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LOGARITHMIC ORDER AND DUAL LOGARITHMIC ORDER

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Abstract. We shall define the following four orders for strictly positive operators A and B on a Hilbert space H such that $1 \notin \sigma(A), \sigma(B)$.

Strictly logarithmic order (denoted by $A \succ_{sl} B$) is defined by $\frac{A-I}{\log A} > \frac{B-I}{B \log B}$.

Logarithmic order (denoted by $A \succ_{l} B$) is defined by $\frac{A-I}{\log A} \geq \frac{B-I}{\log B}$.

Strictly dual logarithmic order (denoted by $A \succ_{sdl} B$) is defined by $\frac{A\log A}{A-I} > \frac{B \log B}{B-I}$.

Dual Logarithmic order (denoted by $A \succ_{dl} B$) is defined by $\frac{A\log A}{A-I} \geq \frac{B \log B}{B-I}$.

Firstly we shall show direct and simplified proofs of operator monotonicity of logarithmic function $f(t) = \frac{t-1}{\log t}$ and dual logarithmic function $f^*(t) = \frac{t \log t}{t-1}$.

In what follows, let A and B be strictly positive operators on a Hilbert space H such that $1 \notin \sigma(A), \sigma(B)$. Secondary we shall show the followimg:

($\star$) $\log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

($\ddagger$) $\log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

By using these two results ($\star$) and ($\ddagger$), we summarize the following interesting contrast among $A > B > 0$, $A \geq B > 0$, $\log A > \log B$ and $\log A \geq \log B$.

(l-i) $A > B > 0 \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(l-ii) $A \geq B > 0 \Longrightarrow A^\alpha \succ_{l} B^\alpha$ for all $\alpha \in (0,1]$.

(l-iii) $\log A \geq \log B \Longrightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(l-iv) $\log A \geq \log B \Longrightarrow$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^{p\alpha} \succ_{l} B^{p\alpha}$ for all $\alpha \in (0,1]$.

(dl-i) $A > B > 0 \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(dl-ii) $A \geq B > 0 \Longrightarrow A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0,1]$.

(dl-iii) $\log A \geq \log B \Longrightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(dl-iv) $\log A \geq \log B \Longrightarrow$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$.

Finally we cite a counterexample related to (l-iii) and (dl-iii).

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. *The strictly chaotic order is defined by* $\log A > \log B$ *for strictly positive operators* A *and* B.

It is well known that the usual order $A \geq B > 0$ ensures the chaotic order $\log A \geq \log B$ since $logt$ is operator monotone function.

Also it is known by [Theorem,6] and [Example 5.1.12 and Corollary 5.1.11, 5] that

$$A \geq B > 0 \text{ ensures } \frac{A-I}{\log A} \geq \frac{B-I}{\log B}$$

and

$$A \geq B > 0 \text{ ensures } \frac{A\log A}{A-I} \geq \frac{B\log B}{B-I}$$

since $f(t) = \dfrac{t-1}{\log t}$ $(t > 0, t \neq 1)$ and $f^*(t) = \dfrac{t\log t}{t-1}$ $(t > 0, t \neq 1)$ are both operator monotone functions (see Theorem A underbelow). The function $f(t) = \dfrac{t-1}{\log t}$ $(t > 0, t \neq 1)$ is said to be "*logarithmic function*" which is widely used in the theory of heat transfer of the heat engineering and fluid mechanics. Also the function $f^*(t) = \dfrac{t\log t}{t-1}$ $(t > 0, t \neq 1)$ is said to be "*dual logarithmic function*". Related to these two operator inequalities, we shall define the following four orders for strictly positive operators A and B such that $1 \notin \sigma(A), \sigma(B)$.

Definition 1. *Let* A *and* B *be strictly positive operators on a Hilbert space* H *such that* $1 \notin \sigma(A), \sigma(B)$.

