

Fock Space Representation of the Virasoro Algebra I.

Yukihiro Kanie 蟹江 幸博 (三重大教育)

§1. The Virasoro algebra $\underline{L}$ is the Lie algebra over the complex number field $\mathbb{C}$ of the following form:

$$\underline{L} = \sum_{n \in \mathbb{Z}} \mathbb{C} e_n \oplus \mathbb{C} e'_0,$$

with the relations: for any $n, m \in \mathbb{Z}$

$$\begin{cases} [e_n, e_m] = (m-n)e_{n+m} + \frac{m^3-m}{12} \delta_{n+m,0} e'_0, \\ [e'_0, e_n] = 0. \end{cases}$$

This is a unique (up to isomorphisms) central extension of the Lie algebra $\underline{L}'$ of trigonometric polynomial vector fields on the circle:

$$\underline{L}' = \sum_{n \in \mathbb{Z}} \mathbb{C} \ell_n; \quad [\ell_n, \ell_m] = (m-n) \ell_{n+m} \quad (n, m \in \mathbb{Z}).$$

Let $\underline{h} = \mathbb{C}e_0 + \mathbb{C}e'_0$ be the abelian subalgebra of $\underline{L}$ of maximal dimension. For each $(h, c) \in \mathbb{C}^2 \cong \underline{h}^*$ the dual of $\underline{h}$, we can define the Verma module $M(h, c)$ and its dual $M^\dagger(h, c)$ as follows: $M(h, c)$ and $M^\dagger(h, c)$ are the left and right $\underline{L}$ -module with a cyclic vector $|h, c\rangle$ and $\langle c, h|$ with the following fundamental relations respectively:

$$\begin{aligned} e_{-n}|h, c\rangle &= 0 \quad (n \geq 1); \quad e_0|h, c\rangle = h|h, c\rangle, \quad e'_0|h, c\rangle = c|h, c\rangle, \\ \langle c, h|e_n &= 0 \quad (n \geq 1); \quad \langle c, h|e_0 = \langle c, h|h, \quad \langle c, h|e'_0 = \langle c, h|c. \end{aligned}$$

V.G.Kac[1979] studied these $\underline{L}$ -modules and obtained the formula concerning the determinant of the matrices of their vacuum expectation values, and by this Kac's determinant formula, F.L.Feigin and D.B.Fuks[1983] determined the composition series of $M(h, c)$.

§2. Consider the associative algebra $\underline{A}$ over $\mathbb{C}$ generated by $\{p_n, \Lambda\}$ ($n \in \mathbb{Z}$), with the following Bose commutation relations:

$$[p_n, p_m] = m \delta_{n+m, 0} \quad ; \quad [\Lambda, p_n] = 0 \quad (n, m \in \mathbb{Z}).$$

And consider the following operators in a completion $\hat{\underline{A}}$ of $\underline{A}$:

$$L_n = (p_0 - n \Lambda) p_n + \frac{1}{2} \sum_{j=1}^{n-1} p_j p_{n-j} + \sum_{j \geq 1} p_{n+j} p_{-j} \quad (n \geq 1);$$

$$L_{-n} = (p_0 + n \Lambda) p_{-n} + \frac{1}{2} \sum_{j=1}^{n-1} p_{-j} p_{j-n} + \sum_{j \geq 1} p_j p_{-n-j} \quad (n \geq 1);$$

$$L_0 = \frac{1}{2} (p_0^2 - \Lambda^2) + \sum_{j \geq 1} p_j p_{-j} ;$$

$$L'_0 = (-12\Lambda^2 + 1) \text{ id.}$$

Then by easy but long calculations, we get

Theorem 1. The operators L_n ($n \in \mathbb{Z}$) and L'_0 satisfy the commutation relations of the Virasoro algebra: for $n, m \in \mathbb{Z}$

$$\begin{cases} [L_n, L_m] = (m-n)L_{n+m} + \frac{m^3-m}{12} \delta_{n+m, 0} L'_0 ; \\ [L'_0, L_n] = 0. \end{cases}$$

