

EXTENSIONS OF HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. We give an extension of recent Lin's improvement of a generalized Schwarz inequality, which is based on the Heinz-Kato-Furuta inequality. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality.

1. Introduction.

First of all, we cite a generalized Schwarz inequality which is a base of Lin's recent paper [9]. For a (bounded linear) operator T acting on a Hilbert space H ,

$$(1) \quad |(Tx, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

for all $\alpha \in [0, 1]$ and $x, y \in H$, where $|X|$ is the square root of X^*X for an operator X on H . It implies the Heinz-Kato inequality via the Löwner-Heinz inequality, cf. [3], [10]. On the other hand, Furuta [7] extended the Heinz-Kato inequality, so called the Heinz-Kato-Furuta inequality. Rephrasing it parallel to (1), we have

$$(2) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y)$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

Very recently, Lin [9] sharpened (1) as follows:

Theorem L. *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $Tz \neq 0$ and $(Tz, y) = 0$,*

$$(3) \quad |(Tx, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2(1-\alpha)}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

for all $\alpha \in [0, 1]$ and $x, y \in H$. The equality holds if and only if $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$ and T^*y are proportional, or equivalently, $Tx - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}Tz$ and $|T^*|^{2(1-\alpha)}y$ are proportional.

In this note, we extend Theorem L, which is based on the Heinz-Kato-Furuta inequality (2). Our proof is quite simple, in which we clarify the meaning of the assumption in Theorem L that $Tz \neq 0$ and $(Tz, y) = 0$. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality, and Furuta's further generalization [6; Theorem 3] of the Heinz-Kato inequality via the Furuta inequality [4]. Incidentally we discuss Bernstein type inequality on the line of our result.

2. Heinz-Kato-Furuta inequality.

For the sake of convenience, we first cite the Heinz-Kato-Furuta inequality [7]:

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The Heinz-Kato-Furuta inequality. *Let T be an operator on H . If A and B are positive operators on H such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$(4) \quad |(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\|$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

We here remark that the Heinz-Kato inequality is just the case $\alpha + \beta = 1$ in above and that it corresponds to (1). Thus we have the following extension of Theorem L. Throughout this paper, let $T = U|T|$ be the polar decomposition of an operator T on H .

Theorem 1. *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$(5) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y)$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x, y \in H$. In the case $\alpha, \beta > 0$, the equality in (5) holds if and only if $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$ are proportional, or equivalently, $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$ are proportional.

It is easily seen that Theorem L is the case $\alpha + \beta = 1$ in Theorem 1. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e., $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for $\alpha \in [0, 1]$:

Theorem 2. *Let T be an operator on H . If A and B are positive operators on H such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$(6) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq \|A^\alpha x\|^2 \|B^\beta y\|^2$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (6) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T^*|^{2\beta}y$ and $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$ are proportional; the third condition is equivalent to that $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$ are proportional.

Proof of Theorem 1. We only use the positivity of the Gram matrix

$$G = G(U|T|^\alpha x, |T^*|^\beta y, U|T|^\alpha z).$$

Noting that

$$(|T^*|^\beta y, U|T|^\alpha z) = (y, |T^*|^\beta U|T|^\alpha z) = (y, T|T|^{\alpha+\beta-1}z) = 0$$

by the assumption, we have

$$G = \begin{pmatrix} \| |T|^\alpha x \|^2 & (U|T|^\alpha x, |T^*|^\beta y) & (U|T|^\alpha x, U|T|^\alpha z) \\ (U|T|^\alpha x, |T^*|^\beta y)^* & \| |T^*|^\beta y \|^2 & 0 \\ (U|T|^\alpha x, U|T|^\alpha z)^* & 0 & \| |T|^\alpha z \|^2 \end{pmatrix}.$$

Since $|T|^\alpha z \neq 0$, we have

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y).$$

To prove the equality condition, we set up the following lemma, which is applied to the vectors $u = U|T|^\alpha x$, $v = U|T|^\alpha z$ and $w = |T^*|^\beta y$.

Lemma. (1) If $v \neq 0$ and $(v, w) = 0$, then $\{u, v, w\}$ is linearly dependent if and only if w and $u - \frac{(u, v)}{\|v\|^2}v$ are proportional.

(2) Let $T = U|T|$ be the polar decomposition of an operator T on H , (namely $\ker(U) = \ker(T)$). For $\alpha, \beta > 0$ with $\alpha + \beta \geq 1$ and $y, w \in H$, the following conditions are mutually equivalent; (i) $|T^*|^\beta y$ and $U|T|^\alpha w$ are proportional. (ii) $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}w$ are proportional. (iii) $|T^*|^\beta y$ and $T|T|^{\alpha-1}w$ are proportional.

Proof. (1) Suppose that $au + bv + cw = 0$ for some $(a, b, c) \neq 0$. Then $a(u, v) + b\|v\|^2 = 0$ and so $b = -\frac{a(u, v)}{\|v\|^2}$. Hence we have

$$0 = au + bv + cw = a(u - \frac{(u, v)}{\|v\|^2}v) + cw.$$

Since $a = c = 0$ does not occur by $v \neq 0$, vectors $u - \frac{(u, v)}{\|v\|^2}v$ and w are proportional. The converse is easily checked.

(2) (i) is equivalent to that $U|T|^\beta U^*y$ and $U|T|^\alpha w$ are proportional. Noting that $\alpha, \beta > 0$ and $\ker(U) = \ker(T)$, it is equivalent to (ii). Similarly we have the equivalence between (i) and (iii).

3. Furuta inequality.

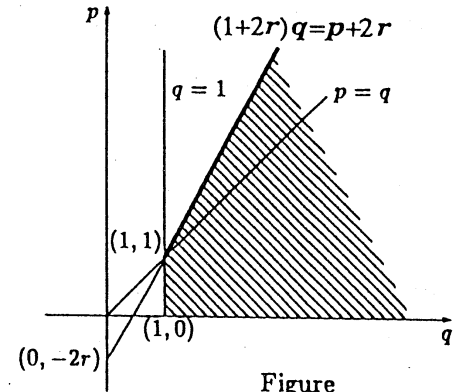
In [6], the Heinz-Kato-Furuta inequality is extended by the use of the Furuta inequality; Theorem 1 also gives us an improvement of the extension due to Furuta. For the sake of convenience, we cite the Furuta inequality [4], see also [2], [5], [8].

The Furuta inequality. If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(*) \quad (1 + 2r)q \geq p + 2r.$$



Figure

The domain representing (*) is drawn in the right and it is shown in [11] that this domain is *best possible one* for the Furuta inequality.

Theorem 3. Let T be an operator on H . If A and B are positive operators on H such that $T^*T \leq A^2$ and $TT^* \leq B^2$. Then for each $r, s \geq 0$

$$(7) \quad \begin{aligned} & |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 + \frac{|(|T|^{2(1+2r)\alpha}x, z)|^2 (|T^*|^{2(1+2s)\beta}y, y)}{(|T|^{2(1+2r)\alpha}z, z)} \\ & \leq ((|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x) ((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y) \end{aligned}$$

for all $p, q \geq 1$, $\alpha, \beta \in [0, 1]$ with $(1 + 2r)\alpha + (1 + 2s)\beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z \neq 0$ and $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (7) holds if and only if $|T|^{2(1+2r)\alpha}x = (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x$, $|T^*|^{2(1+2s)\beta}y = (|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y$ and $|T|^{2(1+2r)\alpha}(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z)$ and $|T|^{2(1+2r)\alpha+2(1+2s)\beta-1}T^*y$

are proportional; the latter is equivalent to that $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z)$ and $|T^*|^{2(1+2s)\beta}y$ are proportional.

