

HYPERBOLIC BESOV FUNCTIONS

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ABSTRACT. In this paper we study bounded holomorphic functions in the unit disk D with given growth of hyperbolic derivative and application of these functions to composition operators.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in $\mathbf{C}$ with pseudohyperbolic metric

$$\rho(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right|$$

and with hyperbolic metric

$$\sigma(a, b) = \frac{1}{2} \log \frac{1 + \rho(a, b)}{1 - \rho(a, b)}.$$

Here $dA(z)$ is a normalized area measure on D and $d\lambda(z)$ is hyperbolic area measure

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}.$$

Let B be the family of bounded holomorphic functions $f(z)$, $|f(z)| < 1$, in D and $f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$ be the hyperbolic derivative of $f(z)$. Let $f_a(z) = f\left(\frac{z + a}{1 + \bar{a}z}\right)$, $a \in D$.

Definition. For $1 \leq p \leq \infty$ hyperbolic analytic Besov class B_p^h consists of functions $f(z) \in B$ which satisfy the condition

$$(1) \quad \|f\|_{B_p^h} = \left(\iint_D ((1 - |z|^2)f^*(z))^p d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

Classes B_p^h , $1 \leq p \leq \infty$, are Möbius invariant. By Schwarz-Pick lemma

$$(1 - |z|^2)f^*(z) \leq 1$$

for any $f \in B$, i.e. $B_\infty^h = B$, but Möbius transforms of D are not p -hyperbolic Besov functions for $1 \leq p < \infty$. It is easy to see that any function $f(z)$, $|f(z)| \leq k < 1$, belongs to all classes B_p^h , $1 \leq p \leq \infty$. Class B_2^h is a class of hyperbolic Dirichlet functions.

Examples.

1. Let $S_\alpha = \{z = x + iy : |x|^\alpha + |y|^\alpha < 1\}$, $0 < \alpha \leq 1$, and $\varphi_\alpha : D \rightarrow S_\alpha$ then $\varphi_1(z) \notin B_2^h$ and $\varphi_1(z) \in B_p^h$ if $p > 2$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B_2^h$.

2. It is known [4] that for hyperbolic Lipschitz functions $\sigma\Lambda_\alpha$, $0 < \alpha \leq 1$, i.e. functions which satisfy the condition $\sup_{|u-v| \leq \tau} \sigma(f(u), f(v)) \leq K\tau^\alpha$, the necessary and sufficient condition that $f(z) \in \sigma\Lambda_\alpha$ is $(1 - |z|^2)f^*(z) = O((1 - |z|^2)^\alpha)$ as $|z| \rightarrow 1$.

Thus we can see that $\sigma\Lambda_{\frac{1}{p}} \subset B_p^h$.

In chapter 2 we establish Lipschitz type properties of p -hyperbolic Besov functions and prove that p -hyperbolic Besov functions don't have angular derivatives for finite values p . In chapter 3 we show that composition of Bloch functions and p -hyperbolic Besov functions are p -analytic Besov functions.

2. Main properties

Classes B_p^h satisfy nesting property $B_p^h \subset B_q^h$ for $p < q$. It follows from the Schwarz-Pick lemma and inequality

$$\begin{aligned} & \iint_D (1 - |z|^2)^{q-2} (f^*(z))^q dA(z) \\ &= \iint_D ((1 - |z|^2)f^*(z))^{q-p} (1 - |z|^2)^{p-2} (f^*(z))^p dA(z) \\ &\leq \iint_D (1 - |z|^2)^{p-2} (f^*(z))^p dA(z). \end{aligned}$$

Theorem 1. If bounded function $f(z)$ satisfies the condition

$$\iint_D \iint_D \frac{\rho(f(z), f(w))^p}{|1 - z\bar{w}|^4} dA(z) dA(w) < \infty$$

then $f(z) \in B_p^h$, $1 \leq p < \infty$.

Proof.

