

SP-property for a pair of C^* -algebras

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Abstract

Recall that C^* -algebra A has the SP-property if every non-zero hereditary C^* -subalgebra of A has a non-zero projection. Let $1 \in A \subset B$ be a pair of C^* -algebras.

In this paper we investigate a sufficient condition for B to have the SP-property under A holds. As an application, we will present the cancellation property for crossed products of simple C^* -algebras by discrete groups.

This paper basically comes from joint works with Ja A Jeong ([7][8]).

1 The SP-Property

In this section we present a sufficient condition for B to have the SP-property under A holds.

The argument in [11, Lemma 10] gives the following general result.

Theorem 1.1 *Let $1 \in A \subset B$ be a pair of C^* -algebras. Suppose that A has the SP-property and there is a conditional expectation E from B to A . If for any non-zero positive element x in B and an arbitrary positive number $\varepsilon > 0$ there is an element y in B such that*

$$\begin{aligned} \|y^*(x - E(x))y\| &< \varepsilon, \\ \|y^*E(x)y\| &\geq \|E(x)\| - \varepsilon \end{aligned}$$

then B has the SP-property. Moreover, every non-zero hereditary C^ -subalgebra of B has a projection which is equivalent to some projection in A in the sense of Murray-von Neumann*

Next we consider the following stronger assumption on a conditional expectation E from B to A .

Definition 1.2 *Let $1 \in A \subset B$ be a pair of C^* -algebras. A conditional expectation E from B to A is called outer if for any element $x \in B$ with*

$E(x) = 0$ and any non-zero hereditary C^* -subalgebra C of A

$$\inf\{\|cxc\|; c \in C^+, \|c\| = 1\} = 0.$$

The following result comes from the same argument as in [10, Lemma 3.2] and Theorem 1.1.

Corollary 1.3 *Let $1 \in A \subset B$ be a pair of C^* -algebras. Suppose that A has the SP-property and there is a conditional expectation E from B to A . If E is outer, then B has the SP-property.*

We present some examples of a pair of C^* -algebras with an outer conditional expectations.

Example 1.4 *Let ρ be a corner endmorphism on a unital C^* -algebra A , and let E be a canonical conditional expectation from a crossed product $A \times_\rho \mathbb{N}$ by ρ to A . Suppose that*

$$\tilde{T}(\rho) = \{\lambda \in \mathbb{T} \mid \hat{\rho}(I) = I \text{ for } \forall I \in \text{Prime}(A \times_\rho \mathbb{N})\} = \mathbb{T}.$$

Then, E is outer.

Proof. See Jeong-Kodaka-Osaka [6]. □

Example 1.5 (Kishimoto[10]) *Let G be a discrete group and let α be a representation of G by automorphisms of a simple unital C^* -algebra A . Suppose α is outer. Then, a canonical conditional expectation from a crossed product $A \times_\alpha G$ to A is outer.*

In the case of a crossed product of a simple unital C^* -algebra with the SP-property by a finite group G , we can deduce the SP-property for the crossed product algebra $A \times_\alpha G$ by any automorphism α on A .

Theorem 1.6 ([7]) *Let A be a simple unital C^* -algebra with the SP-property, and let α be an action by a finite group G . Then, a crossed product algebra $A \times_\alpha G$ has the SP-property.*

2 C^* -Index Theory

In this section, we brief the C^* -index theory by Watatani ([16]).

