

Fine limits of logarithmic potentials

Yoshihiro MIZUTA

(水田 義弘)

(Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Hiroshima 730, Japan)

1. Statement of results

Let R^n ($n \geq 2$) be the n -dimensional euclidean space. For a nonnegative (Radon) measure μ on R^n , we set

$$L\mu(x) = \int \log(1/|x-y|) d\mu(y)$$

if the integral exists at x . We note here that $L\mu$ is not identically $-\infty$ if and only if

$$(1) \quad \int \log(1+|y|) d\mu(y) < \infty.$$

Denote by $B(x,r)$ the open ball with center at x and radius r .

For $E \subset B(0,2)$, define

$$C(E) = \inf \mu(R^n),$$

where the infimum is taken over all nonnegative measures μ on R^n such that S_μ (the support of μ) $\subset B(0,4)$ and

$$\int \log(8/|x-y|) d\mu(y) \geq 1 \quad \text{for every } x \in E.$$

If $E \subset B(x^0,2)$, then we set

$$C(E) = C(\{x-x^0; x \in E\}).$$

We note here that this is well defined.

Fuglede [2] discussed fine differentiability properties of logarithmic potentials in the plane. To state his result, we let

$$L(x) = \log (1/|x|)$$

and set for a nonnegative integer m ,

$$L_m(x,y) = L(x-y) - \sum_{|\lambda| \leq m} (\lambda!)^{-1} (x-x^0)^\lambda \left[\left(\frac{\partial}{\partial x} \right)^\lambda L \right] (x^0-y),$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a multi-index with length $|\lambda| = \lambda_1 +$

$\dots + \lambda_n$, $\lambda! = \lambda_1! \dots \lambda_n!$, $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$ and $(\partial/\partial x)^\lambda = (\partial/\partial x_1)^{\lambda_1}$
 $\dots (\partial/\partial x_n)^{\lambda_n}$.

Theorem 1 (Fuglede [2; Notes 3]). Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |x^0-y|^{-1} \log (2+|x^0-y|^{-1}) d\mu(y) < \infty,$$

then there exists a set E in R^n which is logarithmically thin at x^0 and satisfies

$$\lim_{x \rightarrow x^0, x \in R^n - E} |x-x^0|^{-1} \int L_1(x,y) d\mu(y) = 0.$$

Here a set E in R^n is called logarithmically thin at x^0 if

$$\sum_{j=1}^{\infty} jC(E_j^!) < \infty,$$

where $E_j^! = \{x \in B(x^0, 2) - B(x^0, 1); x^0 + 2^{-j}(x - x^0) \in E\}$. For a proof of Theorem 1, see also Davie and Øksendal [1; Theorem 6]. Our main aim in this paper is to establish the following two theorems.

Theorem 2. Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |x^0 - y|^{-m} d\mu(y) < \infty$$

for a positive integer m smaller than n . Then there exists a set E in R^n such that

$$\lim_{x \rightarrow x^0} \lim_{x \in R^n - E} |x - x^0|^{-m} \int L_m(x, y) d\mu(y) = 0$$

and

$$\sum_{j=1}^{\infty} C(E_j^!) < \infty.$$

Theorem 3. Let μ be a nonnegative measure on R^n satisfying (1) and the following two conditions:

$$(a) \quad \lim_{r \downarrow 0} r^{-n} |\mu - a\Lambda_n|(B(x^0, r)) = 0 \quad \text{for some } a,$$

where Λ_n denotes the n -dimensional Lebesgue measure;

$$(b) \quad A_\lambda = \lim_{r \downarrow 0} \int_{R^n - B(x^0, r)} \left[\left(\frac{\partial}{\partial x} \right)^{\lambda_L} \right] (x^0 - y) d\mu(y)$$

exists and is finite for any λ with length n .

