

非零境界条件下での多成分 Fokas-Lenells 方程式の多重ソリトン公式

Multisoliton formulas for the multi-component Fokas-Lenells equation with nonzero boundary conditions

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Abstract. The multi-component Fokas-Lenells equation is considered. In particular, we present the multisoliton formulas for the system with plane-wave boundary conditions, as well as with mixed zero and plane-wave boundary conditions. A direct approach is employed to construct solutions, showing that for both boundary conditions, the multisoliton solutions have compact determinantal expressions.

1. Introduction

The Lax pair of the integrable multi-component Fokas-Lenells system is given by

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad (1.1a)$$

$$U = \begin{pmatrix} \frac{i}{2}\zeta^2 & -i\zeta\mathbf{u}_x \\ i\zeta\mathbf{v}_x^T & -\frac{i}{2}\zeta^2 I \end{pmatrix} = (u_{jk}), \quad V = \begin{pmatrix} -\frac{i}{2\zeta^2} - i\mathbf{u}\mathbf{v}^T & \frac{1}{\zeta}\mathbf{u} \\ \frac{1}{\zeta}\mathbf{v}^T & I + i\mathbf{v}^T\mathbf{u} \end{pmatrix} = (v_{jk}), \quad (1.1b)$$

where ζ is the spectral parameter, and $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$, $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are n -component row vectors.

It follows from the compatibility condition of the Lax pair that $U_t - V_x + UV - VU = O$. This yields the system of nonlinear PDEs for $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}_{xt} - \mathbf{u} + i(\mathbf{u}_x\mathbf{v}^T\mathbf{u} + \mathbf{u}\mathbf{v}^T\mathbf{u}_x) = \mathbf{0}, \quad (1.2a)$$

$$\mathbf{v}_{xt} - \mathbf{v} - i(\mathbf{v}_x\mathbf{u}^T\mathbf{v} + \mathbf{v}\mathbf{u}^T\mathbf{v}_x) = \mathbf{0}. \quad (1.2b)$$

The system (1.2) can be reduced from the first negative flow of the matrix derivative NLS hierarchy [1-4]. There arise several integrable PDEs from the reductions of the system (1.2). Specifically, if we put $v_j = \sigma_j u_j^*$, $\sigma_j = \pm 1$ ($j = 1, 2, \dots, n$), then the above system reduces to

$$u_{j,xt} = u_j - i \left\{ \left(\sum_{k=1}^n \sigma_k u_{k,x} u_k^* \right) u_j + \left(\sum_{k=1}^n \sigma_k u_k u_k^* \right) u_{j,x} \right\}, \quad (j = 1, 2, \dots, n). \quad (1.3)$$

The system of PDEs (1.3) is the basic equation that we consider here. The two special cases reducing from the system (1.3) are particularly important:

1) $n = 1$: FL equation [5, 6]

$$u_{xt} = u - 2i\sigma|u|^2u_x, \quad (u \equiv u_1, \sigma_1 = \sigma). \quad (1.4)$$

2) $n = 2$: two-component FL system [4, 7]

$$u_{1,xt} = u_1 - i \left\{ (2|u_1|^2 + \sigma|u_2|^2)u_{1,x} + i\sigma u_1 u_2^* u_{2,x} \right\}, \quad (1.5a)$$

$$u_{2,xt} = u_2 - i \left\{ (|u_1|^2 + 2\sigma|u_2|^2)u_{2,x} + i\sigma u_2 u_1^* u_{1,x} \right\}, \quad (1.5b)$$

$$(\sigma_1 = 1, \sigma_2 = \sigma).$$

The N -soliton solutions of the FL equation have been constructed for both zero and plane-wave boundary conditions [8, 9] while for the general n -component system, we have obtained the bright N -soliton solution with zero boundary conditions [10]. The purpose of the current work is to present the N -soliton formulas of the system (1.3) with the following two types of the boundary conditions:

