

Notes on Graph Links

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A 3-manifold M is called a graph manifold, if there is a union T of mutually disjoint tori $T_1, \dots, T_m$ in the interior $\overset{\circ}{M}$ of M such that $M - T$ is a circle bundle over a (possibly disconnected) surface. A link $k = k_1 \cup \dots \cup k_r$ in S^3 or $S^1 \times D^2$ is a graph link, if the complement of an open tubular neighborhood of k , called exterior of k , is a graph manifold.

A fundamental family of graph links is that of solid torus links in $S^1 \times D^2 \subset S^3$ which consist of orbits or the fixed point set of an S^1 -action on $S^1 \times D^2$.

Given a link k , replacing a component of k by a solid torus link in its regular neighborhood, we have a new link k' , which is called a solid cable of k . Starting from a trivial knot and repeating to take solid cables, we have an iterated solid torus link.

Recently, Soma showed that not all graph knots

are iterated (solid) torus knots by showing that a connected sum of graph knots is again a graph knot.

This follows from the fact that an orientable regular neighborhood of a torus with an annulus attached to two parallel meridians is a product of S^1 and a three punctured sphere.

In this note, we shall characterize iterated solid torus links among graph links by means of graphs associated to them.

Let M be a graph manifold with I . Note that ∂M consists of tori. Let T' be a parallel of ∂M in $M - T$, namely, $T' = \partial V - \partial M$ for a collar neighborhood V of ∂M in $M - T$. We associate a graph Γ to $(M; T \cup T')$ as follows:

To each connected component of $M - (T \cup T')$ we associate a vertex v and denote this component M_v .

We color a vertex v in black, if $M_v \cap \partial M = \emptyset$, and in white, otherwise. To each connected component of $T \cup T'$ we associate an edge e and denote this component T_e . We define a vertex v to be incident to an edge e , if T_e is a boundary component of M_v .

Thus we have a graph $\{\mathcal{V}, \mathcal{E}\}$.

Note that

- (1) $\mathcal{V}$ is connected if and only if so is M ,
- (2) a white vertex of $\mathcal{V}$ has valency 1, and
- (3) a black vertex v of $\mathcal{V}$ has valency k_v if and only if M_v is a circle bundle over a connected k_v punctured closed surface.

By making use of the Mayer-Vietoris and the Cysin sequences, we have

Proposition 1. Suppose that M is embedded in a 3-manifold W with $H_1(W; \mathbb{Z}_2) = 0$. Then

- (4) the graph $\mathcal{V}$ is a tree, and
- (5) for each vertex v of $\mathcal{V}$ of valency k_v , M_v is a circle bundle over a k_v punctured sphere.

Now let $k \subset S^3$ be a graph link. Let M be its exterior with T . Then we have a graph $\mathcal{V}$ associated to $(M; T \cup T')$, which satisfies the conditions (1) ~ (5) above. In particular, each edge e of $\mathcal{V}$ has distinct vertices v and v' so that M_v and $M_{v'}$ belong to the distinct components of $M - T_e$.

We direct the edge e as $\overrightarrow{vv'}$ if $M_{v'}$ is contained in a solid torus in S^3 bounded by T_e . Note that if T_e is unknotted, namely, T_e separates S^3 into two solid tori, then e can be directed to both directions $\overrightarrow{vv'}$ and $\overrightarrow{v'v}$. By the Alexander's Theorem, each edge of γ can be directed.

We shall say that a vertex v of γ is minimal, if for any white vertex $v' (\neq v)$, there is a sequence of directed edges $\overrightarrow{v_1 v_2}, \dots, \overrightarrow{v_{m-1} v_m}$ such that $v_1 = v$ and $v_m = v'$.

Theorem 1. A graph link is an iterated solid torus link if and only if there is a graph structure of its exterior whose graph has a minimal vertex.

In particular, a graph knot is an iterated torus knot if and only if there is a graph structure of its exterior whose graph has a minimal vertex.

For the proof of this, we need to study graph links in $S^1 \times D^2$. In this case, we have also a graph associated to a graph structure of its exterior.

However, in general, not all edges can be directed.

In order to specify the boundary $S' \times S'$ of $S' \times D^2$, we color its associated vertex in red.

On the way to prove Theorem 1, we prove
Theorem 2. A graph link in $S' \times D^2$ is an iterated solid torus link if and only if the red vertex of the graph associated to a graph structure of its exterior is minimal.

Note that in the graph associated to an exterior of a graph link, an edge can be "reduced", if it has two black vertices one of which has valency two.

The proof of Theorems 1 and 2 is by induction on the number of vertices of valency ≥ 3 .

The crucial case is the case where the number is 1. In this case, the link must be a solid torus link.