(d1) *Strictly logarithmic order* (denoted by $A \succ_{sl} B$) is defined by $\dfrac{A-I}{\log A} > \dfrac{B-I}{\log B}$.

(d2) *Logarithmic order* (denoted by $A \succ_{l} B$) is defined by $\dfrac{A-I}{\log A} \geq \dfrac{B-I}{\log B}$.

(d3) *Strictly dual logarithmic order* (denoted by $A \succ_{sdl} B$) is defined by $\dfrac{A\log A}{A-I} > \dfrac{B\log B}{B-I}$.

(d4) *Dual Logarithmic order* (denoted by $A \succ_{dl} B$) is defined by $\dfrac{A\log A}{A-I} \geq \dfrac{B\log B}{B-I}$.

2. Simplified proofs of operator monotonicity of logarithmic function and dual logarithmic function

We shall show a direct and simplified proof of the following result [Theorem , 6] and [Example 5.1.12 and Corollary 5.1.11, 5] without use of Löwner general result.

Theorem A. *The function f and f^* given by*

$$f(t) = \begin{cases} \dfrac{t-1}{\log t} & (t>0, t\neq 1) \\ 1 & (t=1) \\ 0 & (t=0) \end{cases}$$

and

$$f^*(t) = \begin{cases} \dfrac{t\log t}{t-1} & (t>0, t\neq 1) \\ 1 & (t=1) \\ 0 & (t=0) \end{cases}$$

are operator monotone functions satisfying the symmetry condition:

$$f(t) = tf(\frac{1}{t}) \text{ and } f^*(t) = tf^*(\frac{1}{t}).$$

Proof. Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. We have only to show the following (i) and (ii) since the latter half is obvious.

(i) If $A \geq B$, then $\dfrac{A-I}{\log A} \geq \dfrac{B-I}{\log B}$.

(ii) If $A \geq B$, then $\dfrac{A\log A}{A-I} \geq \dfrac{B\log B}{B-I}$.

First of all, we cite the following obvious result;

(1) $T - I = (T^{\frac{1}{n}} - I)(T^{1-\frac{1}{n}} + T^{1-\frac{2}{n}} + ... + T^{\frac{1}{n}} + I)$ for $T \geq 0$ and for any natural number n.

(2) $\lim_{n\to\infty} n(T^{\frac{1}{n}} - I) = \log T$ holds for any $T \geq 0$.

(3) If $A \geq B \geq 0$,then $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0,1]$. (Löwner-Heinz inequality)

(i).
$$\begin{aligned} \frac{A-I}{n(A^{\frac{1}{n}}-I)} &= \frac{1}{n}(A^{1-\frac{1}{n}} + A^{1-\frac{2}{n}} + ... + A^{\frac{1}{n}} + I) \text{ by (1) for any natural number } n \\ &\geq \frac{1}{n}(B^{1-\frac{1}{n}} + B^{1-\frac{2}{n}} + ... + B^{\frac{1}{n}} + I) \text{ by (3) for any natural number } n \\ &= \frac{B-I}{n(B^{\frac{1}{n}}-I)} \text{ for any natural number } n \text{ by (1)} \end{aligned}$$

tending n to ∞, so we obtain (i) by (2).

(ii).
$$\begin{aligned} \frac{n(A^{\frac{1}{n}}-I)A}{A-I} &= \frac{n}{(A^{-\frac{1}{n}} + A^{-\frac{2}{n}} + ... + A^{-1})} \text{ by (1) for any natural number } n \\ &\geq \frac{n}{(B^{-\frac{1}{n}} + B^{-\frac{2}{n}} + ... + B^{-1})} \text{ by (3) for any natural number } n \\ &= \frac{n(B^{\frac{1}{n}}-I)B}{B-I} \text{ by (1)} \end{aligned}$$

tending n to ∞, so we obtain (ii) by (2).