Function $g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$ is holomorphic in D . Using an expansion of $g(z)$ on Taylor series we can obtain

$$f^*(0) = \left| \iint_D \bar{z}g(z) dA(z) \right|.$$

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Let $g = f \circ \varphi_a(z)$, where $\varphi_a(z) = \frac{z+a}{1+\bar{a}z}$, $a \in D$. Then

$$((1 - |a|^2)f^*(a))^p \leq \iint_D (\rho(f \circ \varphi_a(z), f(a)))^p dA(z)$$

and thus

$$\begin{aligned} & \iint_D ((1 - |a|^2)f^*(a))^p d\lambda(a) \\ & \leq \iint_D d\lambda(a) \iint_D (\rho(f_a(z), f(a)))^p dA(z) \\ & = \iint_D \iint_D \frac{\rho(f(z), f(w))^p}{|1 - z\bar{w}|^4} dA(z) dA(w) < \infty. \end{aligned}$$

Theorem 2. If $f(z) \in B_p^h$, $1 < p < \infty$, then

$$\iint_D \iint_D \frac{\sigma(f(z), f(w))^p}{|1 - z\bar{w}|^4} dA(z) dA(w) < \infty.$$

Proof.

At first we estimate hyperbolic distance between $f(z)$ and $f(0)$.

$$\begin{aligned} \sigma(f(z), f(0)) &= \left| \int_0^1 f^*(tz)z dt \right| \\ &= \left| \int_0^1 \frac{f^*(tz)}{1 - t^2|z|^2} (1 - t^2|z|^2)z dt \right| \leq \left| \int_0^1 \frac{f^*(tz)}{1 - t^2|z|^2} dt \right| \\ &\leq \left(\int_0^1 \frac{|z|^q}{(1 - t^2|z|^2)^{\frac{1+q}{2}}} dt \right)^{\frac{1}{q}} \left(\int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{(1 - t^2|z|^2)^{\frac{1}{2}}} dt \right)^{\frac{1}{p}} \\ &= \left(\frac{2|z|^{q-1}}{q-1} \left(\frac{1}{(1 - |z|)^{\frac{q-1}{2}}} - 1 \right) \right)^{\frac{1}{q}} \left(\int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{(1 - t^2|z|^2)^{\frac{1}{2}}} dt \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - t^2|z|^2}} dt \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, there exists such C that

$$\sigma(f(z), f(0))^p \leq \frac{C|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - t^2|z|^2}} dt, \quad p > 1.$$

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Then

$$\begin{aligned}
& \iint_D \sigma(f(z), f(0))^p dA(z) \leq \\
& \leq C \iint_D dA(z) \int_0^1 \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2}\sqrt{1-t^2|z|^2}} dt \\
& = C \int_0^1 \iint_D \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2}} \cdot \frac{dA(z) \frac{dt}{t^3}}{\sqrt{1-\frac{|z|^2}{t^2}}} = (\text{by Fubini's theorem}) \\
& = C \iint_D \frac{|z|((1-t^2|z|^2)f^*(z))^p}{\sqrt{1-|z|^2}} dA(z) \int_{|z|}^1 \frac{dt}{t^2 \sqrt{t} \sqrt{t^2-|z|^2}} \\
& \leq 2C \iint_D ((1-|z|^2)f^*(z))^p dA.
\end{aligned}$$

We obtain that for $1 < p < \infty$ and for some C_1

$$\iint_D \sigma(f(z), f(0))^p dA(z) \leq C_1 \iint_D ((1-|z|^2)f^*(z))^p dA(z).$$

A change of variables and some properties of Bergman kernel (see e.g. [5]) give

$$\begin{aligned}
& \iint_D \iint_D \frac{\sigma(f(z), f(w))^p}{|1-z\bar{w}|^4} dA(z) dA(w) \\
& = \iint_D d\lambda(z) \iint_D \sigma^p(f \circ \varphi_z(w), f(z)) dA(w) \\
& \leq C \iint_D d\lambda(z) \iint_D ((1-|w|^2)f \circ \varphi_z^*(w))^p dA(w) \\
& = \iint_D d\lambda(z) \iint_D (1-|w|^2)^p f^*(w)^p |k_z(w)|^2 dA(w) \\
& = \iint_D (1-|w|^2)^p f^*(w)^p dA(w) \iint_D \frac{dA(z)}{|1-z\bar{w}|^4} \\
& = \iint_D (1-|w|^2)^p f^*(w)^p d\lambda(w).
\end{aligned}$$

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Theorem 3. For any $1 < p \leq \infty$ there exists a constant C_p such that for any function $f(z) \in \mathcal{B}_p^h$ and $z, w \in D$, $\sigma(f(z), f(w)) \leq 1$,

$$\sigma(f(z), f(w)) \leq C_p \|f\|_{B_p^h} \sigma(z, w)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof.