Proof. We use Theorem 1 by replacing α (resp. β) to $\alpha_1 = (1+2r)\alpha$ (resp. $\beta_1 = (1+2s)\beta$). Then we have

$$(8) \quad |(T|T|^{\alpha_1+\beta_1-1}x, y)|^2 + \frac{|(|T|^{2\alpha_1}x, z)|^2(|T^*|^{2\beta_1}y, y)}{(|T|^{2\alpha_1}z, z)} \leq (|T|^{2\alpha_1}x, x)(|T^*|^{2\beta_1}y, y).$$

Next we use the Furuta inequality for $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$; namely (for the former) we replace A, B, q in the Furuta inequality to $A^2, |T|^2, \frac{p+2r}{(1+2r)\alpha}$ respectively. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+2r)\alpha} \leq (|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}$$

and similarly

$$|T^*|^{2\beta_1} = |T^*|^{2(1+2s)\beta} \leq ((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}).$$

Combining with (8), we obtain the inequality (7).

The equality condition is showed similarly to Theorem 2.

Remark. (1) We remark that the condition $(1+2r)\alpha + (1+2s)\beta \geq 1$ in Theorem 3 is unnecessary if T is either positive or invertible.

(2) Though Theorem 3 is followed from the Furuta inequality, they are equivalent actually, that is, Theorem 3 is an alternative representation of the Furuta inequality. As a matter of fact, we put $T = B, \alpha = \beta, r = s$ and also $x = y$ in Theorem 3. Thus it follows from the above remark (1) that if $A^2 \geq B^2$, then for $B^{2(1+2r)\alpha}z \neq 0$ and $(B^{2(1+2r)\alpha}z, x) = 0$

$$\begin{aligned} & |(B^{2(1+2r)\alpha}x, x)|^2 + \frac{|(B^{2(1+2r)\alpha}x, z)|^2(B^{2(1+2r)\alpha}x, x)}{(B^{2(1+2r)\alpha}z, z)} \\ & \leq ((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)((B^{2(1+2r)\alpha}x, x), \end{aligned}$$

that is, $A^2 \geq B^2$ ensures

$$(B^{2(1+2r)\alpha}x, x)^2 \leq ((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)$$

for all $p \geq 1, r \geq 0$ and $\alpha \in [0, 1]$. This is nothing but the Furuta inequality.

4. Generalization.

In this section, we generalize Theorem 1 along with a generalization of Theorem L [9; Theorem 4].

Theorem 4. Let T be an operator on H and $0 \neq y \in H$. If $T|T|^{\alpha+\beta-1}z_i \neq 0$ and $(T|T|^{\alpha+\beta-1}z_i, y) = 0$ for $i = 1, 2, \dots, n$, then

$$(9) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha}u_{i-1}, z_i)|^2|||T^*|^{\beta}y||^2}{|||T|^{\alpha}z_i||^2} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y)$$

for $\alpha, \beta > 0$ with $\alpha + \beta \geq 1$, where $u_0 = x$ and $u_i = u_{i-1} - \frac{(|T|^{2\alpha}u_{i-1}, z_i)}{|||T|^{\alpha}z_i||^2}z_i$ for $i = 1, 2, \dots, n$. The equality in (9) holds if and only if $|T^*|^{\beta}y$ and $U|T|^{\alpha}u_n$ are propotional.

Proof. By the definition of u_i , we have

$$\sum u_i = \sum u_{i-1} - \sum \frac{(|T|^{2\alpha}u_{i-1}, z_i)}{|||T|^{\alpha}z_i||^2}z_i$$

and so

$$u_n = x - \sum \frac{(|T|^{2\alpha} u_{i-1}, z_i)}{\| |T|^\alpha z_i \|^2} z_i.$$

Also we have

$$|T|^\alpha u_i = |T|^\alpha u_{i-1} - \frac{(|T|^{2\alpha} u_{i-1}, z_i)}{\| |T|^\alpha z_i \|^2} |T|^\alpha z_i,$$

so that

$$\| |T|^\alpha u_i \|^2 = \| |T|^\alpha u_{i-1} \|^2 - \frac{|(|T|^{2\alpha} u_{i-1}, z_i)|^2}{\| |T|^\alpha z_i \|^2}.$$

