数理解析研究所講究録 1462 巻 2006 年 217-223

On the probabilities associated with unitary matrices

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1 Background

The series of joint works with T. Shirai on fermion point processes, boson point processes and others strongly suggested the following.

Theorem 1. *For a given unitary matrix* $U=(u_{ij})_{1\leq j,k\leq n}$ *there exists a probability* p *on the symmetric group* S_n *such that*

$$|\det U_{AB}|^2=\sum_{\sigma\in S_n,\sigma(A)=B}p(\sigma)\quad (A,B\subset\{1,2,\ldots,n\}).$$

where $U_{AB}=(u_{jk})_{j\in A,k\in B}$ *and we set* $\det U_{AB}=0$ *unless* $|A|\neq|B|$.

This result sharpens the following well-known theorem which shows the existence of an $i.i.d$ sequence of permutations that drives a given symmetric Markov chain.

Theorem 2. *A doubly stochastic matrix* $P=(p_{jk}),\sum_k p_{jk}=\sum_k p_{kj}=1$ *is a convex combination of representation matrices of permutations,* $E_\sigma=(\delta(k=\sigma(j)))_{1\leq j,k\leq n}$.

The proof of Theorem 1 will be published elsewhere. Here we discuss the uniqueness problem for $|\det U_{AB}|^2$ appearing in the L.H.S. of the assertion.

Theorem 3. *Let* X,Y *be matrices of the same type and assume*

$$\det(I+X^*SXT)=\det(I+Y^*SYT)$$

for any diagonal matrices S *and* T.

Then there exist unitary diagonal matrices D_1 *and* D_2 *such that*

$$Y = D_1^* X D_2 \quad or \quad Y = D_1^* \overline{X} D_2$$

where $\overline{X}$ *stands for the component-wise complex conjugate of* X.

It is obvious that the converse of Theorem 3 holds. The determinant $\det(I + X^* SXT)$ is a generating function in components of S and T with coefficients $|\det X_{AB}|^2$. Consequently, it solves the uniqueness problem stated above.

By the way, such a kind of uniqueness problem is not so simple in general. For instance, we have the following

Theorem 4. *Let* X *and* Y *be hermitian matrices and assume*

$$\det(I + XT) = \det(I + YT) \quad \text{for any diagonal matrix } T.$$

Then, "generically", there exists a unitary diagonal matrix D *such that* $Y = D^* X D$ *or* $Y = D^* \overline{X} D$ *but there exist counter-examples if the size* $n \geq 4$.

In deed, the "canonical form" of counter-examples for $n = 4$ is as follows. Consider

$$\begin{bmatrix} c_{11} & c_{12}e^{i\delta\alpha} & c_{13} & c_{14} \\ c_{21}e^{-i\delta\alpha} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34}e^{i\varepsilon\beta} \\ c_{41} & c_{42} & c_{43}e^{-i\varepsilon\beta} & c_{44} \end{bmatrix}.$$

where $c_{jk} \geq 0$, $\alpha, \beta > 0$. If we choose distinct pairs of $\delta, \varepsilon \in \{\pm 1\}$ for X and Y, we can find a counter-example.

2 The proof of Theorem 3

We employ the following notations for matrices $X = (x_{jk})_{1 \leq j \leq m, 1 \leq h \leq n}$ and $Y = (y_{jk})_{1 \leq j \leq m, 1 \leq h \leq n}$ with $x_{jk} \in \mathbb{C}$ and $y_{jk} \in \mathbb{C}$:

(*a*) $\overline{X} = (\overline{x}_{jk})_{1 \leq j \leq m, 1 \leq k \leq n}$.

(*b*) $X \approx Y$ if for any $p = 1, 2, \ldots, \min\{m, n\}$ and for any $j_1 < \cdots < j_p$ and $k_1 < \cdots < k_p$

$$|\det(y_{j_r k_s})_{1 \leq r,s \leq p}| = |\det(x_{j_r k_s})_{1 \leq r,s \leq p}|.$$

(c) $X \sim Y$ if there exist $\theta_1, \dots, \theta_m, \varphi_1, \dots, \varphi_m \in \mathbb{R}$ (precisely, $\mathbb{R}/2\pi\mathbb{Z}$) such that
$$y_{jk} = e^{i(\theta_j - \varphi_k)} x_{j,k}$$
for all $j = 1, \dots, m$ and all $k = 1, \dots, n$.