§3. For each $(\omega, \lambda) \in \mathbb{C}^2$, we consider the left and right $\underline{A}$ -module $\underline{F}(\omega, \lambda)$ and $\underline{F}^\dagger(\omega, \lambda)$ with cyclic vectors $|\omega, \lambda\rangle$ and $\langle \lambda, \omega|$ with the following fundamental relations respectively:

$$\begin{aligned} p_{-n} |\omega, \lambda\rangle &= 0 \quad (n \geq 1); \quad p_0 |\omega, \lambda\rangle = \omega |\omega, \lambda\rangle ; \quad \Lambda |\omega, \lambda\rangle = \lambda |\omega, \lambda\rangle . \\ \langle \lambda, \omega| p_n &= 0 \quad (n \geq 1); \quad \langle \lambda, \omega| p_0 = \langle \lambda, \omega| \omega ; \quad \langle \lambda, \omega| \Lambda = \langle \lambda, \omega| \lambda . \end{aligned}$$

Then by using the canonical homomorphism π (i.e. $\pi(e_n) = L_n$ ($n \in \mathbb{Z}$); $\pi(e'_0) = L'_0$), we get the left $\underline{L}$ -module $(\underline{F}(\omega, \lambda), \pi_{(\omega, \lambda)}, \underline{L})$ which is called the Fock space representation, and by the explicit formulae of L_n and L'_0 ,

$$\begin{cases} L_0 |w, \lambda\rangle = \frac{1}{2}(\omega^2 - \lambda^2) |w, \lambda\rangle ; & L_0^- |w, \lambda\rangle = (1 - 12\lambda^2) |w, \lambda\rangle ; \\ L_{-n} |w, \lambda\rangle = 0 & \text{for } n \geq 1 . \end{cases}$$

By the universal property of the Verma module $M(h, c)$ as an $\underline{L}$ -module, for each $(w, \lambda) \in \mathbb{C}^2$ we get the unique $\underline{L}$ -module mapping

$$\pi_{w, \lambda}: M(h(w, \lambda), c(\lambda)) \longrightarrow \underline{F}(w, \lambda)$$

which sends the vacuum vector $|h(w, \lambda), c(\lambda)\rangle \in M(h(w, \lambda), c(\lambda))$ to the vacuum vector $|w, \lambda\rangle \in \underline{F}(w, \lambda)$, where

$$h(w, \lambda) = \frac{1}{2}(\omega^2 - \lambda^2) \quad \text{and} \quad c(\lambda) = 1 - 12\lambda^2.$$

Then by constructing intertwining operators (Theorem 3) and by showing their nontriviality (Theorem 6), we get the following.

Theorem 2. For each $(w, \lambda) \in \mathbb{C}^2$, let $s_{\pm}$ be the roots of the equation $\lambda = \frac{1}{s} - \frac{s}{2}$.

(1) The canonical $\underline{L}$ -module mapping

$$\pi_{w, \lambda}: M(h(w, \lambda), c(\lambda)) \longrightarrow \underline{F}(w, \lambda)$$

is isomorphic, if and only if the equation

$$(*) \quad \omega + \frac{a}{2}s_+ + \frac{b}{2}s_- = 0$$

has no integral solutions $(a, b) \in \mathbb{Z}^2$ with $a \geq 1$ and $b \geq 1$.

(2) The $\underline{L}$ -module mapping $\pi_{w, \lambda}^+: M^+(h(w, \lambda), c(\lambda)) \longrightarrow \underline{F}^+(w, \lambda)$ is isomorphic, if and only if the equation (*) has no integral solutions $(a, b) \in \mathbb{Z}^2$ with $a \leq -1$ and $b \leq -1$.

(3) $\underline{F}(w, \lambda)$ is irreducible as an $\underline{L}$ -module, if and only if the equation (*) has no integral solutions $(a, b) \in \mathbb{Z}^2$ with $ab \geq 1$.

And this condition (3) is equivalent to the fact that the corresponding Verma module $M(h(w, \lambda), c(\lambda))$ is irreducible.

