

A Refinement to a Theorem of Davenport and Lewis.

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The theorem of H. Davenport and D. J. Lewis cited is in the paper "Exponential Sums in Many Variables" in American Journal of Mathematics, vol. 84, 1962, pp. 649/665. Let F_p be the Galois field with p elements, where p is a large prime. Let $F(X)$ be a cubic polynomial in n variables $X = (X_1, \dots, X_n)$ with coefficients in F_p . Here we suppose $F(X)$ to be non-degenerate, i.e. that $F(X)$ cannot be transformed into a polynomial with fewer variables by any non-singular linear transformations. Let $F(X)$ be expressed as $F(X) = C(X) + Q(X) + L(X) + \text{constant term}$, where C , Q and L are cubic, quadratic and linear part respectively. We define $h = h(C)$ to be the least number for which $C(X)$ is representable identically as

$$L_1 \cdot Q_1 + \dots + L_h \cdot Q_h,$$

where $L_1, \dots$ and $Q_1, \dots$ are linear and quadratic forms respectively with coefficients in F_p . Obviously $h(C)$ is an invariant of C , and $0 \leq h(C) \leq C$, and $h(C) = 0$ if and only if C vanishes identically, and $h(C) = n$ if and only if C does not represent 0 non-trivially in F_p . Then they state, as Theorem 1 of their paper, the

[Theorem] (Davenport-Lewis) For a non-degenerate cubic polynomial $F(X)$ with coefficients in F_p , we have

$$\left| \sum_{x \in F_p^n} e\left(\frac{1}{p} F(x)\right) \right| \ll p^{n - \frac{1}{4} h(C)}.$$

Here and in the followings n is supposed to be fixed and the implied constants depend only on n . As usual, $e(x) = e^{2\pi\sqrt{-1}x}$. For the proof they use a polarization of $Q(X)$.

We show that, for cubic forms, the exponent can be diminished by $\frac{1}{4}$ if $h(C) > 0$, by using Gauss sums, i.e., we have

[Theorem 1] Let $C(X)$ be a non-degenerate cubic form with coefficients in F_p , then we have

$$\left| \sum_{x \in F_p^n} e\left(\frac{1}{p} C(x)\right) \right| \ll p^{n - \frac{1}{4}(h(C)+1)},$$

if $h(C) \geq 1$.

In the followings we suppose $C(X)$ to be non-degenerate in F_p , and $p > 3$.

(Lemma 1) Let $Q(X)$ be a quadratic form with coefficients in F_p , then we have

$$\sum_{x \in F_p^n} e\left(\frac{1}{p} Q(x)\right) = \varepsilon_p^{n-\lambda} \cdot \varepsilon_p^\lambda \times p^{\frac{n-\lambda}{2}}$$

where $n - \lambda$ is the rank Q , $\varepsilon = \pm 1$ in general, $\varepsilon = \left(\frac{\Delta}{p}\right)$ if $\lambda = 0$ with $\Delta = \det Q$, and $\varepsilon_p = 1$ or i according as $p \equiv 1$ or $3 \pmod{4}$.

Proof : Well-known.

Let $C(X)$ be expressed as $\sum_{i,j,k}^n c_{ijk} X_i X_j X_k$, where the coefficients

c_{ijk} are symmetrical in i, j, k . Define $n \times n$ matrix $\mathcal{h}(X)$ with $\sum_k c_{ijk} x_k$ as its (i, j) -th entry. The determinant $H(X)$ of $\mathcal{h}(X)$ is the Hessian of $C(X)$.

(Lemma 2) The number of points $y \in \mathbb{F}_p^n$ for which the matrix $\mathcal{h}(y)$ has the rank $n - \lambda$ in $\mathbb{F}_p$ is of an order $O(p^{2n-h(C)-\lambda})$ if $n > \lambda \geq n-h(C)+1$, 1 if $\lambda = n$, and $O(p^{n-1})$ if $n-h(C) \geq \lambda \geq 1$.

Proof : The first is a restatement of Lemma 3 in the paper of Davenport and Lewis. The case $\lambda = n$ is suggested on the page 662 between the 11th and 8th lines from below. For the last statement we use the fact that, if $H(y) = 0$ as a polynomial and $p \geq n+1$, then $C(X)$ is degenerate in $\mathbb{F}_p$.

(Lemma 3) We have

$$\sum_{x \in \mathbb{F}_p^n} \left(\frac{H(6x)}{p} \right) \cdot e\left(\frac{2}{p} C(x)\right) \ll p^{n-\frac{1}{2}}$$

if $n \geq 1$ and $C(x)$ has non-trivial coefficients. Here $\left(\frac{\star}{p}\right)$ is the Legendre symbol.

Proof : Easy.

Now we proceed to the proof of the Theorem.

We have

$$\begin{aligned} \left| \sum_{x \in \mathbb{F}_p^n} \left(\frac{1}{p} C(x) \right) \right|^2 &= \sum_{x, x} \sum e\left(\frac{1}{p} (C(x) - C(\bar{x}))\right) \\ &= \sum_{x, y} \sum e\left(\frac{1}{p} \sum_{i, j} (6 \cdot \sum_k c_{ijk} y_k) x_i x_j + \frac{2}{p} C(y)\right) \end{aligned}$$

by putting $\bar{x} = x + y$, $\bar{x} = x - y$. If $n > h(C) \geq 1$, the above sum is equal to

$$\begin{aligned}
& \epsilon_p^n \cdot \sum_{y \in F_p^n} \left(\frac{H(6y)}{p} \right) p^{\frac{n}{2}} \cdot e\left(\frac{2}{p} C(y)\right) + \sum_{\lambda=1}^{n-h(C)} 0(p^{\frac{n+\lambda}{2}}) \times 0(p^{n-1}) \\
& + \sum_{\lambda=n-h(C)+1}^{n-1} 0(p^{\frac{n+\lambda}{2}}) \times 0(p^{2n-h(C)-\lambda}) + p^n \\
& = 0(p^{\frac{n}{2}}) \times 0(p^{\frac{n-1}{2}}) + 0(p^{\frac{1}{2}n + \frac{1}{2}(n-h(C))}) \times 0(p^{n-1}) \\
& + 0(p^{\frac{n}{2} + (2n-h(C)) - \frac{1}{2}(n-h(C)+1)}) + p^n \\
& = 0(p^{2n - \frac{1}{2}(h(C)+1)}).
\end{aligned}$$

If $h(C) = n$, the sum is equal to

$$\begin{aligned}
& \epsilon_p^n \cdot \sum_{y \in F_p^n} \left(\frac{H(6y)}{p} \right) p^{\frac{n}{2}} \cdot e\left(\frac{2}{p} C(y)\right) + \sum_{\lambda=1}^{n-1} 0(p^{\frac{n+\lambda}{2}}) \times 0(p^{n-\lambda}) + p^n \\
& = 0(p^{2n - \frac{1}{2}(n+1)}).
\end{aligned}$$

And we have the stated result.