

On Mod-separable extensions of modules

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Abstract

It is well known that separable extensions of noncommutative rings are characterized by the derivation. In this paper, using the generalized derivation, we shall introduce **Mod**-separable extensions of modules as a generalization of separable ring extensions. Moreover, we shall show that the **Mod**-separability is a hereditary property with respect to Morita equivalence in ring extensions.

1 Introduction and Preliminaries

Let A/B be a ring extension and U an A -bimodule. An additive map $\delta : A \rightarrow U$ is called a *derivation* if it satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for any $x, y \in A$. Moreover, a derivation $\delta : A \rightarrow U$ is called a *B-derivation* if $B \subset \ker \delta$. By $\text{Der}(A, U)$ (resp. $\text{Der}_B(A, U)$) we denote the set of all derivations (resp. B -derivations) from A to U . For a fixed element $u \in U$, it is easy to see that a map $\delta_u : A \rightarrow U$ defined by $\delta_u(x) = xu - ux$ ($x \in A$) is a derivation. Such δ_u is called an *inner derivation* by $u \in U$. We denote by $\text{Inn}(A, U)$ the set of all inner derivations from A to U .

A ring extension A/B is called *separable* if the A - A -homomorphism from $A \otimes_B A$ to A defined by $\sum_i x_i \otimes y_i \mapsto \sum_i x_i y_i$ splits. The following is well known.

Proposition 1.1. [1, Satz 4.2] *A ring extension A/B is separable if and only if $\text{Der}_B(A, U) \subset \text{Inn}(A, U)$ for any A -bimodule U .*

Let A be a ring and S, U be A -bimodules. In [4], H. Komatsu introduced the notion of generalized derivations from S to U as follows : an additive map $\delta : S \rightarrow U$ is called a *generalized derivation* if it satisfies $\delta(asb) = \delta(as)b + a\delta(sb) - a\delta(s)b$ for any $s \in S$ and $a, b \in A$. By $\text{GDer}_{(A}S_A, {}_AU_A)$ we denote the set of all generalized derivations of A -bimodules from S to U . It is obvious that $\text{Der}(A, U) \subset \text{GDer}_{(A}A_A, {}_AU_A)$ for any ring A and any A -bimodule U .

Let A/B be a ring extension, S an A -bimodule, and T a submodule of S as a B -bimodule. In this paper, we shall define generalized T -derivations as a generalization of B -derivations (Definition 2.1). Using generalized T -derivations, we shall introduce the notion of **Mod**-separable extensions of modules which extend the notion of separable extensions of rings (Definition 2.6). Moreover, we shall show that the **Mod**-separability is a hereditary property with respect to Morita equivalence in ring extensions (Theorem 3.3).

2 Mod-separable extension of modules

Throughout this section, let A/B be a ring extension and S/T an additive group extension (i.e., T is a subgroup of an additive group S). Noting that there always holds $\text{Der}_B(A, U) \subset \text{Hom}({}_B A_B, {}_B U_B)$ for any A -bimodule U , we give the following definition.

Definition 2.1. (1) An additive group extension S/T is called A/B -extension if S is an A -bimodule and T is a submodule of S as a B -bimodule.

(2) For an A/B -extension S/T and an A -bimodule U , a map $\delta : S \rightarrow U$ is called a *generalized T -derivation* if $\delta \in \text{GDer}({}_A S_A, {}_A U_A) \cap \text{Hom}({}_B S_B, {}_B U_B)$ and $T \subset \ker \delta$. We denote by $\text{GDer}_T({}_A S_A, {}_A U_A)$ the set of all generalized T -derivations from S to U .

Remark 2.2. Any ring extension A/B can be considered as an A/B -extension. It is easy to see that $\text{GDer}_B({}_A A, {}_A U_A) = \text{Der}_B(A, U)$ for any ring extension A/B and any A -bimodule U .