§3. To construct intertwining operators between Fock spaces, we introduce the operators of following type acting on $\underline{F}(w, \lambda)$. Fix $s \in \mathbb{C}^*$, and consider

$$X(s, \zeta) = \exp\left(s \sum_{n=1}^{\infty} \zeta^n \frac{p_n}{n}\right) \exp\left(-s \sum_{n=1}^{\infty} \zeta^{-n} \frac{p_{-n}}{n}\right) \zeta^{sp_0 - \frac{s^2}{2}} T_s,$$

and for any $a \geq 1$

$$Z(s; \zeta_1, \dots, \zeta_a) = F\left(\frac{s^2}{2}; \zeta_1, \dots, \zeta_a\right) \exp\left(s \sum_{n=1}^{\infty} (\zeta_1^n + \dots + \zeta_a^n) \frac{p_n}{n}\right) \exp\left(-s \sum_{n=1}^{\infty} (\zeta_1^{-n} + \dots + \zeta_a^{-n}) \frac{p_{-n}}{n}\right) T_{as},$$

where

$$T_s: \underline{F}(w, \lambda) \longrightarrow \underline{F}(w+s, \lambda)$$

is the operator such that

$$T_s |w, \lambda\rangle = |w+s, \lambda\rangle; \quad [T_s, p_n] = 0 \quad (n \neq 0); \quad [T_s, \Lambda] = 0,$$

and

$$F(\alpha; \zeta_1, \dots, \zeta_a) = \prod_{j=1}^a \zeta_j^{-(a-1)\alpha} \prod_{1 \leq i < j \leq a} (\zeta_i - \zeta_j)^{2\alpha}.$$

Operators of this type are called Vertex Operators.

Then $X(s; \zeta)$ and $Z(s; \zeta_1, \dots, \zeta_a)$ are multi-valued holomorphic functions of $\zeta \in \mathbb{C}^*$ and $(\zeta_1, \dots, \zeta_a) \in M_a$ respectively with values in the operators acting on $\underline{F}(w, \lambda)$'s, where M_a is the manifold defined by

$$M_a = \{(\zeta_1, \dots, \zeta_a) \in (\mathbb{C}^*)^a; \zeta_i \neq \zeta_j \quad (1 \leq i < j \leq a)\}.$$

In order to get intertwining operators, we want integrate these vertex operators $Z(s; \zeta_1, \dots, \zeta_a)$. For the guarantee of the convergence of these integrals, we introduce the homology theory associated to the monodromy structure of the multi-valued function $F(\alpha; \zeta_1, \dots, \zeta_a)$.

For each $\alpha \in \mathbb{C}^*$, denote by $\underline{S}_\alpha^*$ the local coefficient system with values in $\mathbb{C}$ which is determined by the monodromy of the multi-valued holomorphic function $F(\alpha; \zeta_1, \dots, \zeta_a)$ on M_a , and denote by $\underline{S}_\alpha$ the dual local system of $\underline{S}_\alpha^*$.

Fix $s \in \mathbb{C}^*$ and an integer $a \geq 1$, and take an element $\Gamma \in H_a(M_a; \underline{S}_\alpha)$. For each integer $b \in \mathbb{Z}$, we consider the operator

$$O(s, \Gamma; a, b) = \int_{\Gamma} Z(s; \zeta_1, \dots, \zeta_a) \zeta_1^{-b-1} \dots \zeta_a^{-b-1} d\zeta_1 \dots d\zeta_a.$$

Then we get the following.

Theorem 3.

1) For each $(w, \lambda) \in \mathbb{C}^2$, the operator $O(s, \Gamma; a, b)$ acts as

$$O(s, \Gamma; a, b) : \underline{F}(w, \lambda) \longrightarrow \underline{F}(w+as, \lambda).$$

2) Take $s \in \mathbb{C}^*$ and $a, b \in \mathbb{Z}$ with $a \geq 1$. Put $\lambda = \lambda(s) = \frac{1}{s} - \frac{s}{2}$, then the operator

$$O(s, \Gamma; a, b) : \underline{F}\left(-\frac{a}{2}s - \frac{b}{s}, \lambda\right) \longrightarrow \underline{F}\left(\frac{a}{2}s - \frac{b}{s}, \lambda\right).$$

commutes with the action of $\underline{L}$.