Remark 1. It is well known that (i) is equivalent to (ii) in Theorem A. Alternative proof of (i) in the proof of Theorem A is cited in [5]. Related to Theorem A, we remark that the following result in [Corollary 2.6, 4], [Theorem 2, 7] and [Corollary 5.1.11, 5]: let $g(t)$ be a continuous positive function such that $(0,\infty) \to (0,\infty)$. Then $g(t)$ is operator monotone function if and only if $g^*(t) = \dfrac{t}{g(t)}$ is operator monotone function. Actually, $f(t)$ and $f^*(t)$ in Theorem A satisfy this condition $f^*(t) = \dfrac{t}{f(t)}$.

3. Strictly logarithmic order $A \succ_{sl} B$ and logarithmic order $A \succ_l B$

Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Firstly we shall give Theorem 1 asserting the following

$(\star)$ $\log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

Secondary, we shall give Corollary 2 showing that there exists an interesting contrast between $A \geq B > 0$ and $A > B > 0$ related to $A \succ_{sl} B$ and $A \succ_l B$. Thirdly, we shall give some applications of two characterizations (Theorem A and Theorem B under below) of chaotic order to $A \succ_{sl} B$ and $A \succ_l B$ in Corollary 3.

Lemma 1. *Let A and B be invertible self adjoint operators on a Hilbert space H. If $A > B$, then there exists $\beta \in (0,1]$ such that the following inequality holds for all $\alpha \in (0,\beta)$;*

$$\frac{e^{\alpha A} - I}{\alpha A} > \frac{e^{\alpha B} - I}{\alpha B}, \text{ i.e., } e^{\alpha A} \succ_{sl} e^{\alpha B}.$$

Proof. There exists ε such that $A - B \geq \varepsilon > 0$. Choose α and β such that

$$0 < \alpha < Min\{\frac{\varepsilon}{2}\left(\frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||}\right)^{-1}, 1\} = \beta. \tag{4}$$

By an easy calculation, we obtain

$$\begin{aligned}
\frac{e^{\alpha A} - I}{A} - \frac{e^{\alpha B} - I}{B} &= \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} A^{n-1} - \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B^{n-1} \\
&= \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1}) \\
&= \frac{\alpha^2}{2!}(A - B) + \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1}) \\
&\geq \frac{\alpha^2}{2!}\varepsilon - \alpha^3 \left[\sum_{n=3}^{\infty} \frac{1}{n!}(||A||^{n-1} + ||B||^{n-1})\right] \\
&\geq \alpha^2 \left[\frac{\varepsilon}{2!} - \alpha(\frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||})\right] > 0 \qquad \text{by (4),}
\end{aligned}$$

so that $\frac{e^{\alpha A}-I}{\alpha A} - \frac{e^{\alpha B}-I}{\alpha B}$ holds,i.e., there exists $\beta \in (0,1]$ such that $e^{\alpha A} \succ_{sl} e^{\alpha B}$ holds for all $\alpha \in (0,\beta)$ and the proof is complete.

Theorem 1. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. If $\log A > \log B$, then there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.*

Proof. We have only to replace A by $\log A$ and also B by $\log B$ respectively in Lemma 1.

Corollary 2. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Then*

(i) *If $A > B > 0$, then there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.*
(ii) *If $A \geq B > 0$, then $A^\alpha \succ_l B^\alpha$ holds for all $\alpha \in (0,1]$.*

In Corollary 2, It is interesting to point out the contrast between $A > B > 0$ and $A \geq B > 0$.

Proof of Corollary 2. (i). We cite the following obvious and fundamental result (5)

(5) *If $A > B > 0$, then $\log A > \log B$.*

In fact if $A > B > 0$, then $A \geq B + \varepsilon > B$ for some $\varepsilon > 0$, so that $\log A \geq \log(B+\varepsilon) > \log B$, that is, (5) holds. (i) follows by (5) and Theorem 1.

