

Geometric properties of certain analytic functions with real coefficients

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Abstract

Let $\mathcal{T}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and a_k are real numbers. For a function $f(z) \in \mathcal{T}$, some sufficient conditions for starlikeness and convexity are discussed.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of the univalent functions in $\mathbb{U}$. By $\mathcal{S}^*$ and $\mathcal{K}$, we denote the subclasses of $\mathcal{A}$ whose members map $\mathbb{U}$ onto the domain which are starlike and convex.

Further, the function $f(z) \in \mathcal{A}$ is said to be starlike of order α ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U}). \quad (1.2)$$

Similarly, $f(z) \in \mathcal{A}$ is said to be convex of order α ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$1 + \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U}) \quad (1.3)$$

We shall denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively.

It is known that for $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}$ and that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$.

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Furthermore, we define $\mathcal{T}$ the class of analytic functions with real coefficients, that is,

$$\mathcal{T} := \left\{ f(z) \in \mathcal{A} : f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ a_k \in \mathbb{R} \right\}, \quad (1.4)$$

where $\mathbb{R}$ is the set of real numbers.

According to Silverman, we introduce $\mathcal{N}$ the class of analytic functions with negative coefficients, that is,

$$\mathcal{N} := \left\{ f(z) \in \mathcal{A} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0 \right\}. \quad (1.5)$$

We note that

$$\mathcal{N} \subset \mathcal{T} \subset \mathcal{A}.$$

Next, we define the Hadamard product or convolution by

$$(f * g)(z) = f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad (1.6)$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$.

With a view to introducing the Srivastava-Attiya convolution operator $\mathcal{J}_{s,b}$, we begin by recalling a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (1.7)$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C} \text{ when } |z| < 1; \ \operatorname{Re}(s) > 1 \text{ when } |z| = 1).$$

Srivastava and Attiya [3] introduced the linear operator

$$\mathcal{J}_{s,b}(f) : \mathcal{A} \longrightarrow \mathcal{A}$$

defined; in term of the Hadamard product (or convolution), by

$$\mathcal{J}_{s,b}(f)(z) := G_{s,b}(z) * f(z) \quad (z \in \mathbb{U}; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C}), \quad (1.8)$$

where for convenience,

$$G_{s,b}(z) := (1+b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}). \quad (1.9)$$

It is easy to observe from (1.1) and the definition (1.7) and (1.8) that

$$\mathcal{J}_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^s a_k z^k. \quad (1.10)$$

For $f(z) \in \mathcal{A}$, we define the class $\mathcal{S}_{s,b}^*(\alpha)$ by

$$f(z) \in \mathcal{S}_{s,b}^*(\alpha) \iff \operatorname{Re} \left(\frac{z \mathcal{J}'_{s,b}(f)(z)}{\mathcal{J}_{s,b}(f)(z)} \right) > \alpha, \quad (1.11)$$

that is, $\mathcal{J}_{s,b}(f)(z)$ is in $\mathcal{S}^*(\alpha)$ ($z \in \mathbb{U}$; $0 \leq \alpha < 1$; $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$).

Remark 1 For $f(z) \in \mathcal{A}$, we put

$$G(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n$$

is convex ($\operatorname{Re}(c) > -1$). So we have

$$\begin{aligned} \Phi_c(f(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \\ &= (f * G)(z) \\ &= z + \sum_{n=2}^{\infty} \frac{1+c}{n+c} a_n z^n \\ &= \mathcal{J}_{1,c}(f)(z). \end{aligned}$$

2 Preliminaries

We introduce the following lemmas for our results.

Lemma 1 [4] Let $f(z) \in \mathcal{T}$ and $\operatorname{Re}\{f'(z)\} > 0$, then the function

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \quad (2.1)$$

belongs to $\mathcal{K}(-c)$ for all c ($0 \leq -c < 1$).

Lemma 2 [1, Carathéodory] Let $\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{U}$ and $\operatorname{Re}\{\varphi(z)\} > 0$ ($z \in \mathbb{U}$). Then,

$$|c_n| \leq 2 \quad (n = 1, 2, 3, \dots).$$

Lemma 3 [2] Let $f(z) \in \mathcal{T}$ and suppose that

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > 0 \quad (z \in \mathbb{U}) \quad (2.2)$$

where $\alpha \geq 1$. Then, we have

$$1 + \operatorname{Re}\left\{\frac{z f''(z)}{f'(z)}\right\} > \frac{\alpha - 1}{\alpha} \quad (z \in \mathbb{U}),$$

or $f(z)$ is convex of order $\frac{\alpha - 1}{\alpha}$.

3 Main results

Theorem 1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$ and $0 \leq \alpha < 1$.

(i) If $|zf''(z) + (1 - \alpha)(f'(z) - 1)| \leq 1 - \alpha$, then $f(z) \in \mathcal{K}(\alpha)$ ($z \in \mathbb{U}$).

(ii) If $\left| f'(z) + \alpha \left(1 - \frac{f(z)}{z} \right) - 1 \right| \leq 1 - \alpha$, then $f(z) \in \mathcal{S}^*(\alpha)$ ($z \in \mathbb{U}$).

