

The Kodaira dimension of subvarieties of Siegel modular varieties

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1 Introduction

Let t be a positive integer, and $\mathcal{A}_{g,t}$ the moduli space of g -dimensional abelian varieties with polarizations of type $T = (1, \dots, 1, t)$. We write $\tilde{\mathcal{A}}_{g,t}$ for a smooth compactification of $\mathcal{A}_{g,t}$. It is known that $\tilde{\mathcal{A}}_{g,t}$ is of general type in the following cases:

- (1)(Tai, Freitag, Mumford) $t = 1, g \geq 7$,
- (3)(Gritsenko) $t = 2, g \geq 13$,
- (2)(Tai) $t \neq 1, 2, g \geq 16$.

We have the same result for subvarieties in $\mathcal{A}_{g,t}$. To be more precise, Freitag, Weissauer and Tsuyumine showed that in the case where $g \geq 10, t = 1$, all subvarieties of codimension one in $\mathcal{A}_{g,t}$ are of general type. Here they adopted the weakened form of the notion “general type”. The purpose of the present paper is to report a similar result for the case where $t \neq 1$. Our main result is the following:

Theorem (Theorem 7) Assume $g \geq 13$. If $\tilde{\mathcal{A}}_{g,t}$ is of general type, then any irreducible variety in $\mathcal{A}_{g,t}$ is of general type.

Throughout this article, we assume $g \geq 13$.

2 Siegel modular varieties

Let $H_g = \{Z \in M_g(\mathbb{C}) \mid {}^t Z = Z, \operatorname{Im} Z > 0\}$ denote the Siegel upper half space. The symplectic group $Sp_{2g}(\mathbb{R})$ acts on H_g by the usual symplectic substitution

$$\gamma \rightarrow \gamma Z = (AZ + B)(CZ + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R}).$$

For a positive integer t , let

$$\Delta_t = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & t \end{pmatrix}, \quad \Lambda_t = \begin{pmatrix} O & \Delta_t \\ -\Delta_t & O \end{pmatrix}.$$

We now define some kinds of modular groups. Define

$$\tilde{\Gamma}_t = \{\gamma \in GL(2g, \mathbb{Z}) \mid \gamma \Lambda_t {}^t \gamma = \Lambda_t\}.$$

Using $R = \begin{pmatrix} E & O \\ O & \Delta_t \end{pmatrix}$, put $\Gamma_t = R^{-1}\tilde{\Gamma}_t R$. For $L = \mathbb{Z}^{2g} \subset \mathbb{C}^g$, $\langle, \rangle : L \times L \rightarrow \mathbb{C}$ is defined by $(x, y) \mapsto x\Lambda_t^t y$. We denote by $L^\vee$ the dual lattice of L with respect to $\langle, \rangle$. Put

$$\Gamma_t^{\text{lev}} = \{ \gamma \in \Gamma_T \mid M|_{L^\vee/L} = \text{id}|_{L^\vee/L} \}.$$

We call Γ_t and Γ_t^{lev} *Siegel modular groups*. When $t = 1$, we write $\Gamma_g = \Gamma_t$. Concerning Γ_t^{lev} , it is well known that Γ_t^{lev} is a subgroup of $Sp_{2g}(\mathbb{Z})$, and that Γ_t^{lev} is a normal subgroup of finite index in Γ_t .

Let Γ be a Siegel modular group occurring in the above. For a function f on H_g and a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the *slash-operator* is defined by

$$f|_k \gamma(Z) = \det(CZ + D)^{-k} f(\gamma Z).$$

A holomorphic function f on H_g is a Γ -*modular form of weight k* on H_g if for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,

$$f|_k \gamma(Z) = f(Z).$$

Define $\mathcal{A}_{g,t} = \Gamma_t \backslash H_g$, $\mathcal{A}_{g,t}^{\text{lev}} = \Gamma_t^{\text{lev}} \backslash H_g$. In particular, write $\mathcal{A}_g = \mathcal{A}_{g,1}$. The quotient spaces $\mathcal{A}_{g,t}$ and $\mathcal{A}_{g,t}^{\text{lev}}$ are the moduli spaces of g -dimensional abelian varieties of the polarization of type T without or with canonical structure, respectively. Let $\mathcal{A}$ be a one of them. From the theory of the toroidal compactification, we can construct a projective variety $\overline{\mathcal{A}}$ such that $\overline{\mathcal{A}} - \mathcal{A}$ has normal crossing and $\overline{\mathcal{A}}$ has only finite quotient singularities. Resolving these singularities, we obtain a projective nonsingular variety $\tilde{\mathcal{A}}$. We call $\overline{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ *Siegel modular varieties*. These varieties are central objects in this paper.