Then there exists a set E in R^n such that

$$(i) \quad \lim_{x \rightarrow x^0} \lim_{x \in R^n - E} |x - x^0|^{-n} \left\{ \int L_{n-1}(x, y) d\mu(y) - \sum_{|\lambda|=n} (\lambda!)^{-1} C_\lambda (x - x^0)^\lambda \right\} = 0$$

and

$$(ii) \quad \lim_{j \rightarrow \infty} C(E_j) = 0,$$

where $C_\lambda = A_\lambda + aB_\lambda$ for $|\lambda| = n$ and B_λ will be defined later (in Lemma 2).

One may compare these theorems with fine and semi-fine differentiabilitys of Riesz potentials investigated by Mizuta [3] and [4].

Remark. Set $E = \{x; \int |x-y|^{-m} d\mu(y) = \infty\}$ for a nonnegative measure μ on R^n satisfying (1). Then $C_{n-m}(E) = 0$, where C_α denotes the Riesz capacity of order α . Further we note that (a) and (b) in theorem 3 hold for almost every x^0 (cf. [5; Chap. III, 4.1]).

2. Proof of Theorem 2

In this section we prove the following generalization of Theorems 1 and 2.

Theorem 2'. Let h and k be positive and nonincreasing functions on the interval $(0, \infty)$ such that

- (a) $rh(r)$ is nondecreasing on $(0, \infty)$ and $\lim_{r \downarrow 0} rh(r) = 0$;
 (b) $k(r) \leq \text{const. } k(2r)$ for $r > 0$.

Let μ be a nonnegative measure on $\mathbb{R}^n$ satisfying (1) and

$$\int |x^0 - y|^{-m} H(|x^0 - y|) d\mu(y) < \infty,$$

for a positive integer m , where $H(0) = \infty$ and $H(r) = h(r)k(r)$ for $r > 0$. Then there exists a set E in $\mathbb{R}^n$ such that

$$(i) \quad \lim_{x \rightarrow x^0, x \in \mathbb{R}^n - E} |x - x^0|^{-m} h(|x - x^0|) \int L_m(x, y) d\mu(y) = 0;$$

$$(ii) \quad \sum_{j=1}^{\infty} k(2^{-j}) C(E_j^!) < \infty.$$

Proof. Without loss of generality, we may assume that $x^0 = 0$. Let μ be a nonnegative measure on $\mathbb{R}^n$ satisfying (1) and

$$\int |y|^{-m} H(|y|) d\mu(y) < \infty.$$

We write

$$\begin{aligned} \int L_m(x, y) d\mu(y) &= \int_{\mathbb{R}^n - B(0, 2|x|)} L_m(x, y) d\mu(y) \\ &+ \int_{B(0, 2|x|) - B(x, |x|/2)} L_m(x, y) d\mu(y) \\ &+ \int_{B(x, |x|/2)} L_m(x, y) d\mu(y) \end{aligned}$$

$$= u_1(x) + u_2(x) + u_3(x).$$

If $y \in \mathbb{R}^n - B(0, 2|x|)$, then we have by elementary calculations

$$|L_m(x, y)| \leq \text{const. } |x|^{m+1} |y|^{-m-1},$$

so that Lebesgue's dominated convergence theorem gives

$$\begin{aligned} & \limsup_{x \rightarrow 0} |x|^{-m} h(|x|) |u_1(x)| \\ & \leq \text{const. } \limsup_{x \rightarrow 0} |x| h(|x|) \int_{\mathbb{R}^n - B(0, 2|x|)} |y|^{-m-1} d\mu(y) \\ & = \text{const. } \limsup_{x \rightarrow 0} |x| h(|x|) \int_{B(0, 1) - B(0, 2|x|)} |y|^{-m-1} d\mu(y) = 0 \end{aligned}$$

since $\lim_{r \downarrow 0} rh(r) = 0$ and $rh(r) \leq k(1)^{-1} sh(s)k(s)$ whenever $0 < r < s < 1$.