For suitable $s \in \mathbb{C}^*$ and $w \in \mathbb{C}$, the equation (*) $w = \frac{a}{2}s - \frac{b}{s}$ in

Theorem 2 has a countable number of integral solutions $(a, b) \in \mathbb{Z}^2$,

if and only if $\alpha = s^2/2$ is a rational number. This index α characterizes the property of the monodromy of the function

$$F(\alpha; \zeta_1, \dots, \zeta_a) = \prod_{j=1}^a \zeta_j^{-(a-1)\alpha} \prod_{1 \leq i < j \leq a} (\zeta_i - \zeta_j)^{2\alpha}.$$

If α is irrational, then the monodromy of the function F is of logarithmic type, and if α is rational, then the monodromy of F is of algebraic type.

Sketch of the Proof of Theorem 3. The essential points to prove Theorem 3 are the following two propositions which are obtained by rather long calculations:

Proposition 4. For any $(\zeta_1, \dots, \zeta_a) \in M_a$ and $s \in \mathbb{C}^*$,

$$X(s, \zeta_1) \cdots X(s, \zeta_a) = Z(s; \zeta_1, \dots, \zeta_a) (\zeta_1 \cdots \zeta_a)^{sp_0 + \frac{a}{2} s^2}.$$

Proposition 5. (Conformal Covariance)

$$[L_m, X(s, \zeta)] = \zeta^{-m} \left(\zeta \frac{d}{d\zeta} - m \left(s\Lambda + \frac{s^2}{2} \right) \right) X(s, \zeta)$$

for each $m \in \mathbb{Z}$ and $s, \zeta \in \mathbb{C}^*$, and

$$\begin{aligned} [L_m, Z(s; \zeta_1, \dots, \zeta_a)] &= \\ &= \sum_{j=1}^a \zeta_j^{-m} \left[\zeta_j \frac{\partial}{\partial \zeta_j} + \left(sp_0 - \frac{a}{2} s^2 - n \left(s\Lambda + \frac{s^2}{2} \right) \right) \right] Z(s; \zeta_1, \dots, \zeta_a) \end{aligned}$$

for each $m \in \mathbb{Z}$, $s \in \mathbb{C}$ and $(\zeta_1, \dots, \zeta_a) \in M_a$.

Introduce the new coordinates $(k_1, \dots, k_{a-1}, \zeta)$ on the manifold M_a as the product manifold $M_a = Y_{a-1} \times \mathbb{C}^*$, where

$$Y_{a-1} = \{ (k_1, \dots, k_{a-1}) \in (\mathbb{C}^*)^{a-1}; k_i \neq k_j \ (i \neq j), k_i \neq 1 \},$$

and

$$\zeta_i = k_i \zeta \quad (1 \leq i \leq a-1) \text{ and } \zeta_a = \zeta.$$

Then the function $F(\alpha; \zeta_1, \dots, \zeta_a)$ is independent of ζ , and we get

Proposition 6. (Total Momentum Conservation) The integral

$$\int_{\Gamma} \zeta_1^{-\ell_1-1} \cdots \zeta_a^{-\ell_a-1} F(s; \zeta_1, \zeta_2, \dots, \zeta_a) d\zeta_1 \cdots d\zeta_a$$

vanishes unless $\ell_1 + \ell_2 + \cdots + \ell_a = 0$.

This assures the statement 1) of the theorem. And by Proposition 5 and by using integration by parts, we get the following formula.

$$[L_{-m}, O(s, \Gamma; a, b)] = \sum_{j=1}^a \{ b + s p_0 - \frac{a s^2}{2} + m (s \wedge + \frac{s^2}{2} - 1) \} O(s, \Gamma; b, \dots, b_{\hat{j}}, \dots, b).$$

From this formula, we know when $O(s, \Gamma, a, b)$ is intertwining.

§5. We must construct a cycle $\Gamma(\alpha) \in H_a(M_a; \underline{S}_\alpha)$ which gives a nontrivial intertwining operator $O(s, \Gamma; a, b)$.

