

THE $\mathrm{Sp}_n \times \mathrm{Sp}_n$ -PERIOD OF A PSEUDO-EISENSTEIN SERIES ON Sp_{2n}

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ABSTRACT. This is a report on our study of $\mathrm{Sp}_n \times \mathrm{Sp}_n$ -period integrals on the automorphic spectrum of Sp_{2n} . We announce our formula for the period integrals of pseudo-Eisenstein series.

This is a report on our study of period integrals over $\mathrm{Sp}_n \times \mathrm{Sp}_n$ of automorphic forms on Sp_{2n} . In [LO] we suggest a notion of the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ -*distinguished automorphic spectrum* and provide an upper bound in terms of Langlands fine spectral expansion (cf. [MW95, §V]) of the automorphic spectrum of Sp_{2n} . Roughly speaking, it is the orