

Certain subclasses of starlike functions of order α

関根 忠行 (日本大学薬学部) * 尾和 重義 (近畿大学理工学部) †

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let S denote the subclass of A consisting of functions which are univalent in the unit disk U .

A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$$

for some $\alpha (0 \leq \alpha < 1)$ and for all z in U .

We denote by $S^*(\alpha)$ the subclass of S consisting of all starlike functions of order α in the unit disk U .

Let $S_1(\alpha, a)$, $S_2(\alpha, a)$ and $S_3(\alpha, a)$ denote the subclasses of S consisting of functions which satisfy

$$\left| \frac{zf'(z)}{f(z)} - a \right| < a - \alpha \quad (0 \leq \alpha < 1),$$

for $1 \leq a \leq 2$, $a > 2$ and $\frac{1+\alpha}{2} < a < 1$, respectively.

It is clear that $S_i(\alpha, a) \subset S^*(\alpha)$ and $S_i(\alpha, a) \subseteq S_i(\alpha, b) (a \leq b)$ for $i = 1, 2, 3$.

In [1. Theorem 1], Silverman have showed the sufficient condition for a function in S belongs to $S^*(\alpha)$. In this paper, we shall reconsider the sufficient condition by using the subclasses $S_i(\alpha, a) (i = 1, 2, 3)$ of $S^*(\alpha)$ defined above. Further, we determine the distortion theorems for certain subclasses $\tilde{S}_i(\alpha, a)$ of $S_i(\alpha, a) (i = 1, 2, 3)$.

2. Coefficient inequality

We shall now prove the following theorems in a same way of Theorem 1 of Silverman [1].

Theorem 1 Let $f(z) \in S$, $0 \leq \alpha < 1$ and $1 \leq a \leq 2$. If

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

then $f(z) \in S_1(\alpha, a)$.

*Tadayuki Sekine(Nihon University)

†Shigeyoshi Owa(Kinki University)

Proof. We have

$$\begin{aligned}
 \left| \frac{zf'(z)}{f(z)} - a \right| &= \left| \frac{1 - a + \sum_{n=2}^{\infty} (n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\
 &= \left| \frac{a - 1 - \sum_{n=2}^{\infty} (n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\
 &\leq \frac{a - 1 + \sum_{n=2}^{\infty} (n-a)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n||z|^{n-1}} \\
 &\leq \frac{a - 1 + \sum_{n=2}^{\infty} (n-a)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}.
 \end{aligned}$$

This last expression is bounded above by $a - \alpha$ if

$$a - 1 + \sum_{n=2}^{\infty} (n-a)|a_n| \leq (a - \alpha) \left(1 - \sum_{n=2}^{\infty} |a_n| \right). \quad (1)$$

Since the hypothesis of Theorem 1 is equivalent to the coefficient inequality (1), Theorem 1 is proved.

In the proof of Theorem 1 of Silverman [1], only the case of $a=1$ of the Theorem 1 above was considered.

Theorem 2 Let $f(z) \in S$, $0 \leq \alpha < 1$ and $a > 2$. If

$$\sum_{n=2}^j (2a - n - \alpha)|a_n| + \sum_{n=j+1}^{\infty} (n - \alpha)|a_n| \leq 1 - \alpha$$

for $j < a \leq j+1$, then $f(z) \in S_2(\alpha, a)$.

Proof. We have

$$\left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{a - 1 + \sum_{n=2}^j (a-n)a_n z^{n-1} - \sum_{n=j+1}^{\infty} (n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right|$$

$$\begin{aligned}
& \leq \frac{a-1 + \sum_{n=2}^j (a-n)|a_n||z|^{n-1} + \sum_{n=j+1}^{\infty} (n-a)|a_n||z|^{n-1}}{1 - \sum_{n=2}^j |a_n||z|^{n-1} - \sum_{n=j+1}^{\infty} |a_n||z|^{n-1}} \\
& \leq \frac{a-1 + \sum_{n=2}^j (a-n)|a_n| + \sum_{n=j+1}^{\infty} (n-a)|a_n|}{1 - \sum_{n=2}^j |a_n| - \sum_{n=j+1}^{\infty} |a_n|}.
\end{aligned}$$

This last expression is bounded above by $a - \alpha$ if

$$a-1 + \sum_{n=2}^j (a-\alpha)|a_n| + \sum_{n=j+1}^{\infty} (n-\alpha)|a_n| \leq (a-\alpha) \left(1 - \sum_{n=2}^j |a_n| - \sum_{n=j+1}^{\infty} |a_n| \right). \quad (2)$$

(2) is equivalent to the hypothesis of Theorem 2. This completes the proof.

We can prove the following theorem by using the same way of Theorem 1 and Theorem 2.

Theorem 3 Let $f(z) \in S$, $0 \leq \alpha < 1$ and $\frac{1+\alpha}{2} < a < 1$. If

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 2a-1-\alpha,$$

then $f(z) \in S_3(a, \alpha)$.