If we expand

$$\exp[s \sum_{n=1}^{\infty} (\zeta_1^n + \dots + \zeta_a^n) \frac{p_n}{n}] \exp[-s \sum_{n=1}^{\infty} (\zeta_1^{-n} + \dots + \zeta_a^{-n}) \frac{p_{-n}}{n}]$$

as a Laurent series of $(\zeta_1, \dots, \zeta_a)$, then the coefficient of the each term of the operator

$$\int_{\Gamma} Z(s; \zeta_1, \dots, \zeta_a) \zeta_1^{-b-1} \dots \zeta_a^{-b-1} d\zeta_1 \dots d\zeta_a$$

is written as

$$\int_{\Gamma} F(\frac{s^2}{2}; \zeta_1, \dots, \zeta_a) \zeta_1^{-\ell_1-1} \dots \zeta_a^{-\ell_a-1} d\zeta_1 \dots d\zeta_a,$$

and this integral is reduced to

$$\int_{\Gamma_2} k_1^{-\ell_1-1} \dots k_{a-1}^{-\ell_{a-1}-1} G(\alpha; k_1, \dots, k_{a-1}) dk_1 \dots dk_{a-1},$$

where $\Gamma = \Gamma_1 \times \Gamma_2 \in H_a(M_a; \underline{S}_\alpha) \cong H_1(\mathbb{C}^*; \mathbb{C}) \otimes H_{a-1}(Y_{a-1}; \underline{S}'_\alpha)$, Γ_1 is a generator of $H_1(\mathbb{C}^*; \mathbb{C})$, and $\underline{S}'_\alpha$ is the local system on Y_{a-1} similarly associated with the function $G(\alpha; k_1, \dots, k_{a-1})$ as $\underline{S}_\alpha$, and

$$G(\alpha; k_1, \dots, k_{a-1}) = \prod_{j=1}^{a-1} k_j^{-(a-1)\alpha(1-k_j)^{2\alpha}} \prod_{1 \leq i < j \leq a-1} (k_i - k_j)^{2\alpha}.$$

Now we must construct the cycle $\Gamma_2(\alpha) \in H_{a-1}(Y_{a-1}; \underline{S}_{\alpha}')$ which regularizes the divergent integral

$$\int_{\Delta(a-1)} k_1^{-\ell_1-1} \cdots k_{a-1}^{-\ell_{a-1}-1} G(\alpha; k_1, \dots, k_{a-1}) dk_1 \cdots dk_{a-1}.$$

where $\Delta(a-1)$ is the open $(a-1)$ -simplex defined by

$$\Delta(a-1) = \{(k_1, \dots, k_{a-1}) \in \mathbb{R}^{a-1}; 1 > k_1 > \dots > k_{a-1} > 0\}.$$

Let $m=a-1$, and define the set

$$\Omega_m = \{\alpha \in \mathbb{C}; d(d+1)\alpha \notin \mathbb{Z}, d(a-d)\alpha \notin \mathbb{Z} \ (1 \leq d \leq m)\}.$$

Then by using the technique of resolutions of singularities, we get

Theorem 7. There exist cycles $\Gamma_2(\alpha) \in H_m(Y_m; \underline{S}_{\alpha}')$ defined on Ω_m such that

- 1) $\Gamma_2(\alpha)$ is holomorphic on Ω_m .
- 2) If $\alpha \in \Omega_m$ and $(\ell_1, \dots, \ell_m) \in \mathbb{Z}^m$ satisfy the inequalities

$$\operatorname{Re} \alpha > 0 \quad \text{and} \quad \operatorname{Re} 2\alpha > -\min_j \ell_j,$$

then the following equality of integrals holds:

$$\begin{aligned} \int_{\Gamma_2(\alpha)} G(\alpha; k_1, \dots, k_m) k_1^{\ell_1} \cdots k_m^{\ell_m} dk_1 \cdots dk_m &= \\ &= \int_{\Delta(m)} G(\alpha; k_1, \dots, k_m) k_1^{\ell_1} \cdots k_m^{\ell_m} dk_1 \cdots dk_m \end{aligned}$$

And the latter integral is known explicitly as follows:

Theorem 8. (A.Selberg[1944])

Let $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy the inequalities

$$\operatorname{Re} \beta > -1, \quad \operatorname{Re} \gamma > -1, \quad \operatorname{Re} \alpha > -\min\left\{\frac{1}{m}, \frac{\operatorname{Re} \beta + 1}{m-1}, \frac{\operatorname{Re} \gamma + 1}{m-1}\right\},$$

then the improper integral (**) converges absolutely and is explicitly expressed as

$$\begin{aligned}
 (**) \quad & \int_{\Delta(m)} \prod_{1 \leq i < j \leq m} (k_i - k_j)^{2\alpha} \prod_{j=1}^m k_j^\beta (1 - k_j)^\gamma dk_1 \cdots dk_m = \\
 & = \frac{1}{m!} \prod_{j=1}^m \frac{\Gamma(j\alpha+1) \Gamma((j-1)\alpha+\beta+1) \Gamma((j-1)\alpha+\gamma+1)}{\Gamma(\alpha+1) \Gamma((m+j-2)\alpha+\beta+\gamma+2)}.
 \end{aligned}$$

Finally we get

Theorem 9. There exists $\Gamma(\alpha) \in H_a(M_a; \underline{S}_\alpha)$ which depends holomorphically on $\alpha \in \Omega_{a-1}$ such that the operator

$$O(s; a, b) = O(s, \Gamma(\frac{s}{2}); a, b): \underline{F}(w-as, \frac{1-s}{2}) \longrightarrow \underline{F}(w, \frac{1-s}{2})$$

is nontrivial in the sense that for any $w \in \mathbb{C}$

1) for $b \geq 0$, the image $O(s; a, b)|w-as, \frac{1-s}{2}\rangle$ is a nonzero vector.

2) for $b < 0$, there is a vector $|v\rangle \in \underline{F}(w-as, \frac{1-s}{2})$ such that $O(s; a, b)|v\rangle = |w, \frac{1-s}{2}\rangle$.

§6. Case that $\lambda=0$. Assume that $\lambda=0$, i.e., $c=1$, then the solutions $s_\pm$ of $\lambda(s)=0$ are $s_\pm = \pm\sqrt{2}$, and $\alpha = \frac{1}{2}s^2 = 1$. For any $m \in \mathbb{Z}$, denote $q_m = \sqrt{2} p_m$, then

$$[q_m, q_n] = 2n\delta_{n+m,0} L'_0 \quad (\text{note that } L'_0 = \text{id in this case}),$$

and the operators L_n 's are explicitly written as

$$\begin{cases}
 L_n = \frac{1}{2} q_0 q_n + \frac{1}{4} \sum_{j=1}^{n-1} q_j q_{n-j} + \frac{1}{2} \sum_{j \geq 1} q_{n+j} q_{-j} \\
 L_{-n} = \frac{1}{2} q_0 q_{-n} + \frac{1}{4} \sum_{j=1}^{n-1} q_{-j} q_{j-n} + \frac{1}{2} \sum_{j \geq 1} q_j q_{-n-j} \\
 L_0 = \frac{1}{4} q_0^2 + \frac{1}{2} \sum_{j \geq 1} q_j q_{-j}
 \end{cases} \quad (n \geq 1)$$

And denote by $|m\rangle$ and $\langle m|$ the vacuum vectors of the Fock spaces $\underline{F}(m) = \underline{F}(m/\sqrt{2}, 0)$ and $\underline{F}^\dagger(m) = \underline{F}^\dagger(m/\sqrt{2}, 0)$ respectively, that is,

$$q_0|m\rangle = m|m\rangle, \quad \Lambda|m\rangle = 0; \quad \langle m|q_0 = \langle m|m, \quad \langle m|\Lambda = 0.$$