Freitag defined in [1] the following weakened form of the notion “general type”.

Definition 1 A nonsingular compact irreducible algebraic variety X is of type G (of general type) if there exist $n = \dim X$ algebraically independent rational functions $f_1, \dots, f_n$ and a non-zero holomorphic tensor $T \in \Omega^{\otimes d}(X)$ ($d > 0$) such that tensors $f_1 T, \dots, f_n T$ are holomorphic on X

We adopt this notion for subvarieties of Siegel modular varieties.

3 Construction of certain differential forms

Let $Z = (z_{ij})$. Define

$$e_{ij} = \begin{cases} 1 & (i \neq j) \\ 2 & (i = j) \end{cases}.$$

Using it, put

$$\omega_{ij} = (-1)^{i+j} e_{ij} dz_{11} \wedge dz_{12} \wedge \cdots \wedge \widehat{dz_{ij}} \wedge \cdots \wedge dz_{gg} \quad (1 \leq i \leq j \leq g),$$

where $\widehat{dz_{ij}}$ means that dz_{ij} is omitted. Let $\omega = (\omega_{ij})$. Then for a non-negative integer r , $\omega^{\otimes r}$ stands for the tensor power of ω . The tensor power $\omega^{\otimes r}$ satisfies

$$\gamma \cdot \omega^{\otimes r} = \det(CZ + D)^{-r(g+1)} (CZ + D)^{\otimes r} \omega^{\otimes r} \cdot {}^t(CZ + D)^{\otimes r}$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$.

Now let $A = (a_{ij})$ be a matrix of size g , and I, J the ordered sets of r integers in $\{1, \dots, g\}$, where a repeated choice is allowed.

For $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_r\}$, define

$$A^{(I,J)} = a_{i_1 j_1} \cdots a_{i_r j_r}.$$

Then the (k, l) -entry of $A^{\otimes r}$ is $A^{(I,J)}$ if

$$k = 1 + \sum_{s=1}^r (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^r (j_s - 1)g^{s-1} \quad (1 \leq k, l \leq g^r).$$

Put $\text{sgn}(I) = \prod_{i \in I} (-1)^i$. Suppose $m \geq 2(g-1)$. Let η be a complex $m \times (g-1)$ matrix such that ${}^t \eta \eta = 0$ and $\text{rank } \eta = g-1$. Denote by η_i ($1 \leq i \leq g$) the $(g-1) \times g$ matrix such that

$$\eta_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

We take a fixed positive symmetric matrix F of size m with rational coefficients. Define a theta series associated to F by

$$\begin{aligned} \theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix} (Z) &= \text{sgn}(I) \text{sgn}(J) \sum_G \prod_{i \in I} \det(\eta_i {}^t(G+u) F^{1/2} \eta) \\ &\times \prod_{j \in J} \det(\eta_j {}^t(G+u) F^{1/2} \eta) e \left[\text{tr} \left(\frac{1}{2} Z F [G+u] + {}^t(G+u)v \right) \right], \end{aligned}$$

where G runs through all $m \times g$ integral matrices, and u, v are $m \times g$ matrices with rational coefficients.

Let $\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (Z)$ be the square matrix of size g^r whose (k, l) -entry is $\theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix} (Z)$, where

$$k = 1 + \sum_{s=1}^r (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^r (j_s - 1)g^{s-1}$$

when $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_r\}$. Tsuyumine [9] shows that there exist $l, r' \in \mathbb{N}$ such that for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(l)$,

$$\begin{aligned} &\left(\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (\gamma Z) \right)^{\otimes r'} \\ &= \det(CZ+D)^{(m/2+2r)r'} ({}^t(CZ+D)^{-1})^{\otimes rr'} \left(\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (Z) \right)^{\otimes r'} ((CZ+D)^{-1})^{\otimes rr'}. \end{aligned}$$

Let $\{\gamma_j\}$ be a system of representatives of Γ_g modulo $\Gamma_g(l)$. When $\gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$, put

$$\Psi(Z) = \sum_j \det(C_j Z + D_j)^{-(m/2+2r)r'} \cdot {}^t(C_j Z + D_j)^{\otimes rr'} \left(\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (\gamma_j Z) \right)^{\otimes r'} ((C_j Z + D_j)^{-1})^{\otimes rr'}.$$