(ii). If $A \geq B > 0$, then $A^\alpha \geq B^\alpha$ for all $\alpha \in (0,1]$ by Löwner-Heinz inequality and (ii) follows by the result that the function $f(t) = \frac{t-1}{\log t}$ $(t > 0, t \neq 1)$ is an operator monotone function by Theorem A,i.e., $f(A^\alpha) \geq f(B^\alpha)$ for all $\alpha \in (0,1]$, so we have (ii).

Corollary 3. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$ and $\log A \geq \log B$. Then*
(i) *For any $\delta \in (0,1]$ there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.*
(ii) *For any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^{p\alpha} \succ_l B^{p\alpha}$ for all $\alpha \in (0,1]$.*

We cite the following two results in order to give a proof of Corollary 3.

Theorem A [1][3].*Let A and B be invertible positive operators on a Hilbert space H. $\log A \geq \log B$ holds if and only if for any $\delta \in (0,1]$ there exists $\alpha = \alpha_\delta > 0$ such that $(e^\delta A)^\alpha > B^\alpha$.*

Theorem B [8].*Let A and B be invertible positive operators on a Hilbert space H. $\log A \geq \log B$ if and only if for any $p \geq 0$ there exists a $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^p \geq B^p$.*

Proof of Corollary 3.

(i). As $\log A \geq \log B$ holds, then for any $\delta \in (0,1]$, there exists $\alpha' = \alpha'_\delta > 0$ such that $(e^\delta A)^{\alpha'} > B^{\alpha'}$ by Theorem A. Then $\log e^\delta A > \log B$ by (5), so that there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$ by Theorem 1.

(ii). As $\log A \geq \log B$ holds, then for any $p \geq 0$ there exists a there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^p \geq B^p$ by Theoren B, so we have $(K_p A)^{p\alpha} \geq B^{p\alpha}$ for all $\alpha \in (0,1]$ by (ii) of Corollary 2

4. Strictly dual logarithmic order $A \succ_{sdl} B$ and dual logarithmic order $A \succ_{dl} B$

Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Firstly we shall give Theorem 4 asserting the following

(‡) $\quad \log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

Secondary, we shall give Corollary 5 showing that there exists an interesting contrast between $A \geq B > 0$ and $A > B > 0$ related to $A \succ_{sdl} B$ and $A \succ_{dl} B$. Thirdly, we shall give some applications of Theorem A and Theorem B to $A \succ_{sdl} B$ and $A \succ_{dl} B$ in Corollary 6.

Lemma 2. *Let A and B be invertible self adjoint operators on a Hilbert space H. If $A > B$, then there exists $\beta \in (0,1]$ such that the following inequality holds for all $\alpha \in (0,\beta)$;*

$$\frac{\alpha A e^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha B e^{\alpha B}}{e^{\alpha B} - I}, \text{ i.e., } e^{\alpha A} \succ_{sdl} e^{\alpha B}.$$

Proof. As $-B > -A$ holds, by applying Lemma 1 , there exists $\beta \in (0,1]$ such that

$$\frac{e^{-\alpha B} - I}{-\alpha B} > \frac{e^{-\alpha A} - I}{-\alpha A}.$$

holds for all $\alpha \in (0,\beta)$. That is, $\dfrac{e^{\alpha B} - I}{\alpha B e^{\alpha B}} > \dfrac{e^{\alpha A} - I}{\alpha A e^{\alpha A}}$ holds iff $\dfrac{\alpha A e^{\alpha A}}{e^{\alpha A} - I} > \dfrac{\alpha B e^{\alpha B}}{e^{\alpha B} - I}$ holds ,i.e.,

there exists $\beta \in (0,1]$ such that $e^{\alpha A} \succ_{sdl} e^{\alpha B}$ holds for all $\alpha \in (0,\beta)$ and the proof is complete.

Theorem 4. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$.*
If $\log A > \log B$, then there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

Proof. We have only to replace A by $\log A$ and also B by $\log B$ respectively in Lemma 2.