Let $1 \in A \subseteq B$ be a pair of C^* -algebras. By a conditional expectation $E : B \rightarrow A$ we mean a positive faithful linear map of norm one satisfying

$$E(aba') = aE(b)a', \quad a, a' \in A, b \in B.$$

A finite family $\{(u_1, v_1), \dots, (u_n, v_n)\}$ in $B \times B$ is called a quasi-basis for E if

$$\sum_{i=1}^n u_i E(v_i b) = \sum_{i=1}^n E(b u_i) v_i = b \text{ for } b \in B.$$

We say that a conditional expectation E is of index-finite type if there exists a quasi-basis for E . In this case the index of E is defined by

$$\text{Index} E = \sum_{i=1}^n u_i v_i.$$

Note that $\text{Index} E$ does not depend on the choice of a quasi-basis and every conditional expectation E of index-finite type on a C^* -algebra has a quasi-basis of the form $\{(u_1, u_1^*), \dots, (u_n, u_n^*)\}$ ([16, Lemma 2.1.6]). Moreover, $\text{Index} E$ is always contained in the center of B , so that it is a scalar whenever B has the trivial center, in particular when B is simple.

Let $E : B \rightarrow A$ be a conditional expectation. Then $B_A (= B)$ is a pre-Hilbert module over A with an A -valued inner product

$$\langle x, y \rangle = E(x^* y), \quad x, y \in B_A.$$

Let $\mathcal{E}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|E(x^* x)\|_A^{1/2}, \quad x \in B_A.$$

Then $\mathcal{E}$ is a Hilbert C^* -module over A . Since E is faithful, the canonical map $B \rightarrow \mathcal{E}$ is injective. Let $L_A(\mathcal{E})$ be the set of all (right) A -module homomorphisms $T : \mathcal{E} \rightarrow \mathcal{E}$ with an adjoint A -module homomorphism $T^* : \mathcal{E} \rightarrow \mathcal{E}$ such that

$$\langle T\xi, \zeta \rangle = \langle \xi, T^*\zeta \rangle \quad \xi, \zeta \in \mathcal{E}.$$

Then $L_A(\mathcal{E})$ is a C^* -algebra with the operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. There is an injective $*$ -homomorphism $\lambda : B \rightarrow L_A(\mathcal{E})$ defined by

$$\lambda(b)x = bx$$

for $x \in B_A$, $b \in B$, so that B can be viewed as a C^* -subalgebra of $L_A(\mathcal{E})$. Note that the map $e_A : B_A \rightarrow B_A$ defined by

$$e_A x = E(x), \quad x \in B_A$$

is bounded and thus it can be extended to a bounded linear operator, denoted by e_A again, on $\mathcal{E}$. Then $e_A \in L_A(\mathcal{E})$ and $e_A = e_A^2 = e_A^*$, that is, e_A is a projection in $L_A(\mathcal{E})$.

The (reduced) C^* -basic construction is a C^* -subalgebra of $L_A(\mathcal{E})$ defined to be

$$C^*(B, e_A) = \overline{\text{span}\{\lambda(x)e_A\lambda(y) \in L_A(\mathcal{E}) : x, y \in B\}}^{\|\cdot\|}$$

see [16, Definition 2.1.2].

Then,

Lemma 2.1 ([16, Lemma 2.1.4]) (1) $e_AC^*(B, e_A)e_A = \lambda(A)e_A$.

(2) $\psi : A \rightarrow e_AC^*(B, e_A)e_A$, $\psi(a) = \lambda(a)e_A$, is a $*$ -isomorphism (onto).

Lemma 2.2 ([16, Lemma 2.1.5]) The following are equivalent:

(1) $E : B \rightarrow A$ is of index-finite type

(2) $C^*(B, e_A)$ has an identity and there exists a number c with $0 < c < 1$ such that

$$E(x^*x) \geq c(x^*x) \quad x \in B.$$

The above inequality was shown first in [13] by Pimsner and Popa for the conditional expectation $E_N : M \rightarrow N$ from a type II_1 factor M onto its subfactor N (c can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \rightarrow B$ defined by

$$E_B(\lambda(x)e_A\lambda(y)) = (\text{Index}E)^{-1}xy, x, y \in B$$

is called the dual conditional expectation of $E : B \rightarrow A$. If E is of index-finite type, so is E_B with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\text{Index}E}u_i e_A$, and $\{(u_i, u_i^*)\}$ are quasis-basis for E ([16, Proposition 2.3.4]).