Moreover we write

$$X \overset{+}{\sim} Y \quad \text{if} \quad X \sim Y \quad \text{and} \quad X \overset{-}{\sim} Y \quad \text{if} \quad \overline{X} \sim Y.$$

Under above notations the statement of Theorem can be restated as follows:
$$\text{if} \quad X \approx Y, \quad \text{then} \quad X \overset{+}{\sim} Y \quad \text{or} \quad X \overset{-}{\sim} Y.$$

2.1 Preliminary

Lemma 1. *Let $a, b, \theta, \varphi \in \mathbb{R}$ and assume*

$$|e^{i\theta} a - b| = |e^{i\varphi} a - b|.$$

Then one of the following holds:

$$(a)\ a = 0 \quad (b)\ b = 0 \quad (c)\ \theta = \varphi \pmod{2\pi} \quad (d)\ \theta = -\varphi \pmod{2\pi}.$$

Conversely, if one of (a)-(d) holds then $|e^{i\theta} a - b| = |e^{i\varphi} a - b|$.

Proof. The cases (a) and (b) are trivial. Assume $a \neq 0$ and $b \neq 0$. Then $|z - b| = r$ and $|z| = |a|$ are two distinct circles on the complex plane which are symmetric with respect to the real axis. Hence they interested at most two points which are complex conjugate. □

Lemma 2. *Let $U_{jk} \in \mathbb{C}$, $j, k = 1, 2$. Then the identity*

$$U_{11} + U_{22} = U_{12} + U_{21}$$

holds if and only if there exist $v_1, v_2, w_1, w_2 \in \mathbb{C}$ such that

$$u_{jk} = v_j - w_k.$$

Proof. The "if" part is obvious. To prove the "only if" part set

$$\begin{aligned} U_{11} + U_{22} &= U_{12} + U_{21} = s, \\ U_{21} - U_{11} &= U_{22} - U_{12} = a, \\ U_{12} - U_{11} &= U_{22} - U_{21} = b \end{aligned}$$

and

$$v_1 = \frac{s-a}{2}, v_2 = \frac{s+a}{2}, w_1 = \frac{b}{2}, w_2 = -\frac{b}{2}.$$

Then

$$u_{jk} = v_j - w_k \quad \text{for} \quad j, k = 1, 2.$$

Lemma 3. *Let X and Y be matrices of type (m, n) and set*

$$\begin{aligned} X' &= (x_{jk})_{l \le j \le m, l \le k \le n-1}, \quad X'' = (x_{jk})_{l \le j \le m, 2 \le k \le n}, \\ Y' &= (y_{jk})_{l \le j \le m, l \le k \le n-1}, \quad Y'' = (y_{jk})_{l \le j \le m, 2 \le k \le n}. \end{aligned}$$

Assume that

$$X' \sim Y' \quad \text{and} \quad X'' \sim Y''.$$

In addition, assume that $x_{jk} \neq 0$ for some j and k with $1 \le j \le m$ and $2 \le k \le n-1$. Then

$$X \sim Y.$$

Proof. By the assumption there exist $\theta'_1, \dots, \theta'_m, \varphi'_1, \dots, \varphi_{n-1}$ and $\theta''_1, \dots, \theta''_m, \varphi''_2, \dots, \varphi''_n$ such that

$$\begin{aligned} y_{jk} &= e^{i(\theta'_j - \varphi'_k)} x_{jk} \quad \text{for} \quad l \le j \le m \quad \text{and} \quad 1 \le k \le n-1, \\ y_{jk} &= e^{i(\theta''_j - \varphi''_k)} x_{jk} \quad \text{for} \quad l \le j \le m \quad \text{and} \quad 2 \le k \le n. \end{aligned}$$