If $y \in B(0, 2|x|)$ and $|x-y| \geq |x|/2 > 0$, then

$$|L_m(x, y)| \leq \text{const. } |x|^m |y|^{-m}.$$

Hence we obtain

$$\begin{aligned} & \limsup_{x \rightarrow 0} |x|^{-m} h(|x|) |u_2(x)| \\ & \leq \text{const. } \limsup_{x \rightarrow 0} h(|x|) \int_{B(0, 2|x|)} |y|^{-m} d\mu(y) = 0 \end{aligned}$$

since $h(r) \leq h(s) \leq 2h(2s) \leq 2k(1)^{-1} h(2s)k(2s)$ whenever $0 < s < r < 1/2$.

As to u_3 , we note that

$$\begin{aligned} & |x|^{-m} h(|x|) |u_3(x)| \\ & \leq \text{const. } |x|^{-m} h(|x|) \int_{B(x, |x|/2)} \log(|x|/|x-y|) d\mu(y) \\ & + \text{const. } \int_{B(x, |x|/2)} |y|^{-m} h(|y|) d\mu(y). \end{aligned}$$

The second term of the right hand side tends to zero as $x \rightarrow 0$ by the assumption. What remains is to prove that the first term of the right hand side tends to zero as $x \rightarrow 0$, $x \in \mathbb{R}^n - E$, where E is a set in $\mathbb{R}^n$ satisfying property (ii). To prove this, take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j \rightarrow \infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{B_j} |y|^{-m} h(|y|) d\mu(y) < \infty,$$

where $B_j = B(0, 2^{-j+2}) - B(0, 2^{-j-1})$. Consider the sets

$$E_j = \left\{ x \in A_j; \int_{B_j} \log(2^{-j+3}/|x-y|) d\mu(y) \geq 2^{-mj} h(2^{-j})^{-1} a_j^{-1} \right\}$$

for $j = 1, 2, \dots$, and $E = \bigcup_{j=1}^{\infty} E_j$, where $A_j = B(0, 2^{-j+1}) - B(0,$

$2^{-j})$. By the assumption on h , one sees easily that

$$k(2^{-j})C(E'_j) \leq a_j 2^{mj} H(2^{-j}) \mu(B_j) \leq \text{const. } a_j \int_{B_j} |y|^{-m} H(|y|) d\mu(y).$$

Hence E satisfies property (ii). Further,

$$\begin{aligned} & \limsup_{x \rightarrow 0, x \in \mathbb{R}^n - E} |x|^{-m} h(|x|) \int_{B(x, |x|/2)} \log(|x|/|x-y|) d\mu(y) \\ & \leq \text{const.} \limsup_{j \rightarrow \infty} \sup_{x \in A_j - E_j} 2^{mj} h(2^{-j}) \int_{B_j} \log(2^{-j+3}/|x-y|) d\mu(y) \\ & \leq \text{const.} \limsup_{j \rightarrow \infty} a_j^{-1} = 0, \end{aligned}$$

$$\text{and hence } \lim_{x \rightarrow 0, x \in \mathbb{R}^n - E} |x|^{-m} h(|x|) \int_{B(x, |x|/2)} \log(|x|/|x-y|) d\mu(y) = 0.$$

Thus the proof is complete.

Remark 1. Theorem 2' is best possible as to the size of the exceptional set. In fact, if E is a set in $\mathbb{R}^n$ satisfying property (ii), then one can find a nonnegative measure μ on $\mathbb{R}^n$ with compact support such that

$$\int |x^0 - y|^{-m} H(|x^0 - y|) d\mu(y) < \infty$$

and

$$\lim_{x \rightarrow x^0, x \in E} |x - x^0|^{-m} h(|x - x^0|) \int L_m(x, y) d\mu(y) = \infty.$$

Remark 2. Let μ be a nonnegative measure on R^n satisfying

(1) and $\int |x^0 - y|^{-m} h(|x^0 - y|) d\mu(y) < \infty$. If in addition there exist $M, r_0 > 0$ such that

$$h(|x - x^0|) \mu(B(x, r)) \leq Mr^m$$

for any $x \in B(x^0, r_0)$ and any $r, 0 < r < |x - x^0|/2$, then E appeared in Theorem 2' can be taken to be an empty set and $L\mu$ is m times differentiable at x^0 .

