

ON THE RESTRICTION OF AUTOMORPHIC FORMS ON AN ORTHOGONAL GROUP TO A SMALLER ORTHOGONAL GROUP AND THE GROSS-PRASAD CONJECTURE

ATSUSHI ICHINO AND TAMOTSU IKEDA

In this note, we announce a conjecture which expresses the restriction of automorphic forms on $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$ in terms of special values of automorphic L -functions. Details will appear elsewhere.

1. THE GROSS-PRASAD CONJECTURE

Our conjecture can be regarded as a refinement of the global Gross-Prasad conjecture [4].

Let k be a global field of characteristic not two and $\mathbb{A}$ the ring of adeles of k . Let V_0 and V_1 be quadratic spaces over k of dimension n and $n+1$, respectively. We assume that there exists an embedding $\iota: V_0 \rightarrow V_1$ over k . Let G_i denote the special orthogonal group of V_i for $i = 0, 1$. Then ι induces an embedding $\iota: G_0 \rightarrow G_1$ over k . Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible tempered cuspidal automorphic representation of $G_i(\mathbb{A})$ for $i = 0, 1$. We are interested in the integral

$$\langle \varphi_1|_{G_0}, \varphi_0 \rangle = \int_{G_0(k) \backslash G_0(\mathbb{A})} \varphi_1(\iota(g_0)) \overline{\varphi_0(g_0)} dg_0$$

and consider an element ℓ in

$$\mathrm{Hom}_{G_0(\mathbb{A})}(\pi_1 \otimes \bar{\pi}_0, \mathbb{C})$$

defined by

$$\ell(\varphi_1 \otimes \varphi_0) = \langle \varphi_1|_{G_0}, \varphi_0 \rangle$$

for $\varphi_i \in \pi_i$. Here dg_0 is the Tamagawa measure on $G_0(\mathbb{A})$ and $\bar{\pi}_0$ is the complex conjugate of π_0 . If $\ell \neq 0$, then it is obvious that

$$(1.1) \quad \mathrm{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq 0$$

for all places v of k . Conversely, assume that (1.1) holds for all v . The global Gross-Prasad conjecture [4] asserts that $\ell \neq 0$ if and only if

$$L(\tfrac{1}{2}, \pi_1 \boxtimes \pi_0) \neq 0.$$

Here $L(s, \pi_1 \boxtimes \pi_0)$ is the automorphic L -function associated to $\mathrm{st}_1 \boxtimes \mathrm{st}_0$, where st_i is the standard representation of the L -group ${}^L G_i$ of G_i for $i =$

0, 1. Note that the existence of the local factor $L_v(s, \pi_{1,v} \boxtimes \pi_{0,v})$ and the meromorphic continuation of $L(s, \pi_1 \boxtimes \pi_0)$ have not been proved in general.

It is believed that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1$$

for all places v of k . Hence if $\ell \neq 0$, then there should be a constant C such that

$$\ell = C \cdot \prod_v \ell_v,$$

where ℓ_v is a non-zero element in $\operatorname{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C})$. One can guess this constant C is related to the special value $L(\frac{1}{2}, \pi_1 \boxtimes \pi_0)$, but it does not work unless one knows how to choose ℓ_v .

To remedy this, instead of ℓ , we consider an element L in

$$\operatorname{Hom}_{G_0(\mathbb{A}) \times G_0(\mathbb{A})}((\pi_1 \boxtimes \bar{\pi}_1) \otimes (\bar{\pi}_0 \boxtimes \pi_0), \mathbb{C})$$

defined by

$$L((\varphi_1 \boxtimes \varphi'_1) \otimes (\varphi_0 \boxtimes \varphi'_0)) = \langle \varphi_1 |_{G_0}, \varphi_0 \rangle \overline{\langle \varphi'_1 |_{G_0}, \varphi'_0 \rangle}$$

for $\varphi_i, \varphi'_i \in \pi_i$. As an analogue of L , we will define an element L_v in

$$(1.2) \quad \operatorname{Hom}_{G_0(k_v) \times G_0(k_v)}((\pi_{1,v} \boxtimes \bar{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \pi_{0,v}), \mathbb{C})$$

as follows. Put

$$\langle \varphi_i, \varphi'_i \rangle = \int_{G_i(k) \backslash G_i(\mathbb{A})} \varphi_i(g_i) \overline{\varphi'_i(g_i)} dg_i$$

for $\varphi_i, \varphi'_i \in \pi_i$, where dg_i is the Tamagawa measure on $G_i(\mathbb{A})$. Then there exists an invariant hermitian inner product $\langle \cdot, \cdot \rangle_v$ on $\pi_{i,v}$ such that