Moreover, by the additional assumption $x_{jk} \neq 0$ and $y_{jk} \neq 0$ for some j and k with $l \le j \le m$ and $2 \le k \le n-1$. Hence

$$\theta'_j - \varphi'_k = \theta''_j - \varphi''_k \quad \text{or} \quad \theta'_j - \theta''_j = \varphi''_k - \varphi''_k = \alpha$$

for such (j, k). Consequently,

$$y_{jk} = e^{i(\theta_j - \varphi_k)} x_{jk} \quad \text{for} \quad l \le j \le m \quad \text{and} \quad l \le k \le n$$

with $\theta_j = \theta'_j$ $(l \le j \le m)$, $\varphi_k = \varphi_k$ $(l \le k \le n-1)$ and $\varphi_n = \theta''_n + \alpha$.

2.2 Proof of Theorem 3

Step 1: $m = n = 2$.

Let $X = (x_{jk})_{1\le j,k\le 2}$ and $Y = (y_{jk})_{1\le j,k\le 2}$. Since $X \approx Y$,

$$|x_{jk}| = |y_{jk}| \ (j,k=1,2) \quad \text{and}$$
$$|x_{11}x_{22} - x_{12}x_{21}| = |y_{11}y_{22} - y_{12}y_{21}|.$$

Set $x_{jk} = c_{jm}e^{i\xi_{jk}}$ and $y_{jk} = c_{jk}e^{i\eta_{jk}}$ where $c_{jk} = |x_{jk}|$. Then

$$\begin{aligned}&|e^{i(\xi_{11}+\xi_{22}-\xi_{12}-\xi_{21})}c_{11}c_{22} - c_{12}c_{21}|\\ =&|e^{i(\eta_{11}+\eta_{22}-\eta_{12}-\eta_{21})}c_{11}c_{22} - c_{12}c_{21}|.\end{aligned}$$

By Lemma 1, it follows either $c_{11}c_{22}c_{12}c_{21} = 0$ or

$$\eta_{11} + \eta_{22} - \eta_{12} - \eta_{21} = \pm(\xi_{11} + \xi_{22} - \xi_{12} - \xi_{21}).$$

In the latter case, by Lemma 2 there exist $\theta_1, \theta_2, \varphi_1$ and φ_2 such that

$$\eta_{jk} \mp \xi_{jk} = \theta_j - \varphi_k, \quad j,k = 1,2.$$

Hence

$$Y \sim X \quad \text{or} \quad Y \sim \overline{X}$$

according to the sign $\mp$.

If $c_{11}c_{22}c_{12}c_{21} = 0$, X is one of the following form

$$(a) \quad \begin{pmatrix} 0 & x_{12} \\ x_{21} & x_{22}\end{pmatrix}, x_{12}x_{21}x_{22} \neq 0 \quad (a') \quad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & 0\end{pmatrix}, x_{11}x_{12}x_{21} \neq 0$$

$$(b) \quad \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22}\end{pmatrix}, x_{11}x_{21}x_{22} \neq 0 \quad (b') \quad \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22}\end{pmatrix}, x_{11}x_{12}x_{22} \neq 0$$

$$(c) \quad \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0\end{pmatrix} \qquad (c') \quad \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0\end{pmatrix}$$

$$(c'') \quad \begin{pmatrix} 0 & x_{11} \\ 0 & x_{21}\end{pmatrix} \qquad (c''') \quad \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22}\end{pmatrix}.$$

In case (a) , setting $\varphi_1 = 0, \theta_l = \eta_{11} - \xi_{11}, \theta_2 = \eta_{21} - \xi_{21}$ and $\varphi_2 = \eta_{22} - \xi_{22} - \theta_2$ one finds $\eta_{jk} = \xi_{jk} + \theta_j - \varphi_k$.

In case (b) , setting $\theta_2 = 0, \varphi_l = \xi_{21} - \eta_{21}, \varphi_2 = \xi_{22} - \eta_{22}$ and $\theta_1 = \eta_{12} - \xi_{12} + \varphi_2$ one finds $\eta_{jk} = \xi_{jk} + \theta_j - \varphi_k$.

