq 0$ for $i \neq j$ and $\sum_j q_{ij} = 0$ and $\sigma_{ij} = \sigma_{ji}$. This defines the infinitesimal generator of a Markov process on $\Delta_K = \{x = (x_1, \dots, x_K) : x_1 \geq 0, \dots, x_K \geq 0, x_1 + \dots + x_K = 1\}$, this process is called the Wright-Fisher diffusion model with selection according to Ethier and Kurtz [5]. Here x_i is a gene frequency of type i , q_{ij} is mutation intensity of $i \rightarrow j$, and σ_{ij} is selection intensity of (i,j) -type. In particular the haploid case, we assume that $\sigma_{ij} = \sigma_i + \sigma_j$.

This diffusion can be generalized as followings. Let E be a locally compact separable metric space and $\mathcal{P}(E)$ be the space of all probability measures on E . For $\mu \in \mathcal{P}(E)$ let us denote $\langle f, \mu \rangle = \int_E f d\mu$. For any $f_1, \dots, f_m \in \mathcal{D}(A)$ and $F \in C^2(R^m)$ let $\varphi(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_m, \mu \rangle) = F(\langle f, \mu \rangle)$.

$$\begin{aligned} \mathcal{L}\varphi(\mu) &= \frac{1}{2} \sum_{i,j=1}^m (\langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle) F_{z_i z_j}(\langle f, \mu \rangle) \\ &\quad + \sum_{i=1}^m \{ \langle A f_i, \mu \rangle + \langle f_i h, \mu \rangle - \langle f_i, \mu \rangle \langle h, \mu \rangle \} F_{z_i}(\langle f, \mu \rangle). \end{aligned} \tag{1}$$

Here E is the space of genetic types and A is a mutation operator in $\bar{C}(E) (\equiv$ the space of bounded continuous functions on $E)$ which is the generator for a Feller semigroup $\{T(t)\}$ on $\hat{C}(E) (\equiv$ the space of continuous functions vanishing at infinity). Here we consider of the haploid case and that $h = h(x) \in \bar{C}(E)$ is a selection intensity for type $x \in E$. According

to [5], this operator defines a generator corresponding to a Markov process $\{\mu_t\}$ on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0, \infty)$ martingale problem for $\mathcal{L}$ is well posed. This process is called the Fleming-Viot process. We denote μ^n the n -fold product of μ . The aim of this paper is to consider duality for this process in the form

$$E_\mu[\langle f, \mu_t \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle$$

for any $t \geq 0$, $n \in N$ and $f \in \bar{C}(E^n)$ with sup-norm $\|\cdot\|$. Here $f_k(t) \in \bar{C}(E^k)$ and satisfy $\sum_{k=1}^{\infty} \gamma^k \|f_k(t)\| < \infty$ for some $\gamma > 1$ and $f_n(0) = f$ and $f_k(0) = 0$ for $k \neq n$, and we construct the strongly continuous semigroup for this process.

2 Fleming-Viot processes with selection

According to Ethier and Kurtz [5], the operator (1) can be generalized as following formula.

$$\begin{aligned} \mathcal{L}\varphi(\mu) &= \frac{1}{2} \sum_{i,j=1}^m (\langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle) F_{z_i z_j}(\langle f, \mu \rangle) \\ &+ \sum_{i=1}^m (\langle A f_i, \mu \rangle + \langle B f_i, \mu^2 \rangle) F_{z_i}(\langle f, \mu \rangle) \\ &+ \sum_{i=1}^m \{ \langle (f_i \circ \pi) \sigma, \mu^2 \rangle - \langle f_i, \mu \rangle \langle \sigma, \mu^2 \rangle \} F_{z_i}(\langle f, \mu \rangle). \end{aligned} \quad (2)$$

Here B is a recombination operator defined by

$$Bf(x, y) = \alpha \int_E (f(x') - f(x)) R((x, y), dx')$$

where $\alpha \geq 0$ and $R((x, y), dx')$ is a one step transition function on $E^2 \times \mathcal{B}(E)$, and $\sigma = \sigma(x, y)$ is a bounded symmetric function on $E \times E$ which is selection parameters for types $x, y \in E$ and $(f_i \circ \pi)(x, y) = f_i(x)$. According to [5], this operator defines a generator corresponding to a Markov process on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0, \infty)$ martingale problem for $\mathcal{L}$ is well posed. This process is called the Fleming-Viot process. In the case (1) $\sigma(x, y) = h(x) + h(y)$ and $B = 0$.

3 Construction of semigroups

We consider that E is a locally compact separable metric space, and treat the case of the formula (2) and assume $\{T(t)\}$ is a Feller semigroup on $\hat{C}(E)$ with the generator A .

Denote the semigroup $T_k(t) = \overbrace{T(t) \otimes \cdots \otimes T(t)}^{k \text{ times}}$ on $\bar{C}(E^k)$ and its generator $A^{(k)}$.