Proof. Using Lemma 3, we have (i) and (ii). □

Remark 2. From Theorem 1, we have the following results given by H. Silverman [5].

$$(i) \sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leq 1 - \alpha \implies f(z) \in \mathcal{K}(\alpha).$$

$$(ii) \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq 1 - \alpha \implies f(z) \in \mathcal{S}^*(\alpha).$$

Next, we prove the following theorem.

Theorem 2 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If

$$\operatorname{Re} \{ (1 - \alpha)f'(z) + zf''(z) \} > 0 \quad (0 \leq \alpha < 1), \quad (3.1)$$

then $|a_n| \leq \frac{2(1 - \alpha)}{n(n - \alpha)}$. The result is sharp.

Proof. The coefficient bounds are maximized at the extreme point. Now the extreme point of (3.1) may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)x^{n-1}}{n(n - \alpha)} z^n, \quad |x| = 1 \quad (3.2)$$

and the result follows. □

Remark 3 If $f(z) \in \mathcal{T}$ and $\alpha = 0$, then $|a_n| \leq \frac{2}{n^2}$. So, we have $\sum_{n=2}^{\infty} |a_n| \leq \frac{\pi^2 - 6}{3} = 1.289 \dots$. Moreover, in the case of $f(z) \in \mathcal{T}$, we have $f(z) \in \mathcal{K}(\alpha)$.

Theorem 3 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If

$$\operatorname{Re} \left\{ f'(z) - \alpha \frac{f(z)}{z} \right\} > 0 \quad (0 \leq \alpha < 1), \quad (3.3)$$

then $|a_n| \leq \frac{2(1 - \alpha)}{n - \alpha}$. The result is sharp.

Proof. The coefficient bounds are maximized at the extreme point. The extreme point of (3.3) is

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)x^{n-1}}{n-\alpha} z^n, \quad |x| = 1 \quad (3.4)$$

and the result follows. $\square$

Remark 4 In the case of $f(z) \in \mathcal{T}$, we have $f(z) \in \mathcal{S}^*(\alpha)$.

Next, in Theorem 4 below, we present the coefficient inequalities for functions in the class $\mathcal{K}(\alpha)$.

Theorem 4 Let $0 \leq \alpha < 1$. If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\sum_{n=2}^{\infty} n(n-\alpha) \left| \left(\frac{1+b}{n+b} \right)^s \right| |a_n| \leq 1-\alpha, \quad (3.5)$$

then $f(z) \in \mathcal{K}(\alpha)$.

Proof. Using Silverman's result (Remark 2 (i)), we can prove this theorem. $\square$

Letting $\alpha = 0$ in Theorem 4, we have

Corollary 1 If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\sum_{n=2}^{\infty} n^2 \left| \left(\frac{1+b}{n+b} \right)^s \right| |a_n| \leq 1, \quad (3.6)$$

then $f(z)$ is convex.

Furthermore, we can have

Theorem 5 Let $0 \leq \alpha < 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}(\alpha)$, then

$$|a_n| \leq \frac{2(1-\alpha)}{n(n-1)} \left| \left(\frac{n+b}{1+b} \right)^s \right| \cdot \prod_{j=2}^{n-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (n \in \mathbb{N} \setminus \{1\}). \quad (3.7)$$

Proof. We set

$$p(z) := \frac{1 + \frac{z \mathcal{J}_{s,b}''(f)(z)}{\mathcal{J}_{s,b}'(f)(z)}}{1-\alpha} = 1 + \sum_{n=2}^{\infty} c_n z^n.$$

Then $p(z)$ is analytic with

$$p(0) = 1 \quad \text{and} \quad \operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Since

$$z\mathcal{J}_{s,b}''(f)(z) = [(1-\alpha)(p(z)-1)]\mathcal{J}_{s,b}'(f)(z),$$

by virtue of equation

$$\mathcal{J}_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s a_n z^n, \quad (3.8)$$

we have

$$n(n-1) \left(\frac{1+b}{n+b}\right)^s a_n = (1-\alpha) \left[c_{n-1} + \sum_{m=2}^{n-1} m \left(\frac{1+b}{m+b}\right)^s a_m c_{n-m} \right] \quad (n \in \mathbb{N} \setminus \{1\}). \quad (3.9)$$

By applying Lemma 2, we obtain

$$n(n-1) \left| \left(\frac{1+b}{n+b}\right)^s \right| |a_n| \leq 2(1-\alpha) \left[1 + \sum_{m=2}^{n-1} m \left| \left(\frac{1+b}{m+b}\right)^s \right| |a_m| \right]. \quad (3.10)$$

We shall prove, by using the principle of mathematical induction, that the inequality (3.7) is satisfied for $n \in \mathbb{N} \setminus \{1\}$. $\square$

Putting $\alpha = 0$ in Theorem 5, we have

Corollary 2 If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}$, then

$$|a_n| \leq \left| \left(\frac{n+b}{1+b}\right)^s \right|.$$

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