1) Plane-wave boundary conditions

$$u_j \sim \rho_j \exp i \left(k_j x - \omega_j t + \phi_j^{(\pm)} \right), \quad x \rightarrow \pm\infty, \quad (j = 1, 2, \dots, n), \quad (1.6a)$$

with the linear dispersion relation

$$k_j \omega_j = 1 + \sum_{s=1}^n \sigma_s k_s \rho_s^2 + \sum_{s=1}^n \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, \dots, n). \quad (1.6b)$$

2) Mixed type boundary conditions

$$u_j \sim 0, \quad x \rightarrow \pm\infty, \quad (j = 1, 2, \dots, m), \quad (1.7a)$$

$$u_{m+j} \sim \rho_j \exp i \left(k_j x - \omega_j t + \phi_j^{(\pm)} \right), \quad x \rightarrow \pm\infty, \quad (j = 1, 2, \dots, n - m), \quad (1.7b)$$

with the linear dispersion relation

$$k_j \omega_j = 1 + \sum_{s=1}^{n-m} \sigma_s k_s \rho_s^2 + \sum_{s=1}^{n-m} \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, \dots, n - m). \quad (1.7c)$$

In this short note, we provide the main results only, and the details will be reported elsewhere.

2. The N -soliton formula with plane-wave boundary conditions

2.1. Bilinearization

Here, we present the multisoliton solutions of the system (1.3) with plane-wave boundary conditions (1.6). The direct approach is used to obtain solutions. To this end, we start from the following proposition:

Proposition 1. *Under the dependent variable transformations*

$$u_j = \rho_j e^{i(k_j x - \omega_j t)} \frac{g_j}{f}, \quad (j = 1, 2, \dots, n), \quad (2.1)$$

the multi-component FL system (1.3) can be decoupled into the system of equations

$$D_t f \cdot f^* = i \sum_{s=1}^n \sigma_s \rho_s^2 (g_s g_s^* - f f^*), \quad (2.2a)$$

$$D_x D_t f \cdot f^* - i \sum_{s=1}^n \sigma_s \rho_s^2 D_x g_s \cdot g_s^* + i \sum_{s=1}^n \sigma_s \rho_s^2 D_x f \cdot f^* + 2 \sum_{s=1}^n \sigma_s k_s \rho_s^2 (g_s g_s^* - f f^*) = 0, \quad (2.2b)$$

$$\begin{aligned} f^* \left[g_{j,xt} f - (f_x - i k_j f) g_{j,t} - i \frac{1}{k_j} \left(1 + \sum_{s=1}^n \sigma_s k_s \rho_s^2 \right) D_x g_j \cdot f \right] \\ = f_t^* (g_{j,x} f - g_j f_x + i k_j g_j f), \quad (j = 1, 2, \dots, n), \end{aligned} \quad (2.2c)$$

where $f = f(x, t)$ and $g_j = g_j(x, t)$ are the complexed-valued functions of x and t , and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}$$

with m and n being nonnegative integers.

Remark 1.

1) We can decouple the last equation into a system of bilinear equations

$$g_{j,xt} f - (f_x - i k_j f) g_{j,t} - i \frac{1}{k_j} \left(1 + \sum_{s=1}^n \sigma_s k_s \rho_s^2 \right) D_x g_j \cdot f = h_j f_t^*, \quad (2.3a)$$

$$g_{j,x} f - g_j f_x + i k_j f g_j = h_j f^*, \quad (2.3b)$$

where $h_j = h_j(x, t)$ are the complexed-valued functions of x and t .