We now construct the strongly continuous contraction semigroup for the diffusion. In this section we consider the operator of the form

$$\begin{aligned}\mathcal{L}\varphi(\mu) &= \frac{1}{2} \sum_{i,j=1}^m (\langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle) F_{z_i z_j}(\langle f, \mu \rangle) \\ &\quad + \sum_{i=1}^m (\langle A f_i, \mu \rangle + \langle \tilde{B} f_i, \mu^\infty \rangle) F_{z_i}(\langle f, \mu \rangle).\end{aligned}\quad (3)$$

Here $\tilde{B}$ is an operator from $\hat{C}(E)$ to $\bar{C}(E^\infty)$ with $\tilde{B}f = \sum_{l=1}^\infty B_l f$ and $B_l: \hat{C}(E) \rightarrow \hat{C}(E^l)$ a bounded operator and $\sum_{l=1}^\infty \|B_l\| \gamma^{l-1} < \infty$ for some $\gamma > 1$ and $\langle \tilde{B} f_i, \mu^\infty \rangle = \sum_{k=1}^\infty \langle B_k f_i, \mu^k \rangle$. In the formula (2) we consider $\tilde{B}f(x) = Bf(x_1, x_2) + \sigma(x_1, x_2)f(x_1) - \sigma(x_2, x_3)f(x_1)$ and in this case $\mathcal{L}$ is well defined. Let us define the space $S = \{f = (f_1, f_2, \dots) \in \sum_{k=1}^\infty \hat{C}(E^k) : \|f\|_\gamma \equiv \sum_{k=1}^\infty \gamma^k \|f_k\| < \infty\}$. Let $\mathcal{C} = \{\varphi_f(\mu) = \sum_{k=1}^\infty \langle f_k, \mu^k \rangle, f_k \in \hat{C}(E^k), \|f\|_\gamma < \infty\}$, and $\mathcal{D} = \{\varphi_f(\mu) = \sum_{k=1}^\infty \langle f_k, \mu^k \rangle \in \mathcal{C}, f_k \in \mathcal{D}(A^{(k)})\}$.

Theorem 1. Assume above and $\mathcal{L}$ of (3) defined on $\mathcal{D}$ is well defined, closable, and dissipative, then $\mathcal{L}$ with the domain $\mathcal{D}$ generates a strongly continuous contraction semigroup $T(t)$ on $\mathcal{C}(\mathcal{P}(E))$.

Proof. For $\varphi_f(\mu) = \sum_{k=1}^\infty \langle f_k, \mu^k \rangle \in \mathcal{D}$ and $\varphi_g(\mu) = \sum_{k=1}^\infty \langle g_k, \mu^k \rangle \in \mathcal{C}$, the equation $\mathcal{L}\varphi_f(\mu) = \varphi_g(\mu)$ follows from the formula

$$g_k = (\hat{\mathcal{L}}f)_k \equiv \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + (A^{(k)} - \binom{k}{2}) f_k + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1}$$

for $k \geq 1$, and $B_l^{(k)}: \hat{C}(E^k) \rightarrow \hat{C}(E^{k+l-1})$ defined by

$$B_l^{(k)} f(x_1, \dots, x_{k+l-1}) = \sum_{i=1}^k B_l f(x_1, \dots, x_{i-1}, \cdot, x_i, \dots, x_{k-1})(x_k, \dots, x_{k+l-1})$$

for $f \in \bar{C}(E^k)$, and for $i < j$

$$\Phi_{ij}^{(k)} f_k(x_1, \dots, x_{k-1}) = f_k(x_1, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_{k-1})$$

for $f \in \bar{C}(E^k)$.

For given $g = (g_1, g_2, \dots) \in S$ let us consider the equation on S

$$\lambda f_k - (\hat{\mathcal{L}}f)_k = g_k, \quad k \geq 1. \quad (4)$$

Then

$$(\lambda + \binom{k}{2} - A^{(k)}) f_k = g_k + \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1}$$

holds for $k \geq 1$. This is equivalent to the equation

$$f_k = (\lambda + \binom{k}{2} - A^{(k)})^{-1} \{g_k + \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1}\}$$

for $k \geq 1$. Put $u = (u_1, u_2, \dots)$ by

$$u_k = (\lambda + \binom{k}{2} - A^{(k)})^{-1} \{ \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1} \}.$$

Then we have

$$\|u\|_\gamma \leq \sup_k \left(\frac{\binom{k}{2} \gamma^{k-1}}{(\lambda + \binom{k-1}{2}) \gamma^k} + \sum_{l=1}^{\infty} \frac{\|B_l^{(k)}\| \gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2}) \gamma^k} \right) \|f\|_\gamma.$$