Since $\|\cdot\|_{B_p^h}$, σ , ρ are Möbius invariant it suffices to prove that there is a constant C_p such that

$$\rho(f(z), f(0)) \leq C_p \|f\|_{B_p^h} \sigma(z, 0)^{\frac{1}{q}}$$

for all $f \in B_p^h$ and all $z \in D$.

Let $\varphi_f(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$. Since $\varphi_f(z)$ is holomorphic function in D then for any $f \in B$ and $z \in D$ following Zhu [5]

$$\varphi_f(z) = \iint_D \frac{1 - |w|^2}{\bar{w}(1 - z\bar{w})^2} \varphi'_f(w) dA(w).$$

Thus there exists a constant C_1 such that (see [2], [3])

$$\begin{aligned} \rho(f(z), f(0)) &= |\varphi_f(z)| \leq \iint_D \frac{1 - |w|^2}{|w||1 - z\bar{w}|^2} |\varphi'_f(w)| dA(w) \\ &\leq C_1 \iint_D \frac{1 - |w|^2}{|1 - z\bar{w}|^2} f^*(w) (1 - |\varphi_f(w)|^2) dA(w) \\ &\leq C_1 \iint_D (1 - |w|^2) f^*(w) \left(\frac{1 - |w|^2}{|1 - z\bar{w}|^2} \right)^2 d\lambda(w). \end{aligned}$$

By Hölder inequality and 1.4.10 of [3] we get

$$\rho(f(z), f(0)) \leq C_1 \|f\|_{B_p^h} \left(\iint_D \frac{(1 - |w|^2)^{2q-2}}{|1 - z\bar{w}|^{2q}} dA(w) \right)^{\frac{1}{q}} \leq C_2 \|f\|_{B_p^h} \sigma(z, 0)^{\frac{1}{q}}.$$

When points a and b lie closely one to another the behaviours of hyperbolic distance $\sigma(a, b)$ and pseudohyperbolic distance $\rho(a, b)$ are the same. Thus for sufficiently small τ there is a constant C such that for any $z, w \in D$, $\sigma(z, w) \leq \tau$,

$$\sigma(f(z), f(w)) \leq C \|f\|_{B_p^h} \sigma(z, w)^{\frac{1}{q}}.$$

Let $\varphi : D \rightarrow D$ analytic and $\Gamma = \{|z| = 1\}$.

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Definition. [1] Bounded function $\varphi(z)$ has a finite angular derivative at ζ on unit circle Γ if there is η on Γ so that $\frac{\varphi(z)-\eta}{z-\zeta}$ has a finite nontangential limit as $z \rightarrow \zeta$.

By the Julia-Caratheodory Theorem [1] function $\varphi(z)$ has a finite angular derivative at ζ if and only if

$$\liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty.$$

Let ζ be a point on the unit circle Γ , $0 < \delta < 1$, $0 < \varepsilon < \frac{\pi}{2}$ and $\Delta_\zeta(\delta, \varepsilon) = \{z : |z - \zeta| < \delta, |\arg(\zeta) - \arg(z)| < \varepsilon\}$.

Theorem 4. Functions of B_p^h , $1 < p < \infty$, don't have angular derivatives.

Proof.

Suppose that $\varphi(z) \in B_p^h$, $1 < p < \infty$, and $\varphi(z)$ has an angular derivative at a point $\zeta \in \Gamma$. Then by the Julia-Caratheodory Theorem $\frac{1 - |\varphi(z)|}{1 - |z|} \sim |\varphi'(\zeta)|$ for sufficiently small δ and $z \in \Delta_\zeta(\delta, \varepsilon)$ and moreover, $|\varphi'(z)| \approx |\varphi'(\zeta)|$ for $z \in \Delta_\zeta(\delta, \varepsilon)$. Thus

$$\iint_D ((1 - |z|^2)\varphi^*(z))^p d\lambda(z) \geq \iint_{\Delta_\zeta(\delta, \varepsilon)} ((1 - |z|^2)\varphi^*(z))^p d\lambda(z) \approx \iint_{\Delta_\zeta(\delta, \varepsilon)} d\lambda(z) = \infty.$$

3. Composition operators

Let $\mathcal{B}$ be Bloch space of holomorphic functions in D . By the definition (e.g. [5]) $f(z) \in \mathcal{B}$ if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)|f'(z)| < \infty.$$

Holomorphic in D function $f(z)$ belongs to the analytic Besov space B_p , $1 < p < \infty$, if

$$\|f\|_{B_p} = \left(\iint_D (1 - |z|^2)^p |f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}} < \infty$$

and $B_\infty = \mathcal{B}$.

Let X be a normed subspace of $\mathcal{B}$ and $\varphi(z)$ be a holomorphic self map of the unit disk D . We say that composition operator $C_\varphi : \mathcal{B} \rightarrow X$ is compact if

$$\lim_{r \rightarrow 1-0} \|f \circ \varphi - f \circ (\chi_r \varphi)\|_X = 0$$

where $\chi_r(z)$ is a characteristic function of the disk $D_r = rD$

$$\chi_r(z) = \begin{cases} 1, & |z| \leq r \\ 0, & |z| > r. \end{cases}$$

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Theorem 5. For every $\varphi(z) \in B_p^h$, $1 < p < \infty$, and any $f \in \mathcal{B}$ composition $f \circ \varphi \in B_p$. Composition operator C_φ is a compact.

Proof.

Let $f \in \mathcal{B}$ and $\sup_{z \in D} (1 - |z|^2) |f'(z)| = M_f$. Let $\varphi(z) \in B_p^h$ and $\|\varphi(z)\|_{B_p^h} = M_\varphi$. If $g(z) = f \circ \varphi(z)$ then

$$\begin{aligned} \|g(z)\|_{B_p^h}^p &= \iint_D (1 - |z|^2)^p |g'(z)|^p d\lambda(z) = \\ &= \iint_D (1 - |z|^2)^p |f'_\varphi(z)|^p |\varphi'(z)|^p d\lambda(z) = \\ &= \iint_D (1 - |z|^2)^p (\varphi^*(z))^p (1 - |\varphi(z)|^2)^p |f'_\varphi(z)|^p d\lambda(z) \leq M_f^p \cdot M_\varphi^p < \infty. \end{aligned}$$

Now we prove compactness of operator C_φ .

$$\begin{aligned} &\|f(\varphi(z)) - f(\chi_r \cdot \varphi(z))\|_{B_p}^p = \\ &= \iint_{D \setminus D_r} (1 - |z|^2)^p |(f \circ \varphi(z))'|^p d\lambda(z) \leq \\ &\leq M_f^p \iint_{D \setminus D_r} (1 - |z|^2)^p (\varphi^*(z))^p d\lambda(z). \end{aligned}$$

The last integral tends to zero as $r \rightarrow 1$ because $\varphi(z) \in B_h^p$.

Theorem 6. If for every Bloch function $f(z)$ the composition $f \circ \varphi$ is a p -analytic Besov function, $1 < p < \infty$, then $\varphi(z) \in B_p^h$.

Proof.

Ramey and Ullrich [2] constructed such Bloch functions f and g that

$$|f'(z)| + |g'(z)| \geq \frac{1}{1 - |z|^2}.$$

Then for every $p > 1$

$$|f'(z)|^p + |g'(z)|^p \geq \frac{K_p}{(1 - |z|^2)^p}.$$

and thus

$$K_p \|\varphi\|_{B_p^h}^p \leq \|f \circ \varphi\|_{B_p}^p + \|g \circ \varphi\|_{B_p}^p < \infty.$$

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