Let

$$\hat{h} := \mathbb{C}q_0 \oplus \mathbb{C}L_0 \oplus \mathbb{C}L'_0 \supset \underline{h} := \mathbb{C}q_0 \oplus \mathbb{C}L_0 ,$$

then $\hat{h}$ is the maximal abelian subalgebra of the Lie subalgebra $\underline{L} \oplus \mathbb{C}q_0 \subset \underline{A}$. The sums $\underline{F}$ and $\underline{F}^\dagger$ of Fock spaces $\underline{F}(m)$ and $\underline{F}^\dagger(m)$ can be considered as $(\underline{L} \oplus \mathbb{C}q_0)$ -modules, and have the weight space decompositions w.r.t. the abelian subalgebra $\underline{h}$:

$$\underline{F} = \sum_{m \in \mathbb{Z}} \sum_{d \geq 0} \underline{F}_d(m) \quad \text{and} \quad \underline{F}^\dagger = \sum_{m \in \mathbb{Z}} \sum_{d \geq 0} \underline{F}_d^\dagger(m) ,$$

where $\underline{F}_d(m)$ and $\underline{F}_d^\dagger(m)$ are the weight spaces belonging to the same weight $(m, \frac{m^2}{4} + d)$.

Now decompose the vertex operators $X(\pm\sqrt{2}, \zeta)$ as

$$\begin{aligned} X(\pm\sqrt{2}; \zeta) &= \exp\left(\pm \sum_{n \geq 1} \frac{q_n}{n} \zeta^n\right) \exp\left(\mp \sum_{n \geq 1} \frac{q_{-n}}{n} \zeta^{-n}\right) T_{\pm\sqrt{2}} \zeta^{\pm q_0 + 1} \\ &= \sum_{m \in \mathbb{Z}} X_m^\pm \zeta^m \quad (\zeta \in \mathbb{C}^*) . \end{aligned}$$

Then as operators on the Fock spaces $\underline{F}$ and $\underline{F}^\dagger$, $X_m^\pm$ are well-defined and satisfy the following commutation relations : for $n, m \in \mathbb{Z}$,

$$(\#) \quad \begin{cases} [q_n, X_m^\pm] = \pm 2 X_{n+m}^\pm ; & [q_n, q_m] = 2m\delta_{n+m,0} L'_0 , \\ [X_n^+, X_m^+] = [X_n^-, X_m^-] = 0 ; & [X_n^+, X_m^-] = m \delta_{n+m,0} L'_0 + q_{n+m} , \end{cases}$$

$$(\#2) \quad [L_n, X_m^\pm] = m X_{n+m}^\pm ; \quad [L_n, q_m] = m q_{n+m} ,$$

in particular

$$(\#3) \quad [q_0, X_0^\pm] = \pm 2 X_0^\pm ; \quad [X_0^+, X_0^-] = q_0 ,$$

$$(\#4) \quad [L_n, X_0^\pm] = [L_n, q_0] = 0 .$$

Hence the space $\underline{g}$ spanned by above operators

$$\underline{g} = \sum_{n \in \mathbb{Z}} \mathbb{C}X_n^+ \oplus \sum_{n \in \mathbb{Z}} \mathbb{C}q_n \oplus \sum_{n \in \mathbb{Z}} \mathbb{C}X_n^- \oplus \mathbb{C}L'_0 \oplus \mathbb{C}L_0$$

is a Lie algebra isomorphic to the affine Lie algebra of type $A_1^{(1)}$

In fact, $A_1^{(1)}$ is realized as

$$\begin{aligned} A_1^{(1)} &= \underline{\mathfrak{sl}}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \\ &= \text{span}\{e \otimes t^n, h \otimes t^n, f \otimes t^n \mid (n \in \mathbb{Z})\} \oplus \mathbb{C}c \oplus \mathbb{C}d, \end{aligned}$$

where $\{e, h, f\}$ is a canonical basis of $\underline{\mathfrak{sl}}(2, \mathbb{C})$ with the relations:

$$[e, f] = h ; [h, e] = 2e ; [h, f] = -2f.$$

The isomorphism φ of $\underline{\mathfrak{g}}$ to $A_1^{(1)}$ is given by

$$\begin{cases} \varphi(X_n^+) = e \otimes t^n ; \varphi(\alpha_n) = h \otimes t^n ; \varphi(X_n^-) = f \otimes t^n & (n \in \mathbb{Z}) \\ \varphi(L_0') = -c ; \varphi(L_0) = d = t \frac{d}{dt} . \end{cases}$$