Then for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$,

$$\Psi(\gamma Z) = \det(CZ + D)^{(m/2+2r)r'} ({}^t(CZ + D)^{-1})^{\otimes rr'} \Psi(Z) ((CZ + D)^{-1})^{\otimes rr'}.$$

Moreover we construct the symmetrization of $\Psi(Z)$. Let $\{\delta_i\}$ be a system of representatives of Γ_t modulo Γ_t^{lev} . When $\delta_i = \begin{pmatrix} A'_i & B'_i \\ C'_i & D'_i \end{pmatrix}$, put

$$\Phi(Z) = \sum_i \det(C'_i Z + D'_i)^{-(m/2+2r)r'} \cdot {}^t(C'_i Z + D'_i)^{\otimes rr'} \Psi(\delta_i Z) ((C'_i Z + D'_i))^{-1})^{\otimes rr'}.$$

Then we have

Proposition 2 For any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t$, we have

$$\Phi(\gamma Z) = \det(CZ + D)^{(m/2+2r)r'} ({}^t(CZ + D)^{-1})^{\otimes rr'} \Phi(Z) ((CZ + D)^{-1})^{\otimes rr'}.$$

Proposition 3 Let Z_0 be any fixed point of H_g . Take any non-zero complex symmetric matrix W of size g . Let m be an integer with $m \geq 2(g-1)$. Then for infinitely many r and r' , there exists a symmetric matrix $\Phi(Z)$ occurring in the last proposition such that $\text{tr}(\Phi(Z_0)W^{\otimes rr'}) \neq 0$.

Put

$$\lambda_{m,r,r'} = \text{tr} \left(\Phi(Z) \omega^{\otimes rr'} \right).$$

Combining Proposition 1 with Proposition 2, we conclude

Theorem 4 Let Z_0 be any fixed point in H_g , and m an integer with $m \geq 2(g-1)$. Then for infinitely many r and r' , there exists $\lambda_{m,r,r'}$ such that

- (1) $\lambda_{m,r,r'}$ does not vanish at $Z = Z_0$.
- (2) for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t$,

$$\gamma \cdot \lambda_{m,r,r'} = \det(CZ + D)^{(m/2-r(g-1))r'} \lambda_{m,r,r'}.$$

4 Extensibility of certain differential forms

For a rational boundary component F , let $P(F) \subset Sp_{2g}(\mathbb{R})$ denote the stabilizer of F , $P'(F)$ the center of the unipotent radical of $P(F)$, and $C(F)$ the self-adjoint cone corresponding to F .

Definition 5 Let Γ be a Siegel modular group. A Γ -modular form f *vanishes on the rational boundary component F of order at least l* the following condition are satisfied. If we consider the Fourier-Jacobi expansion of f at F

$$f(Z) = \sum_{x \in (P'(F))^\vee} a_x^F(u, t) e[\langle x, z \rangle],$$

then $a_x^F \neq 0$ implies $\min_{y \in P'(F) \cap \overline{C(F)} - \{0\}} (x, y) \geq l$.

Assume that $r(g-1) > m/2$. For any Γ_t -modular form f of weight $(r(g-1) - m/2)r'$, by Theorem 4, $f\lambda_{m,r,r'}$ is a Γ_t -invariant form in $(\Omega_{H_g}^{N-1})^{\otimes rr'}$.

Proposition 6 *If a Γ_t -modular form f vanishes on all rational corank 1 boundary components of order at least rr'/C_t , then $f\lambda_{m,r,r'}$ extends to $\tilde{\mathcal{A}}_{g,t}$. Here C_t stands for $\min\{1, \sqrt{3}/\sqrt[9]{t^{g-1}}\}$.*

5 The main result

Theorem 7 *If $\tilde{\mathcal{A}}_{g,t}$ is of general type, then any irreducible subvariety of codimension one in $\mathcal{A}_{g,t}$ is of general type.*

Let us give an outline of the proof to the above theorem. Let D be any irreducible subvariety in $\mathcal{A}_{g,t}$ of codimension 1, and $\pi : H_g \rightarrow \mathcal{A}_{g,t}$ the canonical map. It should be noted that we can construct a Γ_g -modular form f whose restriction to $\pi^{-1}(D)$ does not vanish ([1], [10]). For such a weight k modular form f , its symmetrization $\text{Sym}(f) = \prod_{\gamma \in \Gamma_t/\Gamma_t^{\text{lev}}} f|_k \gamma$ is a Γ_t -modular form that does not vanish on $\pi^{-1}(D)$. The following diagram is helpful:

$$\begin{array}{ccc} & \mathcal{A}_{g,t}^{\text{lev}} & \\ \swarrow & & \searrow \\ \mathcal{A}_g & & \mathcal{A}_{g,t} \end{array}$$

