

NOTES ON P-VALENTLY BAZILEVIĆ FUNCTIONS

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ABSTRACT

The object of the present paper is to improve the former results for p -valently Bazilević functions which were recently proved by the author and others.

1. INTRODUCTION

Let A_p denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $E = \{z: |z| < 1\}$. A function $f(z)$ belonging to A_p is said to be p -valently starlike if and only if it satisfies the condition

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in E).$$

We denote by S_p^* the subclass of A_p consisting of functions which are p -valently starlike in E .

A function $f(z)$ belonging to the class A_p is said to be p -valently Bazilević of type β and order γ if there exists a function $g(z)$ belonging to S_p^* such that

$$(1.3) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)^{1-\beta} g(z)^\beta} \right\} > \gamma \quad (z \in E)$$

for some β ($\beta > 0$) and γ ($0 \leq \gamma < p$).

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Also we denote by $B_p(\beta, \gamma)$ the subclass of A_p consisting of all p -valently Bazilevič functions of type β and order γ in E . The concept of Bazilevič functions was first introduced by Bazilevič [6]. Thomas [8] has called a function in the class $B_1(\beta, 0)$ a Bazilevič function of type β . Further, Nunokawa [3] has proved that a function $f(z)$ in the class $B_p(\beta, 0)$ is p -valent in the unit disk E .

In particular, the class $B_p(\beta, \gamma)$ for $g(z) = f(z)$ is called is the class of p -valently starlike functions of order γ . Further, we note that the class $B_p(\beta, 0)$ for $g(z) = f(z)$ is equivalent to S_p^* .

Let $A_p(\alpha, \beta)$ be the subclass of A_p consisting of functions which satisfy the condition

$$(1.4) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[1 + \frac{zf''(z)}{f'(z)} \right] \right\} > \beta \quad (z \in E)$$

for some real α and β .

The class $A_1(\alpha, 0)$ when $p = 1$ and $\beta = 0$ was introduced by Mocanu [2], and was studied by Miller, Mocanu and Reade [1], and Sakaguchi and Fukui [7].

Let $C_p(\alpha, \beta)$ be the subclass of A_p consisting of functions satisfying the condition

$$(1.5) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[1 + \frac{zf''(z)}{f'(z)} \right] \right\} < \beta \quad (z \in E)$$

for some real α and β ($\beta > p$).

The class $C_p(\alpha, \beta)$ was recently introduced by Nunokawa and Owa [4].

2. SOME PROPERTIES

In order to derive our results, we need the following lemmas.

LEMMA I ([5]). If a function $f(z)$ belongs to the class $A_p(\alpha, \beta)$ with $\alpha > 0$ and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha})$, and therefore $f(z)$ is p -valent in the unit disk E .

LEMMA 2 ([4]). Let a function $f(z)$ belong to the class $C_p(\alpha, \beta)$ with $\alpha \neq 0$, $\beta > p$, and $|\beta/\alpha| \leq 1/2$. Then $f(z)$ is p -valent in the unit disk E . Moreover, if $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha})$.

Applying the above lemmas, we prove

THEOREM I. If a function $f(z)$ belongs to the class $A_p(\alpha, \beta)$ with $\alpha > 0$ and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha})$.

PROOF. For a function $f(z)$ in the class A_p , we define the function $g(z)$ by

$$(2.1) \quad \begin{aligned} g(z) &= f(z)^{1/p} \\ &= z + g_2 z^2 + g_3 z^3 + \dots \end{aligned}$$

Then $g(z)$ is in the class A_1 , and satisfies

$$(2.2) \quad \frac{zf'(z)}{f(z)} = p \frac{zg'(z)}{g(z)}$$

and

$$(2.3) \quad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + (p-1) \frac{zg'(z)}{g(z)}.$$

It follows from (2.2) and (2.3) that

$$(2.4) \quad \begin{aligned} (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \\ = (p-\alpha) \frac{zg'(z)}{g(z)} + \alpha \left(1 + \frac{zg''(z)}{g'(z)} \right). \end{aligned}$$

Therefore, we have

$$(2.5) \quad f(z) \in A_p(\alpha, \beta) \iff \operatorname{Re} \left\{ \left(1 - \frac{\alpha}{p} \right) \frac{zg'(z)}{g(z)} + \frac{\alpha}{p} \left(1 + \frac{zg''(z)}{g'(z)} \right) \right\} > \frac{\beta}{p}$$

$$\Longleftrightarrow g(z) \in A_1(\alpha/p, \beta/p).$$

Applying Lemma 1 for $p = 1$, we see that

$$g(z) \in A_1(\alpha/p, \beta/p) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha}).$$

It follows that

$$f(z) \in A_p(\alpha, \beta) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha})$$

$$\Longleftrightarrow \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)^{1-p/\alpha} h(z)^{p/\alpha}} \right\} > 2^{2\beta/\alpha} \quad (h(z) \in S_1^*)$$

$$\Longleftrightarrow \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)^{1-1/\alpha} (h(z)^p)^{1/\alpha}} \right\} > p 2^{2\beta/\alpha} \quad (h(z)^p \in S_p^*)$$

$$\Longleftrightarrow f(z) \in B_p(1/\alpha, p 2^{2\beta/\alpha}).$$

This completes the assertion of Theorem 1.

REMARK I. Noting $p 2^{2\beta/\alpha} \geq 2^{2\beta/\alpha}$, we see that

$$B_p(1/\alpha, p 2^{2\beta/\alpha}) \subseteq B_1(1/\alpha, 2^{2\beta/\alpha}).$$

Thus Theorem 1 is the improvement of Lemma 1 by Nunokawa, Owa, Saitoh, Yaguchi and Lee [5].

Taking $p = 1$ in Theorem 1, we have

COROLLARY I. If $f(z) \in A_1(\alpha, \beta)$ with $\alpha > 0$ and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$.

Using the same manner as in the proof of Theorem 1, we have

THEOREM 2. If a function $f(z)$ belongs to the class $C_p(\alpha, \beta)$ with

$\alpha \neq 0$, $\beta > p$, and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha})$.

Finally, letting $p = 1$ in Theorem 2, we have

COROLLARY 2. If $f(z) \in C_1(\alpha, \beta)$ with $\alpha \neq 0$, $\beta > 1$, and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$.

REMARK 2. We note that Theorem 2 is the improvement of Lemma 2 due to Nunokawa and Owa [4].

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