Corollary 5. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Then*

(i) *If $A > B > 0$, then there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.*
(ii) *If $A \geq B > 0$, then $A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0,1]$.*

In Corollary 5, It is interesting to point out the contrast between $A > B > 0$ and $A \geq B > 0$.

Proof of Corollary 5. By the same way as a proof of Corollary 2, we shall give the following proofs of (i) and (ii).
(i). If $A > B > 0$, then $\log A > \log B$ holds by (5), so that (i) follows by Theorem 4.
(ii). If $A \geq B > 0$, then $A^{\alpha} \geq B^{\alpha}$ for all $\alpha \in (0,1]$ by Löwner-Heinz inequality. The function $f^*(t) = \frac{t \log t}{t-1}$ $(t > 0, t \neq 1)$ is also an operator monotone function by Theorem A, so that $f^*(A^{\alpha}) \geq f^*(B^{\alpha})$ for all $\alpha \in (0,1]$, so we have (ii).

Corollary 6. *Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$ and $\log A \geq \log B$. Then*
(i). *For any $\delta \in (0,1]$ there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^{\delta}A)^{\alpha} \succ_{sdl} B^{\alpha}$ holds for all $\alpha \in (0,\beta)$.*
(ii). *For any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_pA)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$.*

Proof of Corollary 6. We shall obtain Corollary 6 by the same way as one in Corollary 3.
(i). As $\log A \geq \log B$ holds, then for any $\delta \in (0,1]$, there exists $\alpha' = \alpha'_\delta > 0$ such that $(e^{\delta}A)^{\alpha'} > B^{\alpha'}$ by Theorem A. Then $\log e^{\delta}A > \log B$ by (5), so that there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^{\delta}A)^{\alpha} \succ_{sdl} B^{\alpha}$ holds for all $\alpha \in (0,\beta)$ by Theorem 4.

(ii). As $\log A \geq \log B$ holds, then for any $p \geq 0$ there exists a there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_pA)^p \geq B^p$ by Theorem B, so that $(K_pA)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$ by (ii) of Corollary 5.

5. An example related to strictly logarithmic order $A \succ_{sl} B$ and strictly dual logarithmic order $A \succ_{sdl} B$

Related to (i) of Corollary 3, we consider the following problem:
(Q1) "Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^{\alpha} \succ_l B^{\alpha}$?"

Also related to (i) of Corollary 6, we consider the following problem too;
(Q2) "Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^{\alpha} \succ_{dl} B^{\alpha}$?"

In fact, we cite a counterexample to (Q1) and (Q2) as follows.

Example 1. *Take A and B as follows:*

$$\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \qquad \textit{and} \qquad \log B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}.$$

Then $\log A \geq \log B$ *holds , but*

(i) $A^\alpha \succ_l B^\alpha$ *does not hold for any* $\alpha > 0$.

(ii) $A^\alpha \succ_{dl} B^\alpha$ *does not hold for any* $\alpha > 0$.

(iii) $A^\alpha \geq B^\alpha$ *does not hold for any* $\alpha > 0$.

In fact, $\log A$ is diagonalized by $U = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ as follows;

$$U(\log A)U = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \text{ and } UAU = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^3 \end{pmatrix},$$

so that we have

$$A^\alpha = U \begin{pmatrix} e^{-2\alpha} & 0 \\ 0 & e^{3\alpha} \end{pmatrix} U \text{ and } B^\alpha = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-5\alpha} \end{pmatrix}.$$

