

PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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Abstract

Let $A(p)$ denote the class of functions $f(z)$ which are analytic and p -valent in the unit disk U . A new subclass $\Omega(\alpha, \beta; \gamma)$ of $A(p)$ consisting of analytic and p -valent functions $f(z)$ associated with the certain integral operator Q_β^α which is the generalization of the integral operator studied by I.B.Jung, Y.C.Kim and H.M.Srivastava (J. Math. Anal. Appl. **248**(2000), 475 - 481) is introduced. Some interesting properties of the operator Q_β^α for functions $f(z)$ belonging to $A(p)$ are investigated.

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1. Introduction.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic and p -valent in the unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Let $S_p^*(\gamma)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > p\gamma$$

for $0 \leq \gamma < 1$ and $z \in U$. A function in $S_p^*(\gamma)$ is called p -valent starlike of order γ in U .

Let $f(z)$ and $g(z)$ be analytic in U . Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1$ ($z \in U$) and $g(z) = f(w(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator:

$$Q_{\beta}^{\alpha} f(z) = \left(\begin{matrix} \alpha + \beta \\ \beta \end{matrix} \right) \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha > 0, \beta > -1; f \in A(1)). \quad (1.2)$$

They also showed that

$$Q_{\beta}^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n) \Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n) \Gamma(\beta + 1)} a_n z^n.$$

It follows from (1.3) that one can define the operator Q_{β}^{α} for $\alpha \geq 0$ and $\beta > -1$. Some interesting subclasses of analytic function, associated with the operator Q_{β}^{α} , have been considered recently by Jung et al.[3], Aouf et al.[1], Li[5], Liu[6] and others.

Motivated by Jung, Kim and Srivastava's work [3], we now consider a linear operator $Q_{\beta}^{\alpha} : A(p) \rightarrow A(p)$ as following:

$$Q_{\beta}^{\alpha} f(z) = \left(\begin{matrix} p + \alpha + \beta - 1 \\ p + \beta - 1 \end{matrix} \right) \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha \geq 0, \beta > -1; f \in A(p)). \quad (1.3)$$

We note that

$$Q_{\beta}^{\alpha} f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p + n + \beta) \Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta) \Gamma(p + \beta)} a_{p+n} z^{p+n}$$

$$(\alpha \geq 0, \beta > -1; f \in A(p)). \quad (1.4)$$

It is easily verified from the definition (1.4) that

$$z(Q_\beta^\alpha f(z))' = (\alpha + \beta + p - 1)Q_\beta^{\alpha-1}f(z) - (\alpha + \beta - 1)Q_\beta^\alpha f(z). \quad (1.5)$$

When $p = 1$, the identity (1.5) is given in [3]. One can easily see that the operator Q_β^α has an inverse operator $Q_{\beta+\alpha}^{-\alpha}$ and Q_β^0 is an unit operator.

A function $f(z) \in A(p)$ is said to be in the class $\Omega(\alpha, \beta; \gamma)$ if it satisfies the condition

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p} \prec \frac{p+p(1-2\gamma)z}{1-z} \quad (1.6)$$

for all $z \in U$ and $0 \leq \gamma < 1$.

In this paper, we shall show the extreme points of the closed convex hull of the class $\Omega(\alpha, \beta; \gamma)$. It is then used to determine the coefficient bounds.

In the sequel, we denote the closed convex hull of a class H by coH . Also, let $E(coH)$ denote the set of all extreme points of H .

2. Main Results.

In order to derive our main results, we shall need the following lemmas.

Lemma 1 ([4]). $E(coS_p^*(\alpha))$ consists of the functions given by

$$\frac{z^p}{(1-xz)^{2p(1-\gamma)}} = z^p + \sum_{n=1}^{\infty} \frac{(2p-2p\gamma)_n}{n!} x^n z^{p+n} \quad (z \in U), \quad (2.1)$$

where $(a)_n = a(a+1) \cdots (a+n-1)$, $x \in C$ and $|x| = 1$.

Lemma 2 ([9]). The function $(1-z)^\rho \equiv e^{\rho \log(1-z)}$, $\rho \neq 0$, is univalent in U if and only if ρ is either in the closed disk $|\rho-1| \leq 1$ or in the closed disk $|\rho+1| \leq 1$.

Lemma 3 ([7]). Let $q(z)$ be univalent in U and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that

(1) $Q(z)$ is starlike (univalent) in U ;

(2) $Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in U)$.

If $p(z)$ is analytic in U , with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \quad (2.2)$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1. A function $f(z) \in A(p)$ is in $\Omega(\alpha, \beta; \gamma)$ if and only if $f(z)$ can be expressed as

$$f(z) = Q_{\beta+\alpha}^{-\alpha} \left\{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\}, \quad (2.3)$$

where μ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$.

