

A statistical Theory of Wave-Propagation in Random Medium and the Power Distribution Function-Theory of Cumulants

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1. Introduction

The wave equation considered here is not a particular wave equation but an equation of the form of

$$[L - bq] \psi(x) = \eta(x). \quad (1)$$

Here $\psi(x)$ is the wave function of the space coordinates (x) and $\eta(x)$ is the (given) external source; L is an operator operating on ψ , and b is a constant; $q = q(x)$ is a random function following some statistical distribution. For instance, in the case of an ordinary scalar wave equation,

$$L = -\Delta - k^2, \quad bq(x) = k^2 \Delta \epsilon(x), \quad (2)$$

where $\Delta \epsilon$ is the fluctuation part of dielectric constant of the medium.

It is easy to find that all the statistical informations of the wave can be derived from the generalized characteristic function $\langle 0|0 \rangle$ defined by

$$\langle 0|0 \rangle = \langle \exp[\int dx \{ \bar{\eta}(x) \psi(x) + \bar{\eta}^*(x) \psi^*(x) \}] \rangle, \quad (3)$$

where $\langle Q \rangle$ stands for the average value of Q over all possible functions of $q(x)$, and $\bar{\eta}(x)$ and $\bar{\eta}^*(x)$ are arbitrarily given

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(complex) functions; for instance,

$$\langle \psi(x) \rangle = \{ \delta / \delta \bar{\eta}(x) \} \langle 0|0 \rangle \big|_{\bar{\eta}=\bar{\eta}^*=0},$$

$$\langle \psi^*(x_1) \psi(x_2) \rangle = \{ \delta / \delta \bar{\eta}^*(x_1) \} \{ \delta / \delta \bar{\eta}(x_2) \} \langle 0|0 \rangle \big|_{\bar{\eta}=\bar{\eta}^*=0}$$

Generally, if $f[\psi, \psi^*]$ is a functional of $\psi(x)$ and $\psi^*(x)$, its expectation value is given by

$$\langle f[\psi, \psi^*] \rangle = f[\delta / \delta \bar{\eta}, \delta / \delta \bar{\eta}^*] \langle 0|0 \rangle \big|_{\bar{\eta}=\bar{\eta}^*=0}, \quad (4)$$

In order to obtain $\langle 0|0 \rangle$ as a function of $\bar{\eta}$ and $\bar{\eta}^*$, we put

$$\langle 0|0 \rangle = e^{\theta}, \quad (5)$$

and expand θ with respect to $\bar{\eta}$, $\bar{\eta}^*$, and also to the external sources η and η^* :

$$\begin{aligned} \theta &= \int dx dx' [\bar{\eta}(x) \kappa_{01}(x|x') \eta(x') + \bar{\eta}^*(x) \kappa_{10}(x|x') \eta^*(x')] \\ &+ \frac{1}{(2!)} \int dx_1 dx_2 dx_1' dx_2' [\bar{\eta}(x_1) \bar{\eta}(x_2) \kappa_{02}(x_1, x_2|x_1', x_2') \eta(x_1') \eta(x_2')] \\ &+ 4 \bar{\eta}^*(x_1) \bar{\eta}(x_2) \kappa_{11}(x_1, x_2|x_1', x_2') \eta^*(x_1') \eta(x_2') \\ &+ \bar{\eta}^*(x_1) \bar{\eta}^*(x_2) \kappa_{20}(x_1, x_2|x_1', x_2') \eta^*(x_1') \eta^*(x_2')] + \dots \quad (6) \\ &= \sum_{\nu, \mu=1}^{\infty} \frac{1}{(\nu! \mu!)^2} \int \prod_{i=1}^{\nu} dx_i dx_i' \prod_{j=1}^{\mu} dy_j dy_j' \bar{\eta}^*(x_1) \dots \bar{\eta}^*(x_{\nu}) \bar{\eta}(y_1) \\ &\times \dots \bar{\eta}(y_{\mu}) \kappa_{\nu\mu}(x_1, \dots, x_{\nu}; y_1, \dots, y_{\mu} | x_1', \dots, x_{\nu}', y_1', \dots, y_{\mu}') \\ &\times \eta^*(x_1') \dots \eta^*(x_{\nu}') \eta(y_1') \dots \eta(y_{\mu}'). \end{aligned}$$

Here, the expansion coefficients, i. e., the cumulants $\kappa_{\nu\mu}$'s,

have the symmetry

$$\kappa_{\nu\mu}^*(x;y|x';y') = \kappa_{\nu\mu}(y;x|y';x'), \quad (7)$$

and they are also symmetrical with respect to the coordinates in each group of $\{x_i\}$, $\{y_i\}$, $\{x'_i\}$, and $\{y'_i\}$

The cumulants $\kappa_{\nu\mu}(x;y|x';y')$ are independent of the external sources η , η^* and also of $\bar{\eta}$ and $\bar{\eta}^*$, and the complete statistical informations of the wave can be obtained in terms of them.