To prove this, assume that $x^0 = 0$. For the first assertion, in view of the proof of Theorem 2', it suffices to show that

$$(2) \quad \lim_{x \rightarrow 0} |x|^{-m} h(|x|) \int_{B(x, |x|/2)} \log(|x|/|x-y|) d\mu(y) = 0.$$

For $\delta > 0$, set $\varepsilon(\delta) = \sup_{0 < r \leq \delta} r^{-m} h(r) \mu(B(0, r))$. If $0 < \delta < |x|/2$,

then

$$\begin{aligned} & |x|^{-m} h(|x|) \int_{B(x, |x|/2)} \log(|x|/|x-y|) d\mu(y) \\ &= |x|^{-m} h(|x|) \int_{B(x, \delta)} \log(|x|/|x-y|) d\mu(y) \\ &+ |x|^{-m} h(|x|) \int_{B(x, |x|/2) - B(x, \delta)} \log(|x|/|x-y|) d\mu(y) \\ &\leq \text{const.} \left\{ (\delta/|x|)^m \log(|x|/\delta) + |x|^{-m} h(|x|) \mu(B(0, 2|x|)) \log(|x|/\delta) \right\} \end{aligned}$$

$$\leq \text{const.} \{ (\delta/|x|)^m + \varepsilon(2|x|) \} \log (|x|/\delta).$$

Since $\lim_{x \rightarrow 0} \varepsilon(2|x|) = 0$, for x sufficiently close to 0 we can

choose $\delta > 0$ so that

$$\log (|x|/\delta) = [\varepsilon(2|x|) + |x|]^{-1/2}.$$

Since $\lim_{x \rightarrow 0} (\delta/|x|) = 0$, we derive (2).

To prove the second assertion, we first note that

$$\int |x-y|^{-m+1} d\mu(y) < \infty \quad \text{for every } x \in B(0, r_0),$$

and hence $L\mu$ is $m - 1$ times differentiable at $x \in B(0, r_0)$ and

$$(\partial/\partial x)^\lambda L\mu(x) = \int [(\partial/\partial x)^\lambda L](x - y) d\mu(y)$$

for any $x \in B(0, r_0)$ and any multi-index λ with length $m - 1$. As in the proof of Theorem 2', we can prove that

$$\lim_{x \rightarrow 0} |x|^{-1} h(|x|) \left\{ u_\lambda(x) - u_\lambda(0) - \sum_{i=1}^n a_i x_i \right\} = 0,$$

where $x = (x_1, \dots, x_n)$, $u_\lambda = (\partial/\partial x)^\lambda L\mu$ for a multi-index λ with

length $m - 1$ and $a_i = \int [(\partial/\partial x_i)(\partial/\partial x)^\lambda L](-y) d\mu(y)$. This implies that L_μ is m times differentiable at 0.

3. Proof of Theorem 3

We first recall the following results.

Lemma 1 (cf. [4; Lemma 1]). Let μ be a nonnegative measure on $\mathbb{R}^n$ such that $\lim_{r \downarrow 0} r^{\alpha-n} \mu(B(0, r)) = 0$ for some real number α .

Then the following statements hold:

$$(i) \quad \text{If } \beta < 0, \text{ then } \lim_{r \downarrow 0} r^\beta \int_{B(0, r)} |y|^{\alpha-\beta-n} d\mu(y) = 0;$$

$$(ii) \quad \text{If } n - \alpha + 1 > 0 \text{ and } \beta > 0, \text{ then}$$

$$\lim_{r \downarrow 0} r^\beta \int_{B(0, 1)} (r + |y|)^{\alpha-\beta-n} d\mu(y) = 0.$$

Lemma 2 (cf. [4; Lemma 4]). Set $u(x) = \int_{B(x^0, 1)} L(x - y) dy$.