Proof. Let $f(z) \in \Omega(\alpha, \beta; \gamma)$. Then by Herglotz formula [2], we have

$$\frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{pz^p}{1 - z^p} = p(1 - \gamma) \int_X \frac{1 + xz}{1 - xz} d\mu(x) + p\gamma, \quad (2.4)$$

where μ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$. By means of the identity

$$\frac{d}{dz} \log \frac{Q_{\beta}^{\alpha} f(z)}{z^p (1 - z^p)} = \frac{1}{z} \left[\frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{pz^p}{1 - z^p} - p \right], \quad (2.5)$$

(2.4) yields

$$Q_{\beta}^{\alpha} f(z) = z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)]. \quad (2.6)$$

Thus

$$f(z) = Q_{\beta+\alpha}^{-\alpha} \{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \}.$$

Now the proof is complete.

Theorem 2. Let $0 \leq \gamma_1 < \gamma_2 < 1$, then $\Omega(\alpha, \beta; \gamma_2) \subset \Omega(\alpha, \beta; \gamma_1)$.

Proof. We define a linear operator on $\Omega(\alpha, \beta; \gamma)$ as following:

$$T_{\gamma}(f) = \frac{Q_{\beta}^{\alpha} f(z)}{1 - z^p} \quad (z \in U). \quad (2.7)$$

Then T_{γ} is a linear homeomorphism from $\Omega(\alpha, \beta; \gamma)$ to $S_p^*(\gamma)$. It is well-known that $S_p^*(\gamma_2) \subset S_p^*(\gamma_1)$ for $0 \leq \gamma_1 < \gamma_2 < 1$. The result follows immediately.

Theorem 3. (i) The extreme points of $co\Omega(\alpha, \beta; \gamma)$ are given by the functions

$$f_z(z) = Q_{\beta+\alpha}^{-\alpha} \left\{ \frac{z^p (1 - z^p)}{(1 - xz)^{2p(1-\gamma)}} \right\} \quad (x \in C, |x| = 1; z \in U). \quad (2.8)$$

$$(ii) \text{ } Co \Omega(\alpha, \beta; \gamma) = \{f : f(z) = \int_X f_x(z) d\mu(x)\}, \quad (2.9)$$

where μ varies over the probability measures defined on the unit circle X .

Proof. Since T_γ defined by (2.7) is a linear homeomorphism from $\Omega(\alpha, \beta; \gamma)$ to $S_p^*(\gamma)$, it preserves extreme points. By making use of Lemma 1, the results follow at once.

According to Theorem 3 and Lemma 1, we have the following corollaries.

Corollary 1. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma)$. Then

$$|a_{p+n}| \leq \begin{cases} \frac{(2p-2p\gamma)_n}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & 1 \leq n < p, \\ \frac{(2p-2p\gamma)_{n-p} \prod_{k=1}^p (2p-2p\gamma+n-k) - \prod_{k=1}^p (n-p+k)}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & n \geq p. \end{cases}$$

The result is sharp.

Corollary 2. Let $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma)$. Then for $|z| = r < 1$.

$$|f(z)| \leq r^p + \sum_{n=1}^{p-1} \frac{(2p-2p\gamma)_n}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n} \\ + \sum_{n=p}^{\infty} \frac{(2p-2p\gamma)_{n-p} \prod_{k=1}^p (2p-2p\gamma+n-k) - \prod_{k=1}^p (n-p+k)}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n}.$$

The result is sharp.

Theorem 4. Let $f(z) \in \Omega(\alpha, \beta; \gamma)$. Let ρ be a complex number with $\rho \neq 0$ and satisfy either $|2p\rho(1-\gamma) + 1| \leq 1$ or $|2p\rho(1-\gamma) - 1| \leq 1$. Then

$$\left(\frac{Q_{\beta}^{\alpha} f(z)}{z^p(1-z^p)} \right)^{\rho} \prec \frac{1}{(1-z)^{2p\rho(1-\gamma)}} = q(z) \quad (z \in U), \quad (2.10)$$

where $q(z)$ is the best dominant.

Proof. Let

$$p(z) = \left(\frac{Q_{\beta}^{\alpha} f(z)}{z^p(1-z^p)} \right)^{\rho}, \quad (2.11)$$

then $p(z)$ in analytic is U with $p(0) = 1$. Differentiating (2.11) logarithmically we have

$$\frac{zp'(z)}{p(z)} = \rho \left(\frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{pz^p}{1-z^p} - p \right). \quad (2.12)$$

Since $f(z) \in \Omega(\alpha, \beta; \gamma)$, (2.12) is equivalent to

$$p + \frac{zp'(z)}{\rho p(z)} \prec \frac{p + p(1 - 2\gamma)z}{1 - z} = h(z). \quad (2.13)$$

If we take

$$q(z) = \frac{1}{(1 - z)^{2p\rho(1-\gamma)}}, \theta(w) = p \text{ and } \phi(w) = \frac{1}{\rho w}, \quad (2.14)$$

then $q(z)$ is univalent by the condition of the theorem and Lemma 2. It is easy to show that $q(z)$, $\theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 3. Since

$$Q(z) = zq'(z)\phi(q(z)) = \frac{2p(1 - \gamma)z}{1 - z} \quad (2.15)$$

is univalent starlike in U and

$$h(z) = \theta(q(z)) + Q(z) = \frac{p + p(1 - 2\gamma)z}{1 - z}, \quad (2.16)$$

it may be readily checked that the conditions (1) and (2) of Lemma 3 are satisfied. Thus the result follows from (2.13) immediately.

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