Put $x = e^\alpha > 1$ since $\alpha > 0$. At first we show (i). By a slight elaborate calculation, we have

$$\begin{aligned} &\det\Big(\frac{A^\alpha - I}{\log A} - \frac{B^\alpha - I}{\log B}\Big) \\ &= \begin{vmatrix} \frac{5}{6} - \frac{1}{10x^2} - x + \frac{4x^3}{15} & -\frac{1}{3} + \frac{1}{5x^2} + \frac{2x^3}{15} \\ -\frac{1}{3} + \frac{1}{5x^2} + \frac{2x^3}{15} & \frac{2}{15} + \frac{1}{5x^5} - \frac{2}{5x^2} + \frac{x^3}{15} \end{vmatrix} \\ &= \frac{-1}{50x^7} + \frac{1}{6x^5} - \frac{1}{5x^4} - \frac{4}{25x^2} + \frac{2}{5x} - \frac{3x}{10} + \frac{9x^3}{50} - \frac{x^4}{15} \\ &= \frac{-(x-1)^6(10x^5 + 33x^4 + 48x^3 + 38x^2 + 18x + 3)}{150x^7} < 0 \text{ since } x > 1. \end{aligned}$$

Whence $A^\alpha \succ_l B^\alpha$ does not hold for any $\alpha > 0$, so the proof of (i) is complete.

Next we show (ii). By more elaborate calculation than (i), we obtain

$$\det\Big(\frac{A^\alpha \log A}{A^\alpha - I} - \frac{B^\alpha \log B}{B^\alpha - I}\Big)$$

$$= \begin{vmatrix} \dfrac{7x^3+9x^2+x-2}{5(x^3+2x^2+2x+1)} & \dfrac{2(3x^3+6x^2+4x+2)}{5(x^3+2x^2+2x+1)} \\ \dfrac{2(3x^3+6x^2+4x+2)}{5(x^3+2x^2+2x+1)} & \dfrac{3x^4+3x^3+8x^2+8x+8}{5(x^4+x^3-x-1)} + \dfrac{5(x-1)}{-x^6+x^5+x-1} \end{vmatrix}$$

$$= \left(\frac{7x^3+9x^2+x-2}{5(x^3+2x^2+2x+1)}\right)\left(\frac{3x^4+3x^3+8x^2+8x+8}{5(x^4+x^3-x-1)} + \frac{5(x-1)}{-x^6+x^5+x-1}\right)$$

$$-\left(\frac{2(3x^3+6x^2+4x+2)}{5(x^3+2x^2+2x+1)}\right)^2$$

$$= \frac{-(x-1)^2(3x^5+18x^4+38x^3+48x^2+33x+10)}{5(x+1)(x^2+x+1)(x^4+x^3+x^2+x+1)} < 0 \text{ since } x > 1$$

Whence $A^\alpha \succ_{dl} B^\alpha$ does not hold for any $\alpha > 0$, so the proof of (ii) is complete.

Incidentally, we remark that this example also shows that $\log A \geq \log B$ does not ensure $A^\alpha \geq B^\alpha$ for any $\alpha > 0$. Actually we have

$$\det(A^\alpha - B^\alpha)$$

$$= \begin{vmatrix} \dfrac{1}{5x^2} - x + \dfrac{4x^3}{5} & \dfrac{2(x^5-1)}{5x^2} \\ \dfrac{2(x^5-1)}{5x^2} & \dfrac{x^8+4x^3-5}{5x^5} \end{vmatrix}$$

$$= \frac{-1}{5x^7} + x^{-4} - \frac{4}{5x^2} - \frac{4}{5x} + x - \frac{x^4}{5}$$

$$= \frac{-(x-1)^4(x+1)(x^2+x+1)(x^4+2x^3+4x^2+2x+1)}{5x^7} < 0 \text{ since } x > 1,$$

that is, $A^\alpha \geq B^\alpha$ does not hold for any $\alpha > 0$, so (iii) is shown. In [2], there is another nice example that $\log A \geq \log B$ does not ensure $A^\alpha \geq B^\alpha$ for any $\alpha > 0$. In fact, we construct Example 1 inspired by an excellent method in [2].