3 The Stable Rank for C^* -Crossed Products

Let α be an action of a finite group G on a unital C^* -algebra A by automorphisms, and let $A \rtimes_\alpha G$ be its crossed product, that is, it is the universal C^* -algebra generated by a copy of A and implementing unitaries $\{u_g | g \in G\}$ with $\alpha_g(a) = u_g a u_g^*$ for every $g \in G$ and $a \in A$. Then there exists a canonical conditional expectation $E : A \rtimes_\alpha G \rightarrow A$ defined by

$$E\left(\sum_g a_g u_g\right) = a_e,$$

for $a_g \in A$ and $g \in G$, where e denotes the identity of the group G .

Lemma 3.1 Under this situation, the canonical conditional expectation E is of index-finite type with a quasi-basis $\{(u_g, u_g^*) : g \in G\}$ and $\text{Index}(E) = \sum_{g \in G} u_g u_g^* = |G|$, the order of G .

Let $B = A \rtimes_\alpha G$ and $n = |G|$. Then, a dual conditional expectation E_B is of index-finite type with a quasi-basis $\{(w_g, w_g^*) : g \in G\}$, where $w_g = \sqrt{n}u_g e_A$ (see section 2).

The following fact comes from a simple computation.

Lemma 3.2 ([8]) *The expression $x = \sum_{g \in G} w_g b_g$ ($b_g \in B$) is unique for each $x \in C^*(B, e_A)$.*

Let A be a unital C^* -algebra and $Lg_n(A)$ denote the n -tuples $(x_1, \dots, x_n)$ in A^n which generate A as a left ideal. The *topological stable rank* of A ($sr(A)$) is defined to be the least integer for which $Lg_n(A)$ is dense in A^n . If there does not exist such an integer then $sr(A)$ is defined to be ∞ . For a non unital C^* -algebra A we define $sr(A) = sr(\tilde{A})$ where $\tilde{A}$ is the unitization of A . See [15] for details about stable rank. It is not hard to see that for a unital C^* -algebra A $sr(A) = 1$ if and only if the set of invertible elements is dense in A .

Theorem 3.3 ([8]) *Let G be a finite group, and α be an action of G on a unital C^* -algebra A with $sr(A) = 1$. Then $sr(A \rtimes_\alpha G) \leq |G|$.*

Proof. Let $n = |G|$, and $(b_{g_1}, \dots, b_{g_n}) \in B^n$, where $B = A \rtimes_\alpha G$. Put $y = \sum_{g \in G} w_g b_g \in C^*(B, e_A)$. Since $C^*(B, e_A)$ is strong Morita equivalent to A and $sr(A) = 1$, we have $sr(C^*(B, e_A)) = 1$ ([16, Proposition 1.3.4.]). Approximate y by invertible elements x in $C^*(B, e_A)$, and write $x = \sum_{g \in G} w_g c_g$, $c_g \in B$. Then by Lemma 3.2, $(c_{g_1}, \dots, c_{g_n})$ is close to $(b_{g_1}, \dots, b_{g_n})$. Note that

$$x^* x = n \sum_g c_g^* e_A c_g.$$

By Lemma 2.2

$$E_B(x^* x) \geq \frac{1}{n} x^* x, \quad x \in C^*(B, e_A).$$

Since $E_B(x^* x) = \sum_g c_g^* c_g$, it follows that

$$\sum_g c_g^* c_g \geq \frac{1}{n} x^* x$$

which is invertible in $C^*(B, e_A)$. Therefore $\sum_g c_g^* c_g$ is invertible in B , that is, $(c_{g_1}, \dots, c_{g_n}) \in Lg_n(B)$. $\square$

Remark 3.4 *If $sr(A) = m$ then it can be shown that $sr(A \times_\alpha G) \leq |G|m$ whenever A is a simple unital C^* -algebra. Indeed, it can come from the following two facts; (i) $C^*(B, e_A)$ is isomorphic to the matrix algebra $M_n(A)$ ([16]), (ii) $sr(M_n(A)) = \{\frac{sr(A)-1}{n}\} + 1$, where $\{t\}$ denotes the least integer which is greater than or equal to t ([15]).*

