

Difference Equation for A Population Model

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1. Introduction.

Recently many kinds of population models are formulated with difference equations [1,3,4]. Here we consider the following difference equation :

$$(1.1) \quad u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)},$$

seems to be a general statements of a relative socio-spatial dynamics, where $\alpha = 1 + r$, in which r is the net (births minus death) endogenous population (stock) growth rate, and the second term is a function depicting net immigration at $t+1$, in turn a function of a "momentum" to grow from t to $t+1$. So that we assume that $\alpha > 0$ ($r > -1$) and $\beta > 0$ in (1.1).

The model (1.1) was proposed by Prof. Dimitrios Dendrinos [2]. Let

$$u(t+2) = u_1(t+2) + u_2(t+2),$$

where $u_1(t+2) = \alpha\{u_1(t+1) + u_2(t+1)\}$, $u_2(t+2) = \beta \frac{u_2(t+1)}{u_1(t+1)}$. In this model we assume that α and β are constants. $u_1(t+2)$ is a term for endogenous population growth rate from $t+1$ to $t+2$, and $u_2(t+2)$ due to net in-migration rate.

Here we will study it, at first, from the viewpoint of (real) state-space analysis, especially of stability properties as $t \rightarrow \infty$, for positive solutions. Next we study it from the complex analytic view point.

2. Stability properties of positive solutions

In this section, we suppose that $u(t)$ is a solution of (1.1) which is positive for $t \geq t_0$, with the initial values $u(t_0)$ and $u(t_0+1)$.

We say $u(t)$ is *stable* if there are positive constants L_1, L_2 such that

$$0 < L_1 \leq u(t_0+n) \leq L_2 \quad \text{for } n \in \mathbb{N}.$$

$u(t)$ is *asymptotically stable* if there is b_0 such that, for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$|u(t_0 + n) - b_0| < \epsilon \quad \text{for } n \geq n_0, \quad \text{i.e.} \quad u(t_0 + n) \rightarrow b_0 \quad (n \rightarrow \infty).$$

We write (1.1) as

$$(2.1) \quad u(t+2) - \alpha u(t+1) = \frac{c}{u(t)}(u(t+1) - \alpha u(t)), \quad c = \frac{\beta}{\alpha}.$$

2.1. When $\alpha > 1$

Suppose $u(t_0 + 1) - \alpha u(t_0) \geq 0$ for some t_0 . Then we have $u(t_0 + n) \rightarrow \infty$ ($n \rightarrow \infty$) and the solution $u(t)$ is not stable. So if the solution $u(t)$ is stable, we assume that $u(t+1) - \alpha u(t) < 0$ for any $t \geq 0$.

(i) Suppose $u(t_0) \geq c$. If $u(t_0 + 1) > u(t_0)$, then we have $u(t) \rightarrow +\infty$. Therefore we must have that $u(t+1) \leq u(t)$ whenever $u(t) \geq c$. If $u(t)$ is stable and $u(t) \geq c$ for any t , then c is an asymptotically stable equilib. point : $u(t) \downarrow c$.

(ii) Suppose $u(t_0) < c$. If $u(t_0 + 1) < u(t_0)$, then $u(t_0 + 2) < u(t_0 + 1) < u(t_0) \leq c$, and $u(t_0 + n) \rightarrow -\infty$. So that $u(t)$ is not stable. Therefore, if $u(t)$ is stable, we must have

$$(2.2) \quad u(t+1) \leq u(t) \text{ whenever } u(t) > c, \quad u(t+1) \geq u(t) \text{ whenever } u(t) < c.$$

Conversely, if (2.2) holds, then $u(t) \rightarrow c$ as $t \rightarrow \infty$. Therefore we see that $u(t)$ is stable if and only if $u(t)$ is asymptotically stable, and c is the only stable equilib. point.

Indeed we can show the existence of a positive asymptotically stable solution from the section 4.

2.2. When $\alpha < 1$

Suppose $u(t_0 + 1) - \alpha u(t_0) \leq 0$, for some t_0 . Then we have $u(t_0 + n) \rightarrow 0$ and the population tends to death. So we may assume that $u(t+1) - \alpha u(t) > 0$ for any t .

(i) When $u(t_0) \leq c$ for some t_0 . We have

$$\alpha(u(t_0 + 1) - u(t_0)) \leq u(t_0 + 2) - u(t_0 + 1).$$

(i-a) Suppose $u(t_0 + 1) - u(t_0) \geq 0$ and there is k_0 such that $u(t_0 + k) \leq c$ ($0 \leq k < k_0$) and $u(t_0 + k_0) > c$. Furthermore if we assume $u(t_0 + k_0 + n + 1) - u(t_0 + k_0 + n) \geq 0$ for any $n \geq 1$, we would have

$$u(t_0 + k_0 + n) \uparrow u_0 < \infty (u_0 > c).$$

This is contradiction. Hence there is n_0 such that $u(t_0 + k_0 + n) \leq u(t_0 + k_0 + n + 1)$ ($n < n_0$) and $u(t_0 + k_0 + n_0) > u(t_0 + k_0 + n_0 + 1)$.

(i-b) Suppose $u(t_0 + 1) < u(t_0) \leq c$. If $u(t_0 + n) \leq u(t_0 + n - 1)$ ($n \geq 1$), then $u(t_0 + n) \downarrow 0$.

If there is k_0 such that $u(t_0 + k) \leq u(t_0 + k - 1)$ ($k < k_0$) and $u(t_0 + k_0) > u(t_0 + k_0 - 1)$, we suppose there is h_0 such that $u(t_0 + k_0 + h) \leq c$ ($h < h_0$) and $u(t_0 + k_0 + h_0) > c$. Then we have $u(t_0 + k_0 + h_0 + 1) > u(t_0 + k_0 + h_0) > c$. Now we proceed to the following case (ii).

(ii) When $u(t_0) > c$. Then we have

$$(2.3) \quad \alpha\{u(t_0 + 1) - u(t_0)\} > u(t_0 + 2) - u(t_0 + 1).$$

(ii-a) Suppose $u(t_0 + 1) > u(t_0) > 0$. If $u(t_0 + n) > u(t_0 + n - 1)$ ($n \geq 1$), we have

$$c < u(t_0 + n) \uparrow u_0 < \frac{u(t_0 + 1)}{1 - \alpha},$$

which is a contradiction. Thus there is n_0 such that

$$u(t_0 + n) > u(t_0 + n - 1) \quad (n < n_0) \quad \text{and} \quad u(t_0 + n + 1) < u(t_0 + n).$$

Thus we proceed to the following case (ii-b).

(ii-b) Suppose $u(t_0 + 1) < u(t_0)$. Then $u(t_0 + 2) < u(t_0 + 1)$ by (2.3). If $u(t_0 + n) \geq c$, then we get $u(t_0 + n) \downarrow c$. If $u(t_0 + n) \geq c$ ($n < n_0$) and $u(t_0 + n_0) < c$, then we come back to the case (i).

Thus we see that, when $0 < \alpha < 1$, we have either $u(t + n) \rightarrow 0$ or $u(t)$ is stable.

2.3. When $\alpha = 1$

We have in this case

$$u(t + 2) - u(t + 1) = \frac{c}{u(t)}(u(t + 1) - u(t))$$

Note that,

$$\text{if } \lim_{n \rightarrow \infty} u(t_0 + n) = u_0 \neq \pm\infty, \quad \text{then } \lim_{n \rightarrow \infty} \frac{u(t_0 + n + 1)}{u(t_0 + n)} = 1.$$

(i) When $u(t_0 + 1) > u(t_0)$. If $u(t_0) \geq c$. Then we have

$$u(t_0 + n) \uparrow u_0 \leq \frac{(2 - b)u(t_0 + 1) - u(t_0)}{1 - b}, \quad (u_0 > c).$$

If $u(t_0) < c$. Hereafter $u(t_0 + n)$ behaves as above and we get $u(t_0 + n) \uparrow u_0 < \infty$.

(ii) When $u(t_0 + 1) < u(t_0)$.

If $u(t_0) \leq c$, and $u(t_0 + n) > 0$. $u(t_0 + n)$ oscillate.

If $u(t_0) > c$ and $u(t_0 + n_1) \leq c$ for some n_1 , then $u(t_0 + n)$ oscillate.

If $u(t_0) > c$ and $u(t_0 + n) > c$ for all n , then there is a u_0 such that $u(t_0 + n) \downarrow u_0 \geq c$. Therefore, when $\alpha = 1$, we get that

either $u(t_0 + n_0) \leq 0$ ($\exists n_0$), or $u(t_0 + n) \downarrow u_0 \geq c$, or $u(t_0 + n) \uparrow u_0 > c$.

3. Analytic Solution

3.1. When $\alpha > 1$

We have showed that, if $u(t_0 + n)$ is stable, then $u(t_0 + n) \rightarrow c$ as $n \rightarrow \infty$. Then $v(t) = u(t) - c$ satisfies $v(t_0 + n) \rightarrow 0$ and

$$(3.1) \quad v(t+2) = (\alpha+1)v(t+1) - v(t) + F(v(t), v(t+1)),$$

where $F(v_1, v_2)$ is the sum of terms of higher degree with respect to v_1, v_2 . Let λ_1, λ_2 , be roots of the characteristic equation

$$(3.2) \quad \lambda^2 - (\alpha+1)\lambda + 1 = 0, \quad (0 < \lambda_2 < 1 < \lambda_1).$$

If we suppose $u(t+n)$ is stable when $n \rightarrow \infty$. Therefore we consider a particular solution of (1.1) in the form

$$(3.3) \quad u(t_0 + t) = c + \sum_{k=1}^{\infty} a_k \lambda_2^{kt}.$$

If we suppose $u(t+n)$ is stable when $n \rightarrow -\infty$, then we also consider solutions in the form

$$(3.4) \quad u(t_0 + t) = c + \sum_{k=1}^{\infty} a_{-k} \lambda_2^{-kt}.$$

Furthermore, if $u(t_0 - n) \rightarrow 0$ ($n \rightarrow \infty$), then we have

$$\lim_{n \rightarrow -\infty} \frac{u(t+n+2)}{u(t+n+1)} = \alpha.$$

And we have a solution $u(t) = a_1 \alpha^t$ (a_1 : arbitrary). Suppose $x(t) = \phi(t) \alpha^t$ be a solution of (1.1) such that $x(t-n) \rightarrow 0$ as $n \rightarrow +\infty$, and ϕ satisfies $\lim_{n \rightarrow \infty} |\phi(t-n) - a_1| < M$ uniformly on any compact set for some constant M . Then we have $\lim_{n \rightarrow \infty} (\phi(t-n+1) - \phi(t-n)) = 0$.

3.2. When $\alpha < 1$

There is no asymptotically stable solution.

3.3. When $\alpha = 1$

Suppose $u(t_0 + n) \uparrow u_0$ or $u(t_0 + n) \downarrow u_0$ with $u_0 > c$, then we have

$$u(t_0 + t + 1) - u(t_0 + t) = \sum_{k=1}^{\infty} d_k \left(\frac{c}{u_0} \right)^{kt}, \quad \frac{c}{u_0} < 1$$

and $u(t_0 + t)$ will be obtained.

4 Existence of analytic stable solutions in the case $\alpha > 1$

In this section, we write $u(t_0 + t)$ simply as $u(t)$. Time t is of course a real variable. But in this section, we consider t to be a complex variable, and we will prove existence of analytic solutions.

When $\alpha > 1$, we can determined a formal solution to (3.1) in the form (4.1),

$$(4.1) \quad v(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}, \quad 0 < \lambda = \lambda_2 < 1,$$

where

$$a_n \beta \{ \lambda^{2n} - (1 + \alpha) \lambda^n + 1 \} = \alpha \sum_{k=1}^{n-1} a_k a_{n-k} \lambda^{n-k} (\alpha - \lambda^{n-k}).$$

4.1. Make a Map T and existence of a fixed point of T .

We rewrite (3.1) as

$$(4.2) \quad v(t) = \beta \frac{(1 + \alpha)v(t+1) - v(t+2)}{\alpha v(t+2) - \alpha^2 v(t+1) + \beta} = f(v(t+1), v(t+2)).$$

Let N be a positive integer. Put $P_N(t) = \sum_{n=1}^N a_n \lambda^{nt}$ and $y(t) = v(t) - P_N(t)$. We rewrite (4.2) as

$$(4.3) \quad y(t) = g(t, y(t+1), y(t+2))$$

Put

$$\begin{cases} S(\rho) = \{t \in \mathbb{C} : |\lambda^t| \leq \rho\} \\ H(A, \rho) = \{y : y(t) \text{ is holomorphic and } |y(t)| \leq A|\lambda^t|^{N+1} \text{ for } t \in S(\rho)\}. \end{cases}$$

Take $A > 0$ and $0 < \rho < 1$, which will be determined later. For $y(t) \in H(A, \rho)$, put

$$(4.4) \quad T[y](t) = g(t, y(t+1), y(t+2)).$$

Then we can show that T has a fixed point $y(t) = y_N(t) \in H(A, \rho)$ by schauder's fixed point theorem in [5].

Furthermore we can show the uniqueness of the fixed point and arbitrariness of N (independence of N). Thus we have proved that a solution $v(t)$ is defined and holomorphic in $S(\rho)$ for a $\rho > 0$, which has the expansion (4.1).

4.2. Analytic General Solutions.

Analytic solutions of some difference equations are investigated in [6]-[9]. In this section, we shall investigate analytic general solutions of (1.1).

From in [6], we have following theorem.

THEOREM 1. *Let $v(s)$ be the solution of (4.2) obtained in section 3. Suppose $x(t)$ be an analytic solution of (4.2) such that $x(t-n) \rightarrow 0$ as $n \rightarrow +\infty$, uniformly on any compact set. Then there is a periodic entire function $\pi(t)$, ($\pi(t+1) = \pi(t)$), such that*

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right).$$

Conversely if we put

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right),$$

where π is a periodic function whose period is one, then $x(t)$ is a solution of (3.1).

Now we have sought general solutions of the population model which is given by the equation (1.1) such that

$$u(t) = \frac{\beta}{\alpha} + v\left(t + \frac{\log \pi(t)}{\log \lambda}\right)$$

where $\pi(t)$ is an arbitrarily periodic function whose period is one.

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