6. Concluding remarks

Let A and B be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. We can obtain the following interesting contrast among $A > B > 0$, $A \geq B > 0$, $\log A > \log B$ and $\log A \geq \log B$ by summarizing our results in this paper.

($\star$) $\log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

($\ddagger$) $\log A > \log B \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(l-i) $A > B > 0 \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(l-ii) $A \geq B > 0 \Longrightarrow A^\alpha \succ_l B^\alpha$ for all $\alpha \in (0,1]$.

(l-iii) $\log A \geq \log B \Longrightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(l-iv) $\log A \geq \log B \Longrightarrow$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_pA)^{p\alpha} \succ_l B^{p\alpha}$ for all $\alpha \in (0,1]$.

(dl-i) $A > B > 0 \Longrightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.
(dl-ii) $A \geq B > 0 \Longrightarrow A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0,1]$.

(dl-iii) $\log A \geq \log B \Longrightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(dl-iv) $\log A \geq \log B \Longrightarrow$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$.

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References

[1] M.Fujii,J.F.Jiang, and E.Kamei, Characterization of chaotic order and its application to Furuta inequality, Proc. Amer. Math. Soc.,**125**(1997),3655-3658.

[2] M.Fujii,J.F.Jiang,E.Kamei and K.Tanahashi, A characterization of chaotic order and a problem, J. of Inequality and Appl., **2** (1998),149-156.

[3] M.Fujii,J.F.Jiang, and E.Kamei, Characterization of chaotic order and its application to Furuta's type operator inequalities, Linear and Multilinear Algebra, **43**(1998),339-349.

[4] F.Hansen and G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann., **258**(1982),229-241.

[5], F.Hiai and K.Yanagi, Hilbert space and linear operators,(in Japanese) (1995).

[6] F.Kubo, On logarithmic operator means, Tenth Symposium on Applied Functional Analysis (1987),47-61.

[7] M.K.Kwong, Some results on matrix monotone functions, Linear Alg and Appl., **118**(1989), 129-153.

[8] T.Yamazaki and M.Yanagida, Characterizations of chaotic order associated with Kantorovich inequality, Scienticae Mathematicae, **2**(1999),37-50.

7. Appendix

Simple proof of the concavity on operator entropy $f(A) = -AlogA$

A capital letter means a bounded linear and *strictly positive* operator on a Hilbert space. Here we shall give a simple proof of the following well known and excellent result obtained by [1] and [2] independently.

Theorem A. *$f(A) = -AlogA$ is concave function for any $A > 0$.*

Proof. Firstly we recall the following obvious result

$$(*) \qquad \lim_{n\to\infty} (T^{\frac{-1}{n}} - I)n = -logT \qquad \text{for any } T > 0.$$

As $g(t) = t^q$ is operator concave for $q \in [0,1]$, then for $A > 0$, $B > 0$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$

$$\Big(\alpha A + \beta B\Big)^{1-\frac{1}{n}} \geq \alpha A^{1-\frac{1}{n}} + \beta B^{1-\frac{1}{n}} \qquad \text{for any natural number } n$$

so we obtain

$$(\alpha A + \beta B)\Big((\alpha A + \beta B)^{-\frac{1}{n}} - I\Big)n \geq \alpha A(A^{-\frac{1}{n}} - I)n + \beta B(B^{-\frac{1}{n}} - I)n$$

tending $n \to \infty$, we have

$$-(\alpha A + \beta B)log(\alpha A + \beta B) \geq (-\alpha AlogA - \beta BlogB) \qquad \text{by } (*)$$

that is,

$$f(\alpha A + \beta B) \geq \alpha f(A) + \beta f(B)$$

so the proof is complete.

References

[1] Ch.Davis, Operator-valued entropy of a quantum mechanical measurement, Proc. Japan Acad., **37** (1961),533-538.

[2] M.Nakamura and H.Umegaki, A note on the entropy for operator algebras, Proc. Japan Acad.,**37** (1961),149-154.

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