Under this identification φ , fix the notations of the following Lie algebras

$$\begin{aligned} (\#\#) \quad \dot{\underline{\mathfrak{g}}} &:= \mathbb{C}X_0^+ + \mathbb{C}\alpha_0 + \mathbb{C}X_0^- \cong \underline{\mathfrak{sl}}(2, \mathbb{C}) \\ \underline{\mathfrak{g}} &= \dot{\underline{\mathfrak{g}}} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}L_0' + \mathbb{C}L_0 = A_1^{(1)} \\ \underline{\tilde{\mathfrak{g}}} &:= \underline{\mathfrak{g}} + \sum_{n \neq 0} \mathbb{C}L_n = \dot{\underline{\mathfrak{g}}} \otimes \mathbb{C}[t, t^{-1}] + \underline{\mathfrak{L}} . \end{aligned}$$

Note that $L_0' = -c$ belongs to the center of $\underline{\tilde{\mathfrak{g}}}$, and $L_n = t^{n+1} \frac{d}{dt}$ is the derivation of the loop algebra $\underline{\mathfrak{g}}/(\mathbb{C}L_0 + \mathbb{C}L_0')$ of $\underline{\mathfrak{sl}}(2, \mathbb{C})$.

By (#4), $\dot{\underline{\mathfrak{g}}} = \underline{\mathfrak{sl}}(2, \mathbb{C})$ commutes with the Virasoro algebra $\underline{\mathfrak{L}}$:

$$[\dot{\underline{\mathfrak{g}}}, \underline{\mathfrak{L}}] = 0 ; \quad \text{hence} \quad [U(\dot{\underline{\mathfrak{g}}}), \underline{\mathfrak{L}}] = 0 .$$

The operators $X_0^\pm : \underline{\mathbb{F}}(m) \longrightarrow \underline{\mathbb{F}}(m \pm 2)$ are intertwining operators for any $m \in \mathbb{Z}$, and q_0 acts on $\underline{\mathbb{F}}(m)$ as the scalar operator $m \text{ id}$.

Decompose the Fock space $\underline{\mathbb{F}}$ as

$$\underline{\mathbb{F}} = \underline{\mathbb{F}}^{\text{even}} \oplus \underline{\mathbb{F}}^{\text{odd}} ,$$

where

$$\underline{\mathbb{F}}^{\text{even}} = \sum_{m \in \mathbb{Z}} \underline{\mathbb{F}}(2m) ; \quad \underline{\mathbb{F}}^{\text{odd}} = \sum_{m \in \mathbb{Z}} \underline{\mathbb{F}}(2m+1) .$$

Then these two components $\underline{F}^{\text{even}}$ and $\underline{F}^{\text{odd}}$ are irreducible $\underline{L}$ -modules, furthermore these are also irreducible as $A_1^{(1)}$ -modules. Remark that $\underline{F}^{\text{even}}$ is isomorphic to the basic representation of the affine algebra $A_1^{(1)}$.

References

- Date E., Jimbo M., Kashiwara M., Miwa T. [1983] Transformation groups for soliton equations, in Proc. of RIMS symposium (ed. M. Jimbo & T. Miwa), World Scientific, Singapore (1983), 39-120.
- Feigin F.L., Fuks D.B. [1982] Skew-symmetric invariant differential operators on a straight line and Virasoro algebra, Func. Anal. & its Appl., 16-2 (1982), 47-63 (in Russian).
- Feigin F.L., Fuks D.B. [1983] Verma modules over Virasoro algebra, Func. Anal. & its Appl., 17-3 (1983), 91-92 (in Russian).
- Kac V.G. [1979] Contravariant form for infinite dimensional Lie algebras and superalgebras, Lect. Notes in Physics, 94 (1979), 441-445.
- Selberg A. [1944] Bemerkninger om et Multipliet Integral, Norsk Mat. Tidsskrift, 26 (1944), 71-78.
- Segal G. [1981] Unitary representations of some infinite dimensional groups, Comm. Math. Phys., 80 (1981), 301-342.
- Tsuchiya A., Kanie Y. [1984] Fock space representations of the Virasoro algebra --- Intertwining operators ---, preprint.