2. Relations between the Cumulants and the Green's Functions

The cumulants $\kappa_{\nu\mu}$ can be given in terms of the statistical Green's functions defined as follows:

$$G_{01}(x|x') = \{\delta/\delta\eta(x')\} \langle \psi(x) \rangle \big|_{\eta=\eta^*=0},$$

$$G_{10}(x|x') = \{\delta/\delta\eta^*(x')\} \langle \psi^*(x) \rangle \big|_{\eta=\eta^*=0}, \quad (8)$$

$$G_{11}(x;y|x';y') = \{\delta/\delta\eta^*(x')\} \{\delta/\delta\eta(y')\} \langle \psi^*(x) \psi(y) \rangle \big|_{\eta=\eta^*=0},$$

$$G_{\nu\mu}(x_1, \dots, x_\nu; y_1, \dots, y_\mu | x'_1, \dots, x'_\nu; y'_1, \dots, y'_\mu)$$

$$= \prod_{i=1}^{\nu} \{\delta/\delta\eta^*(x'_i)\} \prod_{j=1}^{\mu} \{\delta/\delta\eta(y'_j)\} \langle \psi^*(x_1) \dots \psi^*(x_\nu) \psi(y_1) \dots$$

$$\dots \psi(y_\mu) \rangle \big|_{\eta=\eta^*=0}$$

$$= \prod_{i=1}^{\nu} \{\delta/\delta\bar{\eta}^*(x_i)\} \{\delta/\delta\eta^*(x'_i)\} \prod_{j=1}^{\mu} \{\delta/\delta\bar{\eta}(y_j)\} \{\delta/\delta\eta(y'_j)\}$$

$$\times \langle 0|0 \rangle \big|_{\eta=\eta^*=\bar{\eta}=\bar{\eta}^*=0},$$

where the last expression is obtained by the use of (4).

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Thus, the Green's functions thus defined are the expansion coefficients of $\langle 0|0 \rangle$ with respect to η , η^* , $\bar{\eta}$ and $\bar{\eta}^*$, and have the same symmetries as in (7).

Using the definition (6) of the cumulants, it is straightforward to obtain the following expressions of $\kappa_{\nu\mu}$ in terms of the Green's functions:

$$\begin{aligned}\kappa_{10}(x|x') &= G_{10}(x|x'), & \kappa_{01}(x|x') &= G_{01}(x|x'), \\ \kappa_{11}(x;y|x';y') &= G_{11}(x;y|x';y') - G_{10}(x|x')G_{01}(y|y'), \\ \kappa_{20}(x_1, x_2|x_1', x_2') &= G_{20}(x_1, x_2|x_1', x_2') \\ &\quad - G_{10}(x_1|x_1')G_{10}(x_2|x_2') - G_{10}(x_1|x_2')G_{10}(x_2|x_1'), \text{ etc.} \\ &\hspace{25em} (9)\end{aligned}$$

It is also useful to express the Green's functions in terms of the $\kappa_{\nu\mu}$'s:

$$\begin{aligned}G_{11}(x;y|x';y') &= \kappa_{10}(x|x')\kappa_{01}(y|y') + \kappa_{11}(x;y|x';y'), \\ G_{21}(x_1, x_2;y|x_1', x_2';y') &= \kappa_{21}(x_1, x_2;y|x_1', x_2';y') \\ &\quad + \kappa_{11}(x_2;y|x_2';y')\kappa_{10}(x_1|x_1') + \kappa_{11}(x_2;y|x_1';y')\kappa_{10}(x_1|x_2') \\ &\quad + \kappa_{11}(x_1;y|x_1';y')\kappa_{10}(x_2|x_2') + \kappa_{11}(x_1;y|x_2';y')\kappa_{10}(x_2|x_1') \\ &\quad + \kappa_{20}(x_1, x_2|x_1', x_2')\kappa_{01}(y|y') \\ &\quad + \{\kappa_{10}(x_1|x_1')\kappa_{10}(x_2|x_2') + \kappa_{10}(x_1|x_2')\kappa_{10}(x_2|x_1')\}\kappa_{01}(y|y'), \\ &\hspace{25em} \text{etc.} \hspace{5em} (10)\end{aligned}$$

The cumulants $\kappa_{\nu\mu}$ can be considered to be more basic statistical quantities than the Green's functions; in the perturbation

A Statistical theory of Wave-Propagation theory, the Feynman diagram representing $\kappa_{\nu\mu}$ is a connected graph which does not contain any disconnected graph.