$$(1.3) \quad \langle \varphi_i, \varphi'_i \rangle = \prod_v \langle \varphi_{i,v}, \varphi'_{i,v} \rangle_v$$

and such that $\langle \varphi_{i,v}, \varphi'_{i,v} \rangle_v = 1$ for almost all v . We also write

$$(1.4) \quad dg_0 = \prod_v dg_{0,v},$$

where $dg_{0,v}$ is a Haar measure on $G_0(k_v)$. We may assume that the volume of a hyperspecial maximal compact subgroup of $G_0(k_v)$ is equal to one for almost all v . Put

$$\begin{aligned} & I_v((\varphi_{1,v} \boxtimes \varphi'_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi'_{0,v})) \\ &= \int_{G_0(k_v)} \langle \pi_{1,v}(\iota(g_{0,v})) \varphi_{1,v}, \varphi'_{1,v} \rangle_v \overline{\langle \pi_{0,v}(g_{0,v}) \varphi_{0,v}, \varphi'_{0,v} \rangle_v} dg_{0,v} \end{aligned}$$

for $\varphi_{i,v}, \varphi'_{i,v} \in \pi_{i,v}$. Using Harish-Chandra's estimate of matrix coefficients, we can prove that this integral is absolutely convergent under the assumption that $\pi_{i,v}$ is tempered. Hence I_v defines an element in (1.2).

To relate L and I_v , we must do the unramified calculation. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4) \cdots \zeta(2m) & \text{if } \dim V_1 = 2m + 1, \\ \zeta(2)\zeta(4) \cdots \zeta(2m-2)L(m, \chi) & \text{if } \dim V_1 = 2m, \end{cases}$$

where χ is the quadratic character of $\mathbb{A}^\times/k^\times$ associated to the discriminant of V_1 . We also put

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + \frac{1}{2}, \pi_1, \text{Ad})L(s + \frac{1}{2}, \pi_0, \text{Ad})},$$

where Ad is the adjoint representation of ${}^L G_i$ on the Lie algebra of the dual group $\hat{G}_i$ of G_i . Let $\Delta_{G_1, v}$ and $\mathcal{P}_{\pi_1, v, \pi_0, v}(s)$ be the local components of Δ_{G_1} and $\mathcal{P}_{\pi_1, \pi_0}(s)$, respectively. Then

$$I_v((\varphi_{1,v} \boxtimes \varphi'_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi'_{0,v})) = \Delta_{G_1, v} \cdot \mathcal{P}_{\pi_1, v, \pi_0, v}(\frac{1}{2})$$

for almost all v . The proof of this formula is based on the unramified calculation by Ginzburg, Piatetski-Shapiro, and Rallis [3] and an explicit formula for Shintani functions by Kato, Murase, and Sugano [8]. Put

$$L_v = \Delta_{G_1, v}^{-1} \cdot \mathcal{P}_{\pi_1, v, \pi_0, v}(\frac{1}{2})^{-1} \cdot I_v.$$

Then L_v is an element in (1.2) such that

$$L_v((\varphi_{1,v} \boxtimes \varphi'_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi'_{0,v})) = 1$$

for almost all v .

Conjecture A. *There exists an integer β such that*

$$L = 2^\beta \cdot \Delta_{G_1} \cdot \mathcal{P}_{\pi_1, \pi_0}(\frac{1}{2}) \cdot \prod_v L_v.$$

We will state Conjecture A in a different way to apply it to the study of periods of automorphic forms. We drop the assumptions (1.3) and (1.4) and choose an arbitrary invariant hermitian inner product $\langle \cdot, \cdot \rangle_v$ on $\pi_{i,v}$ and an arbitrary Haar measure $dg_{0,v}$ on $G_0(k_v)$. Let C_0 be a positive real number such that

$$dg_0 = C_0 \cdot \prod_v dg_{0,v}.$$

Put

$$\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = L_v((\varphi_{1,v} \boxtimes \varphi_{1,v}) \otimes (\varphi_{0,v} \boxtimes \varphi_{0,v}))$$

for $\varphi_{i,v} \in \pi_{i,v}$. Then Conjecture A is equivalent to the following.

Conjecture A'. *There exists an integer β such that*

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\|\varphi_1\|^2 \cdot \|\varphi_0\|^2} = 2^\beta \cdot C_0 \cdot \Delta_{G_1} \cdot \mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) \cdot \prod_v \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|_v^2 \cdot \|\varphi_{0,v}\|_v^2}$$

for $\varphi_i = \otimes_v \varphi_{i,v} \in \pi_i$.

2. THE ARTHUR CONJECTURE

It is more delicate to determine the quantity 2^β in Conjecture A. We believe that this quantity 2^β is related to the Arthur conjecture [1].