If f is a Γ_g -modular form, then we have

$$\frac{(g-1)\text{ord}(\text{Sym}(f))}{\text{weight}(\text{Sym}(f))} = \frac{(g-1)\text{ord}(f)}{\text{weight}(f)}.$$

Here $\text{ord}(\text{Sym}(f))$ and $\text{ord}(f)$ are vanishing orders at rational corank 1 boundary components.

If f is a non-trivial Γ_t -modular form such that $(g-1)\text{ord}(f)/\text{weight}(f) > 1$, then for certain integers a, b , each modular form f' in $f^{ak} M_{bk}(\Gamma_t)$ ($k \geq 1$) has enough vanishing order at the cusp. If $\text{weight}(f') = (r(g-1) - m/2)r'$, then $f'\lambda_{m,r,r'}$ extends to a section of $(\Omega_{\tilde{\mathcal{A}}_{g,t}}^{N-1})^{\otimes rr'}$, where $N = g(g+1)/2$.

Generalizing Lemma 2.2 in [2], it is possible to find generators for Γ_t . Using them, we see that $[\Gamma_t, \Gamma_t]$, the commutator subgroup of Γ_t , is Γ_t itself. Moreover, Γ_t satisfies $b_1(\Gamma_t) = 0$, $b_2(\Gamma_t) = 1$. Hence by Theorem 1' in [8], any effective divisor on $\mathcal{A}_{g,t}$ is

defined by some Γ_t -modular form. Furthermore, the ring $\bigoplus_{k \geq 0} M_k(\Gamma_t)$ of Γ_t -modular forms is factorial.

There exists a Γ_t -modular form h such that its divisor (h) on H_g is $\pi^{-1}(D)$. By Theorem 4, for infinitely many $r, r', \lambda_{m,r,r'} | \pi^{-1}(D) \neq 0$. We can take such r, r' from the set of multiples of any fixed integer. For a suitable integer k , there exist weight k Γ_t -modular forms $g_1, \dots, g_N$ such that they do not vanish on D , that each $g_i \lambda_{m,r,r'}$ extends to $\tilde{\mathcal{A}}_{g,t}$, and that $g_2/g_1, \dots, g_N/g_1$ are algebraically independent. Therefore D is of general type.

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References

- [1] E. Freitag, Holomorphic tensors on subvarieties on the Siegel modular variety. In: *Automorphic Forms of Several Variables, Taniguchi Symposium, Katata, 1983*, 93-113, Birkhäuser, 1984.
- [2] V. Gritsenko, Irrationality of the moduli spaces of polarized abelian surfaces. In: *Abelian Varieties. Proc. the Egloffstein Conf. on Abelian Varieties*, de Gruyter, 63-81, 1995.
- [3] V. Gritsenko and K. Hulek, Commutator coverings of Siegel threefolds. *Duke Math. J.* **94** (1998), 509-542.
- [4] I. Reiner, Real linear characters of the symplectic modular group. *Proc. Amer. Math. Soc.* **6** (1955), 987-990.
- [5] E. Schellhammer, The Kodaira dimension of Siegel modular varieties of genus 3 or higher. *Bollettino U.M.I.* **9** (2006), 749-776.
- [6] Y.-S. Tai, On the Kodaira dimension of the moduli spaces of abelian varieties. *Invent. Math.* **68** (1982), 425-439.
- [7] Y.-S. Tai, On the Kodaira dimension of moduli spaces of abelian varieties with non-principal polarizations. In: *Abelian Varieties. Proc. the Egloffstein Conf. on Abelian Varieties*, de Gruyter, 293-302, 1995.
- [8] S. Tsuyumine, Factorial property of a ring of automorphic forms. *Trans. Amer. Math. Soc.* **296** (1986), 111-123.
- [9] S. Tsuyumine, Multi-tensors of differential forms on the Siegel modular variety and on its subvarieties. *Tsukuba J. Math.* **11** (1987), 107-119.
- [10] R. Weissauer, Untervarietäten der Siegelschen Modulmannigfaltigkeiten von allgemeinem Typ. *Math. Ann.* **275** (1986), 207-220.