3. Distortion theorems

We denote by $\tilde{S}_1(\alpha, a)$ the subclass of $S(\alpha, a)$ ($0 \leq \alpha < 1, 1 \leq a \leq 2$) consisting of functions which satisfy

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1-\alpha. \quad (3)$$

Since a function

$$f(z) = z + \frac{1-\alpha}{n-\alpha} z^n \quad (4)$$

in S belongs to $S(\alpha, a)$ and satisfies the coefficient inequality (3), this function (4) belongs to $\tilde{S}_1(\alpha, a)$. Therefore, the subclass $\tilde{S}_1(\alpha, a)$ is not empty.

Theorem 4 If $f(z) \in \tilde{S}_1(\alpha, a)$ ($0 \leq \alpha < 1, 1 \leq a \leq 2$), then

$$|z| - \frac{1-\alpha}{2-\alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{2-\alpha} |z|^2.$$

Equality holds for the function

$$f(z) = z + \frac{1-\alpha}{2-\alpha} z^2.$$

Proof. By the assumption, note that

$$(2 - \alpha) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

that is, that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \alpha}{2 - \alpha}.$$

Thus, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq |z| + |z|^2 \frac{1 - \alpha}{2 - \alpha}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\geq |z| - |z|^2 \frac{1 - \alpha}{2 - \alpha}. \end{aligned}$$

We denote by $\tilde{S}_2(\alpha, a)$ the subclass of $S(\alpha, a)$ ($0 \leq \alpha < 1, a > 2$) consisting of functions which satisfy

$$\sum_{n=2}^j (2a - n - \alpha) |a_n| + \sum_{n=j+1}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

where $j < a \leq j + 1$. Just as in the case of $\tilde{S}_1(\alpha, a)$, by taking a function

$$f(z) = z + \frac{1 - \alpha}{2(2a - k - \alpha)} z^k + \frac{1 - \alpha}{2(l - \alpha)} z^l,$$

where $2 \leq k \leq j$ and $j + 1 \leq l < \infty$, we see that $\tilde{S}_2(\alpha, a)$ is not empty.

Theorem 5 If $f(z) \in \tilde{S}_2(\alpha, a)$ ($0 \leq \alpha < 1, a > 2$), then

$$|z| - \frac{1 - \alpha}{2a - j - \alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{1 - \alpha}{2a - j - \alpha} |z|^2, \quad (5)$$

for $a - \frac{1}{2} \leq j < a$

and

$$|z| - \frac{1-\alpha}{j+1-\alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{j+1-\alpha}|z|^2, \quad (6)$$

for $a-1 \leq j < a - \frac{1}{2}$.

These results are sharp.

Proof. Since $2a-n-\alpha$ and $n-\alpha$ are decreasing and increasing for n , respectively, by the assumption we note that

$$(2a-j-\alpha) \sum_{n=2}^j |a_n| + (j+1-\alpha) \sum_{n=j+1}^{\infty} |a_n| \leq 1-\alpha.$$

Then, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1-\alpha}{2a-j-\alpha} \quad (a - \frac{1}{2} \leq j < a), \quad (7)$$

and

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1-\alpha}{j+1-\alpha} \quad (a-1 \leq j < a - \frac{1}{2}). \quad (8)$$

By using (7), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq |z| + |z|^2 \frac{1-\alpha}{2a-j-\alpha}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\geq |z| - |z|^2 \frac{1-\alpha}{2a-j-\alpha}. \end{aligned}$$

Equality of (5) holds for the function

$$f(z) = z + \frac{1-\alpha}{2a-j-\alpha} z^2.$$

Using coefficient inequality (8) we can obtain the latter of Theorem 1 similarly. Equality of (6) holds for the function

$$f(z) = z + \frac{1-\alpha}{j+1-\alpha}z^2.$$

Let $\tilde{S}_3(\alpha, a)$ be the subclass of $S(\alpha, a)$ ($0 \leq \alpha < 1, \frac{1+\alpha}{2} < a < 1$) consisting of functions which satisfy

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 2a-1-\alpha.$$

Then, the subclass $\tilde{S}_3(\alpha, a)$ is not empty, because there exists a function

$$f(z) = z + \frac{2a-1-\alpha}{n-\alpha}z^n$$

in $\tilde{S}_3(\alpha, a)$.

We can prove the following theorem in the same way of Theorem 4 and 5.

Theorem 6 If $f(z) \in \tilde{S}_3(\alpha, a)$ ($0 \leq \alpha < 1, \frac{1+\alpha}{2} < a < 1$), then

$$|z| - \frac{2a-1-\alpha}{2-\alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{2a-1-\alpha}{2-\alpha}|z|^2.$$

Equality holds for the function

$$f(z) = z + \frac{2a-1-\alpha}{2-\alpha}z^2.$$

References

- [1] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1975), 109-116.

Tadayuki Sekine
Department of Mathematics
College of Pharmacy
Nihon University
7-1, Narashinodai 7-Chome, Funabashi-shi, Chiba 274
Japan

Shigeyoshi Owa
Department of Mathematics
College of Science and Technology
Kinki University
Higashi-Osaka, Osaka 577
Japan