

Singular difference integrals, hypersingular integrals and their applications

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§1. Singular difference integrals and hypersingular integrals

For a function u on the n -dimensional Euclidean space R^n , the difference $\Delta_t^\ell u$ and the remainder R_t^ℓ of order ℓ with increment t are defined by

$$\Delta_t^\ell u(x) = \sum_{j=0}^{\ell} (-1)^j C_j^\ell u(x + (\ell - j)t),$$

$$R_t^\ell u(x) = u(x + t) - \sum_{|\gamma| \leq \ell-1} \frac{D^\gamma u(x)}{\gamma!} t^\gamma$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi-index, $D^\gamma = D_1^{\gamma_1} \cdots D_n^{\gamma_n}$, $t^\gamma = t_1^{\gamma_1} \cdots t_n^{\gamma_n}$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. The following integral transforms $D^{\alpha, \ell} u$ and $H^{\alpha, \ell} u$ ($\alpha > 0$, ℓ a positive integer)

$$D^{\alpha, \ell} u(x) = \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}} dt,$$

$$H^{\alpha, \ell} u(x) = \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{R_t^\ell u(x)}{|t|^{n+\alpha}} dt$$

are called singular difference integral and hypersingular integral, respectively. The Schwartz space S is the set of infinitely differentiable functions which decrease at infinity faster than any power. For $u \in S$, $D^{\alpha, \ell} u(x)$ exists for $\alpha < 2[(\ell + 1)/2]$, and $H^{\alpha, \ell} u(x)$ exists for $\ell - 1 < \alpha < 2[(\ell + 1)/2]$ where $[r]$ denotes the integral part of r . For $u \in S'$ (the dual of S), the Fourier transform of u is denoted by Fu . If u is an integrable function, then the Fourier transform of u is defined by

$$Fu(\xi) = \int u(x) e^{-x \cdot \xi} dx$$

with $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$. S.G. Samko calculated the Fourier transform of $D^{\alpha, \ell} u$ for $u \in S$.

PROPOSITION 1.1. ([6]) Let $u \in S$ and $0 < \alpha < 2[(\ell + 1)/2]$. Then

$$F(D^{\alpha, \ell} u)(\xi) = d_{\alpha, \ell} |\xi|^\alpha Fu(\xi)$$

with

$$d_{\alpha, \ell} = (-1)^\ell \frac{\pi^{(n/2)+1} \sum_{j=0}^{\ell-1} (-1)^{j+1} C_j^\ell (\ell - j)^\alpha}{2^{\alpha+1} \Gamma(1 + (\alpha/2)) \Gamma((n + \alpha)/2) \sin \frac{\pi}{2} \alpha}.$$

We calculate the Fourier transform of $H^{\alpha,\ell}u$.

LEMMA 1.2.([4]) If $2[(\ell - 1)/2] < \alpha < 2[(\ell + 1)/2]$, then

$$\psi(\xi) = \lim_{\epsilon \rightarrow 0, \delta \rightarrow \infty} \int_{\epsilon \leq |t| \leq \delta} \frac{e^{it \cdot \xi} - \sum_{|\gamma| \leq \ell-1} \frac{t^\gamma}{\gamma!} (i\xi)^\gamma}{|t|^{n+\alpha}} dt$$

exists and

$$\psi(\xi) = e_{\alpha,\ell} |\xi|^\alpha$$

with

$$e_{\alpha,\ell} = \frac{2^{1-\alpha} \pi^{(n/2)+1}}{\alpha \Gamma(\alpha/2) \Gamma((n+\alpha)/2) \sin \frac{\pi}{2} \alpha}.$$