2) If we introduce the variables $q_j = u_{j,x}$, then

$$q_j = \left(\rho_j e^{i(k_j x - \hat{\omega}_j t)} \frac{g_j}{f} \right)_x = \rho_j e^{i(k_j x - \hat{\omega}_j t)} \frac{h_j f^*}{f^2}, \quad \hat{\omega}_j = k_j^2 + 2 \sum_{s=1}^n \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, \dots, n), \quad (2.4)$$

solve the n -component derivative NLS system

$$i q_{j,t} + q_{j,xx} + 2i \left[\left(\sum_{s=1}^n \sigma_s |q_s|^2 \right) q_j \right]_x = 0, \quad (j = 1, 2, \dots, n). \quad (2.5)$$

2.2. N -soliton solution

Theorem 1. *The N -soliton solution of the system of bilinear equations (2.2) is given in terms of the following determinants.*

$$f = |D|, \quad g_s = |G_s|, \quad (s = 1, 2, \dots, n), \quad (2.6a)$$

$$D = (d_{jk})_{1 \leq j, k \leq N}, \quad d_{jk} = \delta_{jk} - \frac{i p_j}{p_j + p_k^*} z_j z_k^*, \quad (2.6b)$$

$$G_s = (g_{jk}^{(s)})_{1 \leq j, k \leq N}, \quad g_{jk}^{(s)} = \delta_{jk} - \frac{i p_k^*}{p_j + p_k^*} \frac{p_j - i k_s}{p_k^* + i k_s} z_j z_k^*, \quad (2.6c)$$

$$z_j = \exp \left[p_j x + \frac{1}{p_j} \left(1 + \sum_{s=1}^n \sigma_s k_s \rho_s^2 \right) t + \zeta_{j0} \right], \quad (j = 1, 2, \dots, N). \quad (2.6d)$$

Here, p_j and ζ_{j0} ($j = 1, 2, \dots, N$) are arbitrary complex parameters. The former parameters are imposed on N constraints

$$\sum_{s=1}^n \sigma_s (k_s \rho_s)^2 \frac{i(p_j - p_j^*) + k_s}{(p_j - i k_s)(p_j^* + i k_s)} = -1, \quad (j = 1, 2, \dots, N). \quad (2.7)$$

The expressions (2.1) with the tau-functions (2.6) give the dark soliton solutions with plane-wave boundary conditions. The analysis of the one-component system (i.e., FL equation) has been performed in [9] where the detailed description of the dark soliton solutions has been given.

Remark 2.

1) The proof of the N -soliton solution can be done by means of an elementary calculation using the basic formulas of determinants, i.e.,

$$\frac{\partial}{\partial x} |D| = \sum_{j,k=1}^N \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (D_{jk} : \text{cofactor of } d_{jk}),$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{j,k=1}^N D_{jk}a_jb_k,$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||D| = |D(\mathbf{a}; \mathbf{c})||D(\mathbf{b}; \mathbf{d})| - |D(\mathbf{a}; \mathbf{d})||D(\mathbf{b}; \mathbf{c})|, \quad (\text{Jacobi's identity}),$$

with the notation

$$\begin{vmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{vmatrix} = |D(\mathbf{a}; \mathbf{b})|, \quad \begin{vmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{vmatrix} = |D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})|.$$

2) The tau-functions h_s are given by

$$h_s = ik_s |H_s|, \quad H_s = (h_{jk}^{(s)})_{1 \leq j,k \leq N}, \quad h_{jk}^{(s)} = \delta_{jk} + \frac{ip_j}{p_j + p_k^* p_k^* + ik_s} z_j z_k^*.$$

2.3. Derivation of constraints (2.7)

In the case of plane-wave boundary conditions, the n constraints must be imposed among the complex parameters p_j ($j = 1, 2, \dots, N$). We derive these constraints from the Lax pair (1.1) of the system. The spatial part of the Lax pair with seed solutions

$$u_j = \rho_j e^{i\theta_j}, \quad \theta_j = k_j x - \omega_j t, \quad (j = 1, 2, \dots, n),$$

are given by

$$\Psi_x = U\Psi, \quad U = \begin{pmatrix} \frac{i}{2}\zeta^2 & k_1\rho_1\zeta e^{i\theta_1} & \cdots & k_n\rho_n\zeta e^{i\theta_n} \\ \sigma_1 k_1 \rho_1 \zeta e^{-i\theta_1} & -\frac{i}{2}\zeta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n \rho_n \zeta e^{-i\theta_n} & 0 & \cdots & -\frac{i}{2}\zeta^2 \end{pmatrix}. \quad (2.8)$$