Example 2.3. By $\mathbb{M}_{k \times \ell}(R)$ we denote the set of all $k \times \ell$ matrices over a commutative ring R , where k and ℓ are positive integers. Let B be a commutative ring, $A = B[x]$, $D = \frac{d}{dx}$, $S = \mathbb{M}_{k \times \ell}(A)$, and $T = \mathbb{M}_{k \times \ell}(B)$. We regard S (resp. T) as an A -bimodule (resp. a B -bimodule) by the usual scalar multiplication of matrices. We define a map $\delta : S \rightarrow S$ by

$$\delta((s_{ij})) = ((Ds_{ij})_{ij}) \quad ((s_{ij}) \in S),$$

where (s_{ij}) is a $k \times \ell$ matrix whose (i, j) entry is s_{ij} . We can see that δ is a generalized T -derivation from S to S .

Let S, U be A -bimodules, $\varphi \in \text{Hom}({}_A S_A, {}_A A_A)$, and $u \in U$. We define a map $\delta_{\varphi, u} : S \rightarrow U$ by

$$\delta_{\varphi, u}(s) = \varphi(s)u - u\varphi(s) \quad (s \in S).$$

So it is easy to see that $\delta_{\varphi, u}$ is a generalized derivations. We give then the following definition.

Definition 2.4. For A -bimodules S and U , a map $\delta : S \rightarrow U$ is called a *generalized inner derivation* if there exist $\varphi \in \text{Hom}({}_A S_A, {}_A A_A)$ and $u \in U$ such that

$$\delta(s) = \varphi(s)u - u\varphi(s) \quad (s \in S).$$

By $\text{GIIn}({}_A S_A, {}_A U_A)$ we denote the set of all generalized inner derivations from S to U .

Remark 2.5. There always holds $\text{Inn}(A, U) \subset \text{GIIn}({}_A A_A, {}_A U_A)$ for any ring A and any A -bimodule U .

Using the generalized derivaions, we introduce the following definition which is a generalization of Proposition 1.1.

Definition 2.6. An A/B -extension S/T is called **Mod-separable** if, for any A -bimodule U , there holds

$$\text{GDer}_T({}_A S_A, {}_A U_A) \subset \text{GInn}({}_A S_A, {}_A U_A).$$

Remark 2.7. By Remark 2.2 and Remark 2.5, we see that every separable ring extension is Mod-separable.

Example 2.8. An A/B -extension S/T in Example 2.3 is not Mod-separable. Because a generalized T -derivation δ in Example 2.3 is not a generalized inner derivation.

3 Mod-separability and Morita equivalence in ring extensions

For two rings A, A' , an A - A' -bimodule ${}_A M_{A'}$ is called a *Morita module* if ${}_A M$ is a progenerator and $\text{End}({}_A M) \cong A'$ as a ring. Two ring extensions A/B and A'/B' are *Morita equivalent* if there exist Morita modules ${}_A M_{A'}$ and ${}_B N_{B'}$ such that ${}_A A \otimes_B N_{B'} \cong {}_A M_{B'}$. The following proposition shows that the separability is a hereditary property with respect to Morita equivalence in ring extensions.

Proposition 3.1. [3, Theorem 5] Assume that two ring extensions A/B and A'/B' are Morita equivalent. If A/B is separable then A'/B' is also separable.

Now we assume that two ring extensions $A/B, A'/B'$ are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \text{Hom}^r({}_B N, {}_B B)$. As was shown in [3] or [6], there holds $A' \cong N^* \otimes_B A \otimes_B N$ and $B' \cong N^* \otimes_B B \otimes_B N (\cong N^* \otimes_B N)$ as rings. So, Proposition 3.1 is equivalent to the following.

Proposition 3.2. Assume that two ring extensions $A/B, A'/B'$ are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \text{Hom}^r({}_B N, {}_B B)$. If A/B is separable then $N^* \otimes_B A \otimes_B N / N^* \otimes_B B \otimes_B N$ is also separable.

