

Conjecture about Regularity of Prefix Square Roots of Regular Languages

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Zsolt Fazekas, Robert Mercas, Daniel Reidenbach gave the conjecture in [2] which gives necessary and sufficient condition for the primitive prefix square root of a regular language L to be regular. The author gives a counterexample of their conjecture and gives a new conjecture.

1. Preliminary

An *alphabet* V is a finite and nonempty set of symbols, called *letters*. Every finite sequence of letters of V is called a *word* over V . Words over V together with the operation of concatenation with the *empty* word ε form a free monoid V^* . We denote $V^+ = V^* - \{\varepsilon\}$.

Let $w = a_1 a_2 \cdots a_n$ where $a_1, a_2, \dots, a_n \in V$. The *length* of a word w is n and denoted by $|w|$ and the length of the empty word ε is 0.

For a positive integer p ,

$$V^{\leq p} = \{w \in V^* \mid |w| \leq p\},$$

$$V^p = \{w \in V^* \mid |w| = p\}.$$

For a word $w = xyz$ for $x, y, z \in V^*$, a *prefix* of w is x , a *factor* of w is y and a *suffix* of w is z .

For a word $w \in V^+$, the following operations are defined in [1]:

- prefix square reduction: $\square(w) = \{uv \mid w = uvv, \text{ for } u \in V^+, v \in V^*\}$

- suffix square reduction: $\sqcap(w) = \{uv \mid w = vuuv, \text{ for } u \in V^+, v \in V^*\}$
- prefix-suffix square reduction: $\square\square(w) = \square(w) \cup \sqcap(w)$

For simplicity, we restrict the argument to prefix square reduction.

We define the bounded version for a fixed positive integer p :

- p -prefix square reduction: ${}_p\square(w) = \{uv \mid w = uv, \text{ for } u \in V^{\leq p}, v \in V^*\}$

For a language L , we have language: $\square(L) = \bigcup_{w \in L} \square(w)$.

The following languages are defined:

$$\begin{aligned}\square^0(w) &= \{w\}, \\ \square^{k+1}(w) &= \square(\square^{k+1}(w)) \quad \text{for any } k \geq 0 \\ \square^*(w) &= \bigcup_{k \geq 0} \square^k(w).\end{aligned}$$

For a word w , the *primitive prefix square root* of w is the set $\{u \mid u \in \square^*(w) \text{ and } \square(u) = u\}$ and it is denoted by $\sqrt[\square]{w}$. The *primitive bounded prefix square root* of w is the set $\{u \mid u \in {}_p\square^*(w) \text{ and } {}_p\square(u) = u\}$ and it is denoted by ${}^p\sqrt[\square]{w}$. For a language L , we define $\sqrt[\square]{L} = \bigcup_{w \in L} \sqrt[\square]{w}$ and ${}^p\sqrt[\square]{L} = \bigcup_{w \in L} {}^p\sqrt[\square]{w}$.

2. Conjectures

Zsolt Fazekas, Robert Mercas, Daniel Reidenbach gave the following conjecture in [2].

Conjecture (in [2]). Let L be a regular language. The primitive prefix square root of L is regular if and only if there exists some positive integer p such that $\sqrt[\square]{L} = {}^p\sqrt[\square]{L}$.

But, I give here the following counterexample and new conjecture.

Example. Let $L = aab^+aab^+c$ where $a, b, c \in V$. The language L is regular. On the other hand, the primitive prefix square root of L is $\sqrt[\square]{L} = ab^+aab^+c \cup ab^+c$ and this language is regular.

But, there is no positive integer p such that $\sqrt[p]{L} = \sqrt[p]{L}$.

Now, we define a new term to describe our new conjecture: For a word w , if xx is a non-trivial prefix of w and x is prefix square free, then we say that xx is the *minimal* prefix square of w .

Conjecture . Let L be a regular language. The primitive prefix square root of L is regular if and only if there exists positive integer N such that, for every word $w \in \square^*(L)$, the length of the minimal prefix square of w is smaller than N .

References

- [1] P. Bottoni, A. Labella and V. Mitrana, Theor. Comput. Sci., **682**, (2017), pp 49-56.
- [2] Szilárd Zsolt Fazekas, Robert Mercas, Daniel Reidenbach, On the Prefix-Suffix Duplication Reduction, International Journal of Foundations of Computer Science (in print).

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