Let $\mathcal{L}$ be the hypothetical Langlands group of k . Then there should exist a one-to-one correspondence between the set of irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and the set of equivalence classes of irreducible representations of $\mathcal{L}$ of dimension n . The Arthur conjecture [1] asserts that there exists a unique equivalence class of homomorphisms

$$\psi_i : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G_i$$

associated to π_i . Note that the restriction of ψ_i to $\mathrm{SL}_2(\mathbb{C})$ is expected to be trivial since π_i is tempered. Let $\mathcal{S}_{\psi_i}$ be the centralizer of the image of ψ_i in $\hat{G}_i$. Then $\mathcal{S}_{\psi_i}$ is expected to be a finite group since π_i is square integrable. Moreover the order of $\mathcal{S}_{\psi_i}$ should be a power of two since G_i is a special orthogonal group.

Conjecture B. *The quantity 2^β in Conjecture A is equal to*

$$\frac{1}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|}.$$

3. EXAMPLES

Let $n = 2$. Then there exist a quaternion algebra B over k and a quadratic extension K of k which splits B such that $G_0 = \ker N_{K/k}$ and $G_1 = B^\times / \mathbb{G}_m$. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) = \frac{L(\tfrac{1}{2}, \pi_{1,K} \otimes \omega)}{L(1, \pi_1, \mathrm{Ad}) L(1, \chi_{K/k})}.$$

Here $\pi_{1,K}$ is the base change of π_1 to $\mathrm{GL}_2(\mathbb{A}_K)$, ω is the character of $\mathbb{A}_K^\times / K^\times$ with $\omega|_{\mathbb{A}^\times} = 1$ determined by π_0 , and $\chi_{K/k}$ is the quadratic character of $\mathbb{A}^\times / k^\times$ associated to K/k . In this case, Conjecture A is proved by Waldspurger [9]. We remark that $\beta = -2$ and $|\mathcal{S}_{\psi_i}| = 2$ for $i = 0, 1$.

Let $n = 3$ and $k = \mathbb{Q}$. We assume that $G_0 = \mathrm{PGL}_2$ and $G_1 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2) / \mathbb{G}_m$, where

$$\mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2) = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}.$$

Let κ_j be an even positive integer for $j = 1, 2, 3$ such that $\kappa_1 + \kappa_2 = \kappa_3$ and $f_j \in S_{\kappa_j}(\mathrm{SL}_2(\mathbb{Z}))$ a normalized Hecke eigenform. Then $f_1 \otimes f_2$ and f_3 define π_1 and π_0 , respectively. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) = \frac{L(\tfrac{1}{2}, \sigma_1 \times \sigma_2 \times \sigma_3)}{\prod_{j=1}^3 L(1, \sigma_j, \mathrm{Ad})}.$$

Here σ_j is the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ determined by f_j . We consider the integral

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}} f_1(\tau) f_2(\tau) \overline{f_3(\tau)} \mathrm{Im}(\tau)^{\kappa_3-2} d\tau.$$

In this case, Conjecture A follows from the result of Harris and Kudla [5]. We remark that $\beta = -2$ and $|\mathcal{S}_{\psi_i}| = 2$ for $i = 0, 1$. Also, using the result of Watson [10], we can prove Conjecture A for more general cases.

Let $n = 4$. Let k be a global field of characteristic greater than two and of genus one. We assume that $G_0 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathbb{G}_m$ and $G_1 = \mathrm{PGSp}_2$. Let π_1 be an endoscopic lift of $\sigma_1 \otimes \sigma_2$ and $\pi_0 = \tau_1 \otimes \tau_2$. Here $\sigma_1, \sigma_2, \tau_1$, and τ_2 are irreducible unramified cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with trivial central characters such that $\sigma_1 \not\cong \sigma_2$. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) = \frac{L(\tfrac{1}{2}, \sigma_1 \times \tau_1 \times \tau_2) L(\tfrac{1}{2}, \sigma_2 \times \tau_1 \times \tau_2)}{L(1, \sigma_1 \times \sigma_2) \prod_{j=1}^2 L(1, \sigma_j, \mathrm{Ad}) L(1, \tau_j, \mathrm{Ad})}.$$

In this case, Conjectures A and B are compatible with an analogue of the result of Böcherer, Furusawa, and Schulze-Pillot [2]. We remark that $|\mathcal{S}_{\psi_1}| = 4$ and $|\mathcal{S}_{\psi_0}| = 2, 4, 8$.