In these cases, it is easy to find $\theta_1, \theta_2, \varphi_1$ and φ_2 such that

$$y_{jk} = e^{i(\theta_j - \varphi_k)} x_{jk} \quad \text{for} \quad j, k = 1, 2.$$

For instance,

$$\begin{aligned} &\text{case}(a): \quad \varphi_1 = 0, \theta_1 = \eta_{11} - \xi_{11}, \theta_2 = \eta_{21} - \xi_{21} \text{ and } \varphi_2 = \eta_{22} - \xi_{22} - \theta_2. \\ &\text{case}(b): \quad \theta_2 = 0, \varphi_1 = \xi_{21} - \eta_{21}, \varphi_2 = \xi_{22} - \eta_{22} \text{ and } \theta_1 = \eta_{12} - \xi_{12} + \varphi_2. \end{aligned}$$

Consequently, in these degenerated cases we obtain

$$Y \sim X.$$

□

Step 2: $m = 2, n = 3$.

Let $X = (x_{jk})_{l \le j \le 2, l \le k \le 3}$ and $Y = (y_{jk})_{l \le j \le 2, l \le k \le 3}$ and define

$$\begin{aligned} &X' = (x_{jk})_{l \le j \le 2, l \le k \le 2}, \quad X'' = (x_{jk})_{l \le j \le 2, 2 \le k \le 3}, \\ &X''' = (x_{jk})_{l \le j \le 2, k \in \{1,3\}} \end{aligned}$$

and Y', Y'', Y''' in a similar manner. Since $X \approx Y$ implies $X' \approx Y', X'' \approx Y'', X''' \approx Y'''$ it follows from Step 1 that

$$X' \overset{\varepsilon'}{\sim} Y', X'' \overset{\varepsilon''}{\sim} Y'', X''' \overset{\varepsilon'''}{\sim} Y'''$$

for some $\varepsilon', \varepsilon'', \varepsilon''' \in \{\pm 1\}$. Then at least two of $\varepsilon', \varepsilon''$ and ε''' coincide. For simplicity, assume $\varepsilon' = \varepsilon'' = +$. Then

$$X' \approx Y' \quad \text{and} \quad X'' \approx Y''.$$

By Lemma 3 one can conclude $X \sim Y$ if $x_{12} \neq 0$ or $x_{22} \neq 0$. If $x_{12} = x_{22} = 0$, then relation $X''' \overset{\varepsilon'''}{\sim} Y'''$ is equivalent to the relation $X \overset{\varepsilon'''}{\sim} Y$.

Step 3: $m = 2, n \geq 4$.

We appeal to the induction on n. In Step 2 we proved the assertion for $n = 3$. Let us assume we have proved for $n - 1$ and show the case for n.

If X and Y are matrices of type $(2, n)$ and $X \approx Y$, then we have n submatrices $X_1, \ldots, X_n$ of X and $Y_1, \ldots, Y_n$ of Y of type $(2, n - 1)$.

By induction assumption, we have $X_i \overset{\varepsilon_i}{\sim} Y_i$ for each i with $\varepsilon_i = \pm$. Since $n \geq 4$, we can find at least two i's for which ε_i's coincide with each other. Thus, a similar argument to Step 2 shows that $X \overset{+}{\sim} Y$ or $X \overset{-}{\sim} Y$.

Step 4: $m \geq 3, n \geq 3$.

We appeal to the induction on m fixing n.

Let X and Y be matrices of type (m, n) and $X \approx Y$. Then we can find at least two par submatrices X', X'', Y', Y'' of type $(m-1, n)$ and $\varepsilon \in \{\pm\}$ such that

$$X' \overset{\varepsilon}{\sim} Y' \quad \text{and} \quad X'' \overset{\varepsilon}{\sim} Y''.$$

By Lemma 3 if X' and X'' have a common nonzero entry, we have $X \overset{\varepsilon}{\sim} Y$. If they have no common nonzero entries, then X and Y are essentially of type $(2, n)$. Hence by Step 3 we obtain $X \overset{+}{\sim} Y$ or $X \overset{-}{\sim} Y$. □

References

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