PROPOSITION 1.3. Let $u \in S$ and $\ell - 1 < \alpha < 2[(\ell + 1)/2]$. Then

$$F(H^{\alpha,\ell}u)(\xi) = e_{\alpha,\ell} |\xi|^\alpha F u(\xi).$$

In fact, we have

$$\begin{aligned} F(H_\epsilon^{\alpha,\ell}u)(\xi) &= \int \left(\int_{|t| \geq \epsilon} \frac{u(x+t) - \sum_{|\gamma| \leq \ell-1} \frac{D^\gamma u(x)}{\gamma!} t^\gamma}{|t|^{n+\alpha}} dt \right) e^{-ix \cdot \xi} dx \\ &= \int_{|t| \geq \epsilon} \frac{1}{|t|^{n+\alpha}} \int u(x+t) e^{-ix \cdot \xi} dx dt - \sum_{|\gamma| \leq \ell-1} \int_{|t| \geq \epsilon} \frac{t^\gamma}{\gamma! |t|^{n+\alpha}} dt \int D^\gamma u(x) e^{-ix \cdot \xi} dx \\ &= F u(\xi) \int_{|t| \geq \epsilon} \frac{e^{it \cdot \xi} - \sum_{|\gamma| \leq \ell-1} \frac{t^\gamma}{\gamma!} (i\xi)^\gamma}{|t|^{n+\alpha}} dt. \end{aligned}$$

Hence the proposition follows from Lemma 1.2.

§2. The truncated integrals of the Riesz kernels

For $\alpha > 0$, the Riesz kernel of order α is given by

$$\kappa_\alpha(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} |x|^{\alpha-n}, & \alpha < n, \text{ or } \alpha \geq n, \alpha - n \neq \text{even}, \\ (\delta_{\alpha,n} - \log |x|) |x|^{\alpha-n}, & \alpha \geq n, \alpha - n = \text{even} \end{cases}$$

with

$$\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha)} + \frac{1}{2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{(\alpha-n)/2} - C \right) - \log \pi$$

where C is Euler's constant. With the above normalizing constants $\gamma_{\alpha,n}$ and $\delta_{\alpha,n}$ we have

$$(2.1) \quad F \kappa_\alpha(\xi) = \text{Pf.} |\xi|^{-\alpha}$$

where Pf. stands for the pseudo function [7:section 3 on Chap II]. Let $\alpha > 0$ and ℓ be a positive integer. We consider the truncated integrals of the Riesz kernels:

$$\rho_\epsilon^{\alpha,\ell}(x) = \int_{|t| \geq \epsilon} \frac{\Delta_t^\ell \kappa_\alpha(x)}{|t|^{n+\alpha}} dt,$$

$$\mu_\epsilon^{\alpha,\ell}(x) = \int_{|t| \geq \epsilon} \frac{R_t^\ell \kappa_\alpha(x)}{|t|^{n+\alpha}} dt.$$

We set $\rho^{\alpha,\ell}(x) = \rho_1^{\alpha,\ell}(x)$ and $\mu^{\alpha,\ell}(x) = \mu_1^{\alpha,\ell}(x)$. We note that $\rho^{\alpha,\ell}(x)$ is finite for every x , and $\mu^{\alpha,\ell}(x)$ is finite for $\alpha > \ell - 1$ and $x \neq 0$. Properties of $\rho^{\alpha,\ell}$ and $\mu^{\alpha,\ell}$ are investigated in [2],[3],[4] and [6].

LEMMA 2.1.(i) *Let ℓ be a positive integer, and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\rho_\epsilon^{\alpha,\ell}(x) = \frac{1}{\epsilon^n} \rho^{\alpha,\ell}\left(\frac{x}{\epsilon}\right).$$

(ii) *Let $\alpha > \ell - 1$, and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\mu_\epsilon^{\alpha,\ell}(x) = \frac{1}{\epsilon^n} \mu^{\alpha,\ell}\left(\frac{x}{\epsilon}\right).$$

LEMMA 2.2. (i) *Let $2[(\ell + 1)/2] > \alpha$. Then for $|x| \geq 1$*

$$|\rho^{\alpha,\ell}(x)| \leq C |x|^{\alpha - 2[2(\ell+1)/2] - n}$$

and for $|x| < 1$

$$|\rho^{\alpha,\ell}(x)| \leq C \begin{cases} |x|^{\alpha-n}, & \alpha < n, \\ (1 - \log |x|), & \alpha = n, \\ 1, & \alpha > n. \end{cases}$$

(ii) *Let $\ell - 1 < \alpha < 2[(\ell + 1)/2]$. Then*

$$|\mu^{\alpha,\ell}(x)| \leq C \begin{cases} |x|^{\alpha - [\alpha] - 1 - n}, & |x| \geq 1, \\ |x|^{\alpha - 2[(\ell-1)/2] - n}, & |x| < 1. \end{cases}$$

By Lemma 2.2, if $2[(\ell + 1)/2] > \alpha$, then $\rho^{\alpha,\ell}$ is integrable, and if $\ell - 1 < \alpha < 2[(\ell + 1)/2]$, then $\mu^{\alpha,\ell}$ is integrable. We denote

$$d_{\alpha,\ell}^1 = \int \rho^{\alpha,\ell}(x) dx, \quad e_{\alpha,\ell}^1 = \int \mu^{\alpha,\ell}(x) dx.$$

S.G.Samko[6] remarked

PROPOSITION 2.3. For $2[(\ell + 1)/2] > \alpha$, $d_{\alpha,\ell} = d_{\alpha,\ell}^1$ and hence $d_{\alpha,\ell}^1 \neq 0$ for $2[(\ell + 1)/2] > \alpha$ and $\alpha \neq \text{odd}$.

We note

PROPOSITION 2.4.([4]) For $\ell - 1 < \alpha < 2[(\ell + 1)/2]$, $e_{\alpha,\ell} = e_{\alpha,\ell}^1$ and hence $e_{\alpha,\ell}^1 \neq 0$ for $\ell - 1 < \alpha < 2[(\ell + 1)/2]$.

§3. The spaces of Riesz potentials

For $f \in S$, the Riesz potential of order α of f is defined by

$$U_{\alpha}^f(x) = \int \kappa_{\alpha}(x - y)f(y)dy.$$

By (2.1) for $f \in S$ we have

$$(3.1) \quad F(U_{\alpha}^f)(\xi) = \text{Pf.} |\xi|^{-\alpha} Ff(\xi).$$

In order to define the Riesz potential of an L^p -function, for an integer $k < \alpha$ we introduce

$$\kappa_{\alpha,k}(x, y) = \begin{cases} \kappa_{\alpha}(x - y) - \sum_{|\gamma| \leq k} \frac{D^{\gamma} \kappa_{\alpha}(-y)}{\gamma!} x^{\gamma}, & 0 \leq k < \alpha, \\ \kappa_{\alpha}(x - y), & k \leq -1 \end{cases}$$

We have

PROPOSITION 3.1.([1]) Let $f \in L^p$ and $k = [\alpha - (n/p)]$.

(i) If $\alpha - (n/p)$ is not a nonnegative integer, then

$$U_{\alpha,k}^f(x) = \int \kappa_{\alpha,k}(x, y)f(y)dy$$

exists and satisfies

$$\left(\int |U_{\alpha,k}^f(x, y)|^p |x|^{-\alpha p} dx \right)^{1/p} \leq C \|f\|_p.$$

(ii) If $\alpha - (n/p)$ is a nonnegative integer, then $U_{\alpha,k-1}^{f_1}$ and $U_{\alpha,k}^{f_2}$ exist and satisfy

$$\left(\int |U_{\alpha,k-1}^{f_1}(x)|^p |x|^{-\alpha p} (1 + |\log |x||)^{-p} dx \right)^{1/p} \leq C \|f_1\|_p,$$

$$\left(\int |U_{\alpha,k}^{f_2}(x)|^p |x|^{-\alpha p} (1 + |\log |x||)^{-p} dx \right)^{1/p} \leq C \|f_2\|_p$$

where $f_1 = f|_{B_1}$ is the restriction of f to $B_1 = \{|x| < 1\}$ and $f_2 = f - f_1$.

By Propositions 1.1, 1.3 and (3.1) it seems that the integral transforms $\frac{1}{d_{\alpha,\ell}} D^{\alpha,\ell}$ and $\frac{1}{e_{\alpha,\ell}} H^{\alpha,\ell}$ are the inverse operators of the Riesz potential operator. Precisely speaking

PROPOSITION 3.2. (I)([3]) Let $f \in L^p, k = [\alpha - (n/p)]$ and $\ell > \alpha - (n/p)$.

(i) If $\alpha - (n/p)$ is not a nonnegative integer, then

$$D_{\epsilon}^{\alpha,\ell} U_{\alpha,k}^f = \rho_{\epsilon}^{\alpha,\ell} * f$$

and hence

$$D^{\alpha,\ell} U_{\alpha,k}^f = d_{\alpha,\ell} f$$

where the symbol $*$ stands for the convolution.