Because $\|B_l^{(k)}\| \leq k \|B_l\|$, for any $\delta > 0$ let a positive constant be $L = L(\delta) = \frac{96^2 - 106 + 4}{8\delta}$ such that $k \leq L + \delta \binom{k-1}{2}$, then

$$\begin{aligned} \frac{\binom{k}{2} \gamma^{k-1}}{(\lambda + \binom{k-1}{2}) \gamma^k} + \sum_{l=1}^{\infty} \frac{\|B_l^{(k)}\| \gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2}) \gamma^k} &\leq \frac{(L + (1 + \delta) \binom{k-1}{2}) \gamma^{k-1}}{(\lambda + \binom{k-1}{2}) \gamma^k} \\ &+ \frac{(L + \delta \binom{k-1}{2}) \sum_{l=1}^{\infty} \|B_l\| \gamma^{k+l-1}}{(\lambda + \binom{k-1}{2}) \gamma^k} \end{aligned} \quad (5)$$

for any k . Let

$$d(\gamma) = \sum_{l=1}^{\infty} \|B_l\| \gamma^{l-1},$$

and put $\delta > 0$ so that $\rho = (1 + \delta)/\gamma + \delta d(\gamma) < 1$, then we have that

$$\|u\|_\gamma \leq \rho \|f\|_\gamma$$

for $\lambda \geq L(\gamma^{-1} + d(\gamma))/\rho$. For this λ we conclude that the equation (4) have a unique solution $f \in \mathcal{D}$ satisfying that $\|f\|_\gamma \leq \frac{1}{(1-\rho)\lambda} \|g\|_\gamma$. The equation (4) implies $(\lambda - \mathcal{L})\varphi_f(\mu) = \varphi_g(\mu)$. Because $\mathcal{D}$ is dense in $C(\mathcal{P}(E))$, this implies that the operator $\mathcal{L}$ with the domain $\mathcal{D}$ generates a strongly continuous semigroup by Hille-Yoshida theory. Q.E.D.

Next we will construct a strongly continuous semigroup $\{U(t)\}$ corresponding to $\hat{\mathcal{L}}$ on Banach space S with the norm $\|\cdot\|_\gamma$. For given $h \in S$ we consider $f(t) = (f_1(t), f_2(t), \dots)$ with $f_k(t) \in \bar{C}(E^k)$ and $f(0) = h$ such that

$$\begin{aligned} \frac{d}{dt} f_k(t) &= (\hat{\mathcal{L}} f(t))_k \\ &= \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(t) \\ &\quad + (A^{(k)} - \binom{k}{2}) f_k(t) + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1}(t) \end{aligned} \quad (6)$$

for $k \geq 1$. This is equivalent to

$$\begin{aligned} f_k(t) &= e^{-\binom{k}{2}(t-u)} T_k(t-u) f_k(u) \\ &+ \int_u^t e^{-\binom{k}{2}(t-s)} T_k(t-s) \left\{ \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(s) \right. \\ &\left. + \sum_{l=1}^k B_l^{(k-l+1)} f_{k-l+1}(s) \right\} ds \end{aligned} \quad (7)$$

for $k \geq 1$ and $t > u$, and we have that

$$\begin{aligned} \|f(t)\|_\gamma &\leq \|f(u)\|_\gamma + \int_u^t \sup_k \left(\binom{k}{2} \gamma^{k-1} e^{-\binom{k-1}{2}(t-s)} / \gamma^k \right. \\ &\left. + \sum_{l=1}^\infty \|B_l^{(k)}\| \gamma^{k+l-1} e^{-\binom{k+l-1}{2}(t-s)} / \gamma^k \right) \|f(s)\|_\gamma ds, \end{aligned}$$

then

$$\begin{aligned} \|f(t)\|_\gamma &\leq \|f(u)\|_\gamma \\ &+ \int_u^t \sup_k \left[(L + (1 + \delta) \binom{k-1}{2}) \gamma^{k-1} e^{-\binom{k-1}{2}(t-s)} / \gamma^k \right. \\ &\left. + (L + \delta \binom{k-1}{2}) \sum_{l=1}^\infty \|B_l\| \gamma^{k+l-1} e^{-\binom{k+l-1}{2}(t-s)} / \gamma^k \right] \|f(s)\|_\gamma ds. \end{aligned}$$

Let $r(t) = \sup_{0 \leq s \leq t} \|f(s)\|_\gamma$, then $r(t) \leq r(u) + (L(1 + d(\gamma))(t-u) + \rho)r(t)$. Therefore by $\rho < 1$, we have

$$r(t) \leq (1 - L(1 + d(\gamma))(t-u) - \rho)^{-1} r(u).$$

Therefore

$$r(t) \leq e^{Mt} r(0) \quad \text{for } t > 0 \quad (8)$$

where $M = \frac{L(1+d(\gamma))}{1-\rho}$. By this equation $r(0) = 0$ implies $r(t) = 0$. So the equation (6) has a unique solution for $f(0) = h \in \mathcal{C}$ and implies

$$\frac{d}{dt} \varphi_{f(t)}(\mu) = \mathcal{L} \varphi_{f(t)}(\mu).$$

Therefore $f(t)$ satisfies

$$T(t) \varphi_h(\mu) = \langle f(t), \mu^\infty \rangle.$$

So we have

$$E_\mu[\langle h, \mu_t^\infty \rangle] = \sum_{k=1}^\infty \langle f_k(t), \mu^k \rangle.$$

By the inequality (8) there exists a strongly continuous semigroup $\{U(t)\}$ on S corresponding to $\hat{\mathcal{L}}$ such that

$$\|U(t)\| \leq e^{Mt}.$$

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