Introduce a new wavefunction Ψ_0 by $\Psi = P\Psi_0$, where P is a diagonal matrix $P = \text{diag}(1, e^{i\theta_1}, \dots, e^{i\theta_n})$. Then, Ψ_0 satisfies the matrix equation

$$\Psi_{0,x} = (P_x P^{-1} + P U P^{-1})\Psi_0 \equiv U_0 \Psi_0, \quad U_0 = \begin{pmatrix} \frac{i}{2}\zeta^2 & k_1\rho_1\zeta & \cdots & k_n\rho_n\zeta \\ \sigma_1 k_1 \rho_1 \zeta & ik_1 - \frac{i}{2}\zeta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n \rho_n \zeta & 0 & \cdots & ik_n - \frac{i}{2}\zeta^2 \end{pmatrix}. \quad (2.9)$$

The characteristic equation of U_0 reads $|U_0 - I_{n+1}\mu| = 0$, i.e.,

$$\begin{vmatrix} \frac{i}{2}\zeta^2 - \mu & k_1\rho_1\zeta & \cdots & k_n\rho_n\zeta \\ \sigma_1 k_1 \rho_1 \zeta & ik_1 - \frac{i}{2}\zeta^2 - \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n \rho_n \zeta & 0 & \cdots & ik_n - \frac{i}{2}\zeta^2 - \mu \end{vmatrix} = 0. \quad (2.10)$$

Expanding the above determinant in μ yields

$$\frac{i}{2}\zeta^2 - \mu = -\zeta^2 \sum_{s=1}^n \frac{\sigma_s(k_s \rho_s)^2}{\mu + \frac{i}{2}\zeta^2 - ik_s}. \quad (2.11)$$

Let $\mu + \frac{i}{2}\zeta^2 = p$ and assume ζ^2 be real and p be complex. Then

$$i\zeta^2 - p = -\zeta^2 \sum_{s=1}^n \frac{\sigma_s(k_s \rho_s)^2}{p - ik_s}, \quad -i\zeta^2 - p^* = -\zeta^2 \sum_{s=1}^n \frac{\sigma_s(k_s \rho_s)^2}{p^* + ik_s}. \quad (2.12)$$

It follows from the above two relations that

$$\sum_{s=1}^n \sigma_s(k_s \rho_s)^2 \frac{i(p - p^*) + k_s}{(p - ik_s)(p^* + ik_s)} = -1, \quad (2.13)$$

which yields (2.7) upon putting $p = p_j$.

3. The N -soliton formula with mixed type boundary conditions

3.1. Bilinearization

The bilinearization of the system (1.3) with mixed type boundary conditions (1.7) can be performed by the following proposition.

Proposition 2. *Under the dependent variable transformations*

$$u_j = e^{-i\hat{\lambda}t} \frac{h_j}{f}, \quad \left(j = 1, 2, \dots, m, \quad \hat{\lambda} = \sum_{s=1}^{n-m} \sigma_{m+s} \rho_s^2 \right), \quad (3.1a)$$

$$u_{m+j} = \rho_j e^{i(k_j x - \omega_j t)} \frac{g_j}{f}, \quad (j = 1, 2, \dots, n - m), \quad (3.1b)$$

the multi-component FL system (1.3) can be decoupled into the system of equations

$$D_t f \cdot f^* = i \sum_{s=1}^m \sigma_s h_s h_s^* + i \sum_{s=1}^{n-m} \sigma_{m+s} \rho_s^2 (g_s g_s^* - f f^*), \quad (3.2a)$$