The cumulants and the Green's functions are connected also by the important relation of the following form:

$$\begin{aligned} \kappa_{11}(x; y | x'; y') &= \int dx'' dy'' dx''' dy''' G_{10}(x | x'') G_{01}(y | y'') \\ &\times I_{11}(x'', y'' | x'''; y''') G_{11}(x'''; y''' | x'; y'), \end{aligned} \quad (11)$$

or, more generally,

$$\begin{aligned} \kappa_{\nu\mu}(x; y | x'; y') &\equiv \kappa_{\nu\mu}(x_1, \dots, x_\nu; y_1, \dots, y_\mu | x'_1, \dots, x'_\nu; y'_1, \dots, y'_\mu) \\ &= \int dx'' dy'' dx''' dy''' \prod_{i=1}^{\nu} \prod_{j=1}^{\mu} G_{10}(x_i | x''_i) G_{01}(y_j | y''_j) \\ &\times I_{\nu\mu}(x'', y'' | x'''; y''') G_{\nu\mu}(x'''; y''' | x'; y'). \end{aligned} \quad (12)$$

If the "interaction" functions $I_{\nu\mu}(x; y | x'; y')$ are known, the cumulants or the Green's functions can be obtained in principle by solving eqn. (12) with the relation (9) or (10).

The form of the $I_{\nu\mu}$'s depends on the statistical properties of the random function $q(x)$. A system of equations is obtained to find $I_{\nu\mu}$, assuming the Gaussian multivariate distribution of $q(x)$ with an arbitrary correlation function $D(x - x') = \langle q(x)q(x') \rangle$.

3. Power Distribution Function

The probability density function $P(w)$ of the power $w(x) = \psi^*(x)\psi(x)$ at a particular point x is very important in practical problems, and it is given by the integral

$$P(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{i\lambda w} f(\lambda). \quad (13)$$

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Here, on using (4) with (5), the characteristic function $f(\lambda)$ is given by

$$\begin{aligned} f(\lambda) &= \langle \exp\{-i\lambda\psi^*(x)\psi(x)\} \rangle \\ &= \exp[-i\lambda\{\delta/\delta\bar{\eta}^*(x)\}\{\delta/\delta\bar{\eta}(x)\}]e^{\theta}|_{\bar{\eta}=\bar{\eta}^*=0}, \end{aligned} \quad (14)$$

and θ is given by the expansion (6). If, in this expansion, the higher terms of $v+\mu \geq 3$ can be neglected, as it is the case in many practical problems, the evaluation of (14) yields

$$f(\lambda) = \{(1 + i\sigma\lambda)^2 + \rho^*\rho\lambda^2\}^{-1/2} \quad (15)$$

$$\times \exp[-i\lambda\{(1+i\sigma\lambda)^2 + \rho^*\rho\lambda^2\}^{-1}\{(1+i\sigma\lambda)|\langle\psi\rangle|^2 - \frac{i}{2}(\langle\psi\rangle^2\rho^* + \langle\psi^*\rangle^2\rho)\lambda\}],$$

where

$$\sigma = \langle\psi^*\psi\rangle - \langle\psi^*\rangle\langle\psi\rangle, \quad \rho = \langle\psi^2\rangle - \langle\psi\rangle^2. \quad (16)$$

When the propagation distance is long enough so that $|\rho| \ll \sigma$, the evaluation of the integral of (13) shows that $P(w)$ reduces to the well-known distribution of the signal $\langle\psi\rangle$ plus the Gaussian noise:

$$P(w) = \sigma^{-1} e^{-\{w + |\langle\psi\rangle|^2\}/\sigma} I_0(2\sqrt{w}|\langle\psi\rangle|/\sigma), \quad w \geq 0. \quad (17)$$

A brief survey of a quantum mechanical method to obtain $\langle 0|0 \rangle$ of (3) is also presented as another method; it is an extension of the previous paper* (for the type of equations of the Schrödinger equation) to cover a general case where the stochastic change of $q(x)$ is not a Markov process.

*

K. Furutsu, 'Application of the method of quantum mechanics in the statistical theory of waves in a fluctuation medium', Phys. Rev., 168, pp. 167 - 179 (1968).