Under the conditions of the above proposition, suppose further that S/T is an A/B -extension. We can see then that $N^* \otimes_B S \otimes_B N$ (resp. $N^* \otimes_B T \otimes_B N$) is an $N^* \otimes_B A \otimes_B N (\cong A')$ -bimodule (resp. a $N^* \otimes_B B \otimes_B N (\cong B')$ -bimodule). Hence $N^* \otimes_B S \otimes_B N / N^* \otimes_B T \otimes_B N$ is an A'/B' -extension. The following is the main theorem of this article which is a generalization of Proposition 3.2.

Theorem 3.3. Assume that two ring extensions $A/B, A'/B'$ are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \text{Hom}^r({}_B N, {}_B B)$. If an A/B -extension S/T is Mod-separable then an A'/B' -extension $N^* \otimes_B S \otimes_B N / N^* \otimes_B T \otimes_B N$ is also Mod-separable.

To prove Theorem 3.3, we need the following lemma.

Lemma 3.4. *Assume that two ring extensions A/B , A'/B' are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \text{Hom}^r({}_B N, {}_B B)$. In addition, we suppose that S/T is an A/B -extension. Then, for any A -bimodule U , there exists the following isomorphism as an additive group:*

$$\begin{aligned} \text{GDer}_T({}_A S_A, {}_A U_A) &\xrightarrow{\cong} \text{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'}) \\ \delta &\longmapsto 1 \otimes \delta \otimes 1 \end{aligned}$$

Outline of the proof of Theorem 3.3. Assume that two ring extensions A/B , A'/B' are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \text{Hom}^r({}_B N, {}_B B)$. Moreover, we suppose that an A/B -extension S/T is **Mod**-separable and U' is an A' -bimodule. Then we can see that there is an A -bimodule such that ${}_{A'} U'_{A'} \cong {}_{A'} N^* \otimes_B U \otimes_B N_{A'}$. Hence it suffices to show that

$$\begin{aligned} &\text{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'}) \\ &\subset \text{GInn}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'}). \end{aligned}$$

Let δ' be in $\text{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$. By Lemma 3.4, there exists $\delta \in \text{GDer}_T({}_A S_A, {}_A U_A)$ such that $\delta' = 1 \otimes \delta \otimes 1$. Since S/T is **Mod**-separable, there exist $\varphi \in \text{Hom}({}_A S_A, {}_A A_A)$ and $u \in U$ such that $\delta(s) = \varphi(s)u - u\varphi(s)$ for any $s \in S$. Now we let $s' = \sum_k \rho_k \otimes s_k \otimes n_k \in N^* \otimes_B S \otimes_B N$, $\varphi' = 1 \otimes \varphi \otimes 1 \in \text{Hom}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} A'_{A'})$, and $u' = \sum_j f_j \otimes u \otimes m_j \in N^* \otimes_B U \otimes_B N$, where $\{f_j, m_j\}$ is a dual basis of ${}_B N$. Noting that $(bu - ub)\varphi(s) = \varphi(s)(bu - ub) = 0$ ($\forall s \in S, \forall b \in B$), we can see that $\delta'(s') = \varphi'(s')u' - u'\varphi'(s')$. Therefore $\delta' \in \text{GInn}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$. This complete the proof. $\square$

Finally, we state some open problems.

- What is the condition for **Mod**-separability that corresponds to equivalent conditions of separable ring extensions? (e.q., A/B is separable iff there exists $\sum_i x_i \otimes y_i \in A \otimes_B A$ such that $\sum_i a x_i \otimes y_i = \sum_i x_i \otimes y_i a$ ($\forall a \in A$) and $\sum_i x_i y_i = 1$).
- Can well known extensions (e.q., G -Galois extension, Frobenius extension, Hirata separable extension) be generalized to the case of module extensions?
- Can “Morita equivalence of module extension” be introduced as a generalization of Morita equivalence of ring extension?

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