(ii) If $\alpha - (n/p)$ is a nonnegative integer, then

$$D_{\epsilon}^{\alpha,\ell} (U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = \rho_{\epsilon}^{\alpha,\ell} * f$$

and hence

$$D^{\alpha,\ell} (U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = d_{\alpha,\ell} f$$

with $f_1 = f|_{B_1}$ and $f_2 = f - f_1$.

(II)([2]) Let $f \in L^p, k = [\alpha - (n/p)]$ and $\alpha - (n/p) < \ell < \alpha + 1$.

(i) If $\alpha - (n/p)$ is not a nonnegative integer, then

$$H_{\epsilon}^{\alpha,\ell} U_{\alpha,k}^f = \mu_{\epsilon}^{\alpha,\ell} * f$$

and hence

$$H^{\alpha,\ell} U_{\alpha,k}^f = e_{\alpha,\ell} f.$$

(ii) If $\alpha - (n/p)$ is a nonnegative integer, then

$$H_{\epsilon}^{\alpha,\ell} (U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = \mu_{\epsilon}^{\alpha,\ell} * f$$

and hence

$$H^{\alpha,\ell} (U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = e_{\alpha,\ell} f.$$

Taking Proposition 3.1 into account, we define the Riesz potential spaces of L^p -functions as follows:

$$R_{\alpha}^p = \begin{cases} \{U_{\alpha,k}^f; f \in L^p\}, & \alpha - (n/p) \neq \text{a nonnegative integer}, \\ \{U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}; f \in L^p, f_1 = f|_{B_1}, f_2 = f - f_1\}, & \alpha - (n/p) = \text{a nonnegative integer} \end{cases}$$

with $k = [\alpha - (n/p)]$.

We give characterizations of the Riesz potential spaces using the singular difference integrals and hypersingular integrals.

THEOREM 3.3.([3]) *Let $[(\ell+1)/2] > \alpha$ and $\alpha =$ an odd number. Then $u \in R_\alpha^p + P_k$ if and only if*

$$(i) \quad \int |u(x)|^p (1 + |x|)^{-\alpha p} (\log(e + |x|))^{-p} dx < \infty,$$

$$(ii) \quad \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p$$

where P_k is the set of polynomials of degree k .

For $1 < r_0, r_1, \dots, r_{\ell-1} < \infty$, we denote

$$W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}} = \{u; D^\gamma u \in L^{r_j} \text{ for } |\gamma| = j, j = 0, 1, \dots, \ell-1\}.$$

COROLLARY 3.4. *Let $[(\ell+1)/2] > \alpha$ and $\alpha \neq$ an odd number. Then $u \in (R_\alpha^p + P_k) \cap W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}}$ if and only if*

$$(i) \quad u \in W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}},$$

$$(ii) \quad \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p$$

for $r_0 \geq p$ in case of $\alpha - (n/p) \geq 0, p \leq r_0 \leq p_\alpha$ in case of $\alpha - (n/p) < 0$ where $1/p_\alpha = (1/p) - (\alpha/n)$.

THEOREM 3.5.([4]) *Let $\ell-1 < \alpha < \min(2[(\ell+1)/2], \ell + (n/p))$. Then $u \in (R_\alpha^p + P_k) \cap W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}}$ if and only if*

$$(i) \quad u \in W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}},$$

$$(ii) \quad \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{R_t^\ell u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p$$

for $r_0 \geq p$ in case of $\alpha - (n/p) \geq 0, p \leq r_0 \leq p_\alpha$ in case of $\alpha - (n/p) < 0$.

THEOREM 3.6.([5]) *Let $\alpha - (n/p) < 0$ and $\ell-1 < \alpha < \min(2[(\ell+1)/2], \frac{1}{2}(\ell + (n/p)))$. Then $u \in R_\alpha^p$ if and only if*

$$(i) \quad u \in W_{\ell-1}^{p_\alpha, p_{\alpha-1}, \dots, p_{\alpha-(\ell-1)}},$$

$$(ii) \quad \lim_{\epsilon \rightarrow 0} \int_{|t| \geq \epsilon} \frac{R_t^\ell u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p.$$

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