$$\begin{aligned} D_x D_t f \cdot f^* - i \sum_{s=1}^m \sigma_s D_x h_s \cdot h_s^* - i \sum_{s=1}^{n-m} \sigma_s \rho_s^2 D_x g_s \cdot g_s^* + i \sum_{s=1}^{n-m} \sigma_{m+s} \rho_s^2 D_x f \cdot f^* \\ + 2 \sum_{s=1}^{n-m} \sigma_s k_s \rho_s^2 (g_s g_s^* - f f^*) = 0, \end{aligned} \quad (3.2b)$$

$$f^*(h_{j,xx}f - h_{j,t}f_x - \lambda h_j f) = f_t^*(h_{j,x}f - h_j f_x), \quad (j = 1, 2, \dots, m), \quad (3.2c)$$

$$f^* \left\{ g_{j,xt} f - (f_x - i k_j f) g_{j,t} - \frac{i\lambda}{k_j} D_x g_j \cdot f \right\} = f_t^* (g_{j,x} f - g_j f_x + i k_j g_j f), \quad (j = 1, 2, \dots, n-m), \quad (3.2d)$$

where $\lambda = 1 + \sum_{s=1}^{n-m} \sigma_s k_s \rho_s^2$.

3.2. N -soliton solution

Theorem 2. *The N -soliton solution of the system of bilinear equations (3.2) is given in terms of the following determinants.*

$$f = |D|, \quad D = (d_{jk})_{1 \leq j, k \leq N}, \quad d_{jk} = \frac{z_j z_k^* - i p_k^* c_{jk}}{p_j + p_k^*}, \quad z_j = \exp \left(p_j x + \frac{\lambda}{p_j} t \right), \quad (3.3a)$$

$$h_j = -\frac{1}{\lambda} |D(\mathbf{a}_j^*; \mathbf{z}_t)|, \quad (j = 1, 2, \dots, m), \quad (3.3b)$$

$$g_j = |D| + \frac{i}{\lambda} |D(\mathbf{z}_j^*; \mathbf{z}_t)|, \quad (j = 1, 2, \dots, n-m), \quad (3.3c)$$

$$\mathbf{z} = (z_1, z_2, \dots, z_N), \quad \mathbf{z}_t = \left(\frac{\lambda}{p_1} z_1, \frac{\lambda}{p_2} z_2, \dots, \frac{\lambda}{p_N} z_N \right), \quad (3.3d)$$

$$\mathbf{a}_j = (\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jN}), \quad (j = 1, 2, \dots, m), \quad (3.3e)$$

$$c_{jk} = \frac{\sum_{s=1}^m \sigma_s \alpha_{sj} \alpha_{sk}^*}{1 + \sum_{s=1}^{n-m} \sigma_s (k_s \rho_s)^2 \frac{i(p_j - p_k^*) + k_s}{(p_j - i k_s)(p_k^* + i k_s)}}, \quad (j, k = 1, 2, \dots, N), \quad (3.3f)$$

where p_j ($j = 1, 2, \dots, N$) and α_{jk} ($j = 1, 2, \dots, m; k = 1, 2, \dots, N$) are arbitrary complex parameters.

The components from (3.1a) take the form of the bright solitons with zero background whereas those of (3.1b) represent the dark solitons with plane-wave background. The properties of the bright soliton solutions of the FL equation have been explored in detail in [8]. It should be remarked that unlike purely plane-wave boundary conditions, no constraints are imposed on the parameters p_j . Consequently, the analysis of solutions becomes more easier than that of solutions for plane-wave boundary conditions.

Remark 3.

1) When compared with the soliton solutions with the pure plane-wave boundary conditions, the parameters p_j can be chosen arbitrary. Consequently, the explicit form of the N -soliton solution is available without solving algebraic equations like (2.7).

2) If we put $\rho_j = 0$, ($j = 1, 2, \dots, n-m$), then (3.1a) and (3.3) yield the bright N -soliton solution of the system (1.3) with the zero boundary conditions $u_j \rightarrow 0, |x| \rightarrow \infty$ [10].

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