Then $u \in C^\infty(B(x^0, 1))$. Moreover, if λ is a multi-index with length n , then

$$B_\lambda \equiv [(\partial/\partial x)^\lambda u](x^0) = \int_{\partial B(0, 1)} y^{\lambda'} [(\partial/\partial x)^{\lambda''} L](y) dS(y),$$

where $\lambda = \lambda' + \lambda''$ and $|\lambda'| = 1$.

Now we prove Theorem 3 by assuming that $x^0 = 0$. Let μ be a nonnegative measure on R^n satisfying (1), (a) and (b) with $x^0 = 0$, and set $\nu = \mu - a\Lambda_n$. For $x \in B(0, 1/2) - \{0\}$, we write

$$\begin{aligned}
& |x|^{-n} \left\{ \int L_{n-1}(x, y) d\mu(y) - \sum_{|\lambda|=n} (\lambda!)^{-1} C_\lambda x^\lambda \right\} \\
&= |x|^{-n} \int_{R^n - B(0, 1)} L_n(x, y) d\mu(y) \\
&+ |x|^{-n} \int_{B(0, 1) - B(0, 2|x|)} L_n(x, y) d\nu(y) \\
&- |x|^{-n} \sum_{0 < |\lambda| \leq n} (\lambda!)^{-1} x^\lambda \lim_{r \rightarrow 0} \int_{B(0, 2|x|) - B(0, r)} [(\partial/\partial x)^\lambda L](-y) d\nu(y) \\
&+ a|x|^{-n} \left\{ \lim_{r \rightarrow 0} \int_{B(0, 1) - B(0, r)} L_n(x, y) dy - \sum_{|\lambda|=n} (\lambda!)^{-1} B_\lambda x^\lambda \right\} \\
&+ |x|^{-n} \int_{B(0, 2|x|) - B(x, |x|/2)} L_0(x, y) d\nu(y) \\
&+ |x|^{-n} \int_{B(x, |x|/2)} L_0(x, y) d\nu(y) \\
&= u_1(x) + u_2(x) - u_3(x) + au_4(x) + u_5(x) + u_6(x).
\end{aligned}$$

If $y \in R^n - B(0, 2|x|)$, then $|L_n(x, y)| \leq \text{const. } |x|^{n+1} |y|^{-n-1}$

and hence

$$\lim_{x \rightarrow 0} u_1(x) = 0.$$

For simplicity, set $\tau = |\nu|$. Then $\lim_{r \downarrow 0} r^{-n} \tau(B(0, r)) = 0$ by (a),

and we have

$$\limsup_{x \rightarrow 0} |u_2(x)| \leq \text{const.} \limsup_{x \rightarrow 0} |x| \int_{B(0,1)} (|x| + |y|)^{-n-1} d\tau(y) = 0$$

because of Lemma 1, (ii).

If $0 < |\lambda| < n$, then Lemma 1, (i) yields

$$\begin{aligned} \limsup_{x \rightarrow 0} |x|^{|\lambda|-n} \int_{B(0,2|x|)} |[(\partial/\partial x)^\lambda L](-y)| d\tau(y) \\ \leq \text{const.} \limsup_{x \rightarrow 0} |x|^{|\lambda|-n} \int_{B(0,2|x|)} |y|^{-|\lambda|} d\tau(y) = 0. \end{aligned}$$

If $|\lambda| = n$, then $\int_{B(0,r)-B(0,s)} [(\partial/\partial x)^\lambda L](-y) dy = 0$ for any

$r, s > 0$. Hence by the definition of A_λ ,

$$\lim_{x \rightarrow 0} \left\{ \lim_{r \downarrow 0} \int_{B(0,2|x|)-B(0,r)} [(\partial/\partial x)^\lambda L](-y) d\nu(y) \right\} = 0.$$

Therefore $\lim_{x \rightarrow 0} u_3(x) = 0$.