Summing up this on $i = 1, \dots, n$,

$$\| |T|^\alpha u_n \|^2 = \| |T|^\alpha x \|^2 - \sum \frac{|(|T|^{2\alpha} u_{i-1}, z_i)|^2}{\| |T|^\alpha z_i \|^2}.$$

Hence it follows from the assumption $(T|T|^{\alpha+\beta-1} z_i, y) = 0$ that

$$\begin{aligned} & \| |T^*|^\beta y \|^2 \| |T|^\alpha x \|^2 - \| |T^*|^\beta y \|^2 \sum \frac{|(|T|^{2\alpha} u_{i-1}, z_i)|^2}{\| |T|^\alpha z_i \|^2} \\ &= \| |T^*|^\beta y \|^2 \| |T|^\alpha u_n \|^2 \\ &\geq |(|T^*|^\beta y, U|T|^\alpha u_n)|^2 \\ &= |(|T^*|^\beta y, U|T|^\alpha x - \sum \frac{(|T|^{2\alpha} u_{i-1}, z_i)}{\| |T|^\alpha z_i \|^2} U|T|^\alpha z_i)|^2 \\ &= |(|T^*|^\beta y, U|T|^\alpha x)|^2 \\ &= |(T|T|^{\alpha+\beta-1} x, y)|^2. \end{aligned}$$

The equality condition is obvious by seeing the only inequality in the above.

Another generalization of Theorem 1 is as follows:

Theorem 5. *Under the same conditions as Theorem 4, the following inequality holds;*

$$|(T|T|^{\alpha+\beta-1} x, y)|^2 + \frac{\sum_i |(|T|^{2\alpha} x, z_i)|^2 \| |T^*|^\beta y \|^2}{\sum_i \| |T|^\alpha z_i \|^2} \leq (|T|^{2\alpha} x, x) (|T^*|^{2\beta} y, y)$$

As a matter of fact, since

$$\{ \| |T|^\alpha x \|^2 \| |T^*|^\beta y \|^2 - |(T|T|^{\alpha+\beta-1} x, y)|^2 \} \| |T|^\alpha z_i \|^2 \geq \| |T^*|^\beta y \|^2 |(|T|^{2\alpha} x, z_i)|^2$$

by Theorem 1, we have it by summing up on i .

Remark. Theorems 4 and 5 give us generalizations of Theorems 2 and 3, whose statements and proofs are quite similar to them.

5. A concluding remark.

Lin also discussed Bernstein type inequalities independently on Theorem L, [9; Theorem 3], see [1]. As an application of Theorem 1, we have a generalization of it:

Theorem 6. Let T be an operator on H having a nonzero normal eigenvalue λ with an eigenvector e . If $y \in H$ satisfies $(e, y) = 0$ and $T^*y \neq 0$, then

$$|\lambda|^2 |(x, e)|^2 \leq \frac{\|Tx\|^2 \| |T^*|^\beta T^*y \|^2 - |(T|T|^\beta x, T^*y)|^2}{\| |T^*|^\beta T^*y \|^2}$$

for all $x \in H$ and $\beta \in [0, 1]$.

Proof. We put $\alpha = 1$, $z = e$ and replace y to T^*y in Theorem 1. Since $(|T|^\beta e, T^*y) = 0$ by $(e, y) = 0$, It follows from Theorem 1 that

$$|(T|T|^\beta x, T^*y)|^2 + \| |T^*|^\beta T^*y \|^2 |\lambda|^2 |(x, e)|^2 \leq \|Tx\|^2 \| |T^*|^\beta T^*y \|^2,$$

so that we have the desired inequality.

We obtain Lin's inequality [9; Theorem 3] by taking $\beta = 0$ in Theorem 6.

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