4 The Cancellation Property

A C^* -algebra A is said to have *cancellation of projections* if for any projections p, q, r in A with $p \perp r$, $q \perp r$, $p + r \sim q + r$, we have $p \sim q$. If $M_n(A)$ has cancellation of projections for each $n = 1, 2, \dots$, then we simply say that A has *cancellation*. Note that every C^* -algebra with cancellation is stably finite, that is, every matrix algebra $M_n(A)$ with entries from A contains no infinite projections for $n = 1, 2, \dots$. It can be shown that if A is a C^* -algebra with $sr(A) = 1$ then it has cancellation. In the previous section we proved that the stable rank of the C^* -crossed product $A \times_\alpha G$ is bounded by the order of the group G if $sr(A)=1$, and actually it seems that the crossed product has stable rank 1, and therefore it would be natural to ask if it has cancellation.

Theorem 4.1 ([2, Theorem 4.2.2]) *Let A be a simple unital C^* -algebra. Suppose A contains a sequence (p_k) of projections such that*

1. *for each k there is a projection r_k such that $2p_{k+1} \oplus r_k$ is equivalent to a subprojection of $p_k \oplus r_k$,*
2. *there is a constant K such that $sr(p_k A p_k) \leq K$ for all k .*

Then A has cancellation.

Theorem 4.2 ([8]) *Let A be a simple unital C^* -algebra with $sr(A) = 1$ and SP-property. If G is a finite group and α is an action of G on A then the crossed product $A \times_\alpha G$ has cancellation.*

Sketch of a proof.

We give a proof in the case that $A \times_\alpha G$ is simple.

Since the fixed point algebra A^α can be identified with a hereditary C^* -subalgebra of the crossed product it has the SP-property by Theorem 1.6. Thus there is a sequence of projections $\{p_k\} \in A^\alpha$ such that $2[p_{k+1}] \leq [p_k]$ by [9, Lemma 2.2], where $[p]$ denotes the equivalence class of p . Since

$p_k \in A^\alpha$, $p_k(A \times_\alpha G)p_k$ is isomorphic to $(p_k A p_k) \times_\alpha G$ for each $k \in N$. Note that each $p_k A p_k$ has stable rank one. By Theorem 3.3 $sr(p_k A p_k \times_\alpha G) \leq |G|$. Therefore, the assertion follows from Theorem 4.1 ($K = |G|, r_k = 0$). $\square$

Recall that a unital C^* -algebra A has real rank zero, $RR(A) = 0$, if the set of invertible self-adjoint elements is dense in A_{sa} . It is well known that $RR(A) = 0$ is equivalent to say that every non-zero hereditary C^* -subalgebra contains an approximate identity consisting of projections (HP) ([3]). From [2, Section 4] where the HP-property is studied for simple C^* -algebras we can deduce the following.

Corollary 4.3 ([8]) *Under the assumptions of the above theorem, if $RR(A \times_\alpha G) = 0$ then its stable rank is one.*

For crossed products by the integer group Z we have the following cancellation theorem:

Theorem 4.4 ([8]) *Let A be a simple unital C^* -algebra with $sr(A) = 1$ and SP-property. If α is an outer action of the integer group Z on A such that $\alpha_* = id$ on the K_0 group $K_0(A)$ of A then the crossed product $A \times_\alpha Z$ has cancellation.*

Example 4.5 *If A is a UHF algebra or an irrational rotation algebra then the identity map is the only possible homomorphism on its K_0 group. Therefore the theorem says that any crossed product $A \times_\alpha Z$ has cancellation.*

Corollary 4.6 ([8]) *Under the same assumption of Theorem 3.5 if $RR(A \times_\alpha Z) = 0$, then its stable rank is one.*

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