Since $u(x) \equiv \int_{B(0,1)} L(x-y) dy \in C^\infty(B(0,1))$ and

$$u_4(x) = |x|^{-n} \left\{ u(x) - \sum_{|\lambda| \leq n} (\lambda!)^{-1} x^\lambda [(\partial/\partial x)^\lambda u](0) \right\}$$

in view of Lemma 2, we see that $\lim_{x \rightarrow 0} u_4(x) = 0$.

As to u_5 , we obtain

$$\begin{aligned} |u_5(x)| &\leq \text{const. } |x|^{-n} \int_{B(0, 2|x|)} \log(2 + |x|/|y|) d\tau(y) \\ &\leq \text{const. } |x|^{1-n} \int_{B(0, 2|x|)} |y|^{-1} d\tau(y), \end{aligned}$$

which tends to zero as $x \rightarrow 0$ by Lemma 1, (i).

Finally we can show, in a way similar to the proof of Theorem 2', that $u_6(x)$ tends to zero as $x \rightarrow 0$ except for x in a set satisfying (ii) of the theorem. Thus we conclude the proof of Theorem 3.

Remark 1. If $\lim_{j \rightarrow \infty} C(E_j') = 0$, then we can find a nonnegative measure μ on R^n with compact support such that $\lim_{r \downarrow 0} r^{-n} \mu(B(0, r)) = 0$ and

$$\lim_{x \rightarrow 0, x \in E} |x|^{-n} \int L_{n-1}(x, y) d\mu(y) = \infty.$$

Remark 2. Let μ be a nonnegative measure on R^n satisfying (1), (a), (b) and

(c) There exist $M, r_0 > 0$ such that $\mu(B(x, r)) \leq Mr^n$ for any $x \in B(x^0, r_0)$ and any $r \leq r_0$.

Then the set E in Theorem 3 can be taken to be empty and, moreover, $L\mu$ is n times differentiable at x^0 .

This fact can be proved in the same way as in Remark 2 in Section 2.

Remark 3. We can prove the following result similar to Theorem 2'.

Theorem 3'. Let k be as in Theorem 2', and h be a nondecreasing positive function on $(0, \infty)$ such that $\lim_{r \downarrow 0} h(r) = 0$ and

$$\int_0^1 [(r+s)H(s)]^{-1} ds \leq \text{const.} [h(r)]^{-1} \quad \text{for } r > 0,$$

where $H(r) = h(r)k(r)$ for $r > 0$. Let m be a nonnegative integer and μ be a nonnegative measure on $\mathbb{R}^n$ satisfying (1) and

$$\lim_{r \downarrow 0} r^{-m} H(r) \mu(B(x^0, r)) = 0.$$

Then there exists a set E in $\mathbb{R}^n$ such that

$$(i) \quad \lim_{x \rightarrow x^0} \lim_{x \in \mathbb{R}^n - E} |x - x^0|^{-m} h(|x - x^0|) \int L_{m-1}(x, y) d\mu(y) = 0;$$

$$(ii) \quad \lim_{j \rightarrow \infty} k(2^{-j}) C(E_j^!) = 0,$$

where $L_{-1}(x, y) = L(x - y)$.

Remark 4. Let $\tilde{h}$ be nonincreasing on the interval $(0, 1)$ and

$$E = \{x \in \mathbb{R}^n; \limsup_{r \downarrow 0} \tilde{h}(r) \mu(B(x, r)) > 0\}$$

for a nonnegative measure μ on $\mathbb{R}^n$. If $\mu(E) = 0$, then $\Lambda_{\tilde{h}^{-1}}(E) = 0$, where $\Lambda_{\tilde{h}^{-1}}$ denotes the Hausdorff measure with respect to the measure function $\tilde{h}^{-1}$.

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