4. THE NON-TEMPERED CASE

When π_i is only square integrable, we have not proved that the integral ℓ is convergent. What is worse is that the integral I_v is divergent in general. Nevertheless we expect that Conjecture A can be extended for square integrable automorphic representations under the assumption that (1.1) holds for all v . Note that Conjecture B does not hold. We will provide some examples.

Let $n = 4$ and $k = \mathbb{Q}$. We assume that $G_0 = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathbb{G}_m$ and $G_1 = \mathrm{PGSp}_2$. Let κ be an odd positive integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ a Saito-Kurokawa lift of f . Let $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then $\mathcal{F}$ and $g \otimes g$ define π_1 and π_0 , respectively. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) = \frac{L(\tfrac{1}{2}, \mathrm{Ad}(\sigma) \times \tau)}{\zeta(2) L(\tfrac{3}{2}, \tau) L(1, \tau, \mathrm{Ad})}.$$

Here σ and τ are the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ determined by g and f , respectively. We consider the integral

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}} \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}} \mathcal{F} \left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right) \overline{g(\tau_1)g(\tau_2)} \mathrm{Im}(\tau_1)^{\kappa-1} \mathrm{Im}(\tau_2)^{\kappa-1} d\tau_1 d\tau_2.$$

In this case, Conjecture A is compatible with the result of [6]. We remark that $\beta = -2$ and $|\mathcal{S}_{\psi_i}| = 2$ for $i = 0, 1$.

Let $n = 5$ and $k = \mathbb{Q}$. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and $\mathcal{O}_K$ the ring of integers of K . We assume that $G_0 = \mathrm{Sp}_2 / \{\pm 1\}$ and $G_1 = \mathrm{SU}(2, 2) / \{\pm 1\}$. Let κ be an even positive integer. Let $g \in S_{\kappa-1}(\Gamma_0(D), \chi_{K/k})$ be a primitive form and $\mathcal{G} \in S_{\kappa}(\Gamma_K^{(2)})$ the hermitian Maass lift of g , where $\Gamma_K^{(2)} = \mathrm{SU}(2, 2)(\mathbb{Q}) \cap \mathrm{GL}_4(\mathcal{O}_K)$. We may assume that $\mathcal{G} \neq 0$. Let $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $\mathcal{F} \in S_{\kappa}(\mathrm{Sp}_2(\mathbb{Z}))$ a Saito-Kurokawa lift of f . Then $\mathcal{G}$ and $\mathcal{F}$ define π_1 and π_0 , respectively. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(\tfrac{1}{2}) = -\frac{L(\tfrac{1}{2}, \mathrm{Sym}^2(\tau) \times \sigma) L(\tfrac{3}{2}, \sigma)}{L(2, \mathrm{Sym}^2(\tau)) L(1, \tau, \mathrm{Ad}) L(1, \sigma, \mathrm{Ad})}.$$

Here σ and τ are the irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ determined by f and g , respectively. We consider the integral

$$\int_{\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2} \mathcal{G}(Z) \overline{\mathcal{F}(Z)} \det(\mathrm{Im}(Z))^{\kappa-3} dZ.$$

In this case, Conjecture A is compatible with the result of [7]. We remark that $\beta = -2$ and $|\mathcal{S}_{\psi_i}| = 2$ for $i = 0, 1$.

REFERENCES

- [1] J. Arthur, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71.
- [2] S. Böcherer, M. Furusawa, and R. Schulze-Pillot, *On the global Gross-Prasad conjecture for Yoshida liftings*, Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, 2004, pp. 105–130.
- [3] D. Ginzburg, I. I. Piatetski-Shapiro, and S. Rallis, *L functions for the orthogonal group*, Mem. Amer. Math. Soc. **128** (1997), no. 611.
- [4] B. H. Gross and D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* , Canad. J. Math. **44** (1992), 974–1002.
- [5] M. Harris and S. S. Kudla, *The central critical value of a triple product L-function*, Ann. of Math. **133** (1991), 605–672.
- [6] A. Ichino, *Pullbacks of Saito-Kurokawa lifts*, Invent. Math. **162** (2005), 551–647.
- [7] A. Ichino and T. Ikeda, *On Maass lifts and the central critical values of triple product L-functions*, preprint.
- [8] S. Kato, A. Murase, and T. Sugano, *Whittaker-Shintani functions for orthogonal groups*, Tohoku Math. J. **55** (2003), 1–64.

- [9] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), 173–242.
- [10] T. C. Watson, *Rankin triple products and quantum chaos*, to appear in Ann. of Math.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA CITY
UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN
E-mail address: ichino@sci.osaka-cu.ac.jp

GRADUATE SCHOOL OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA,
KYOTO 606-8502, JAPAN
E-mail address: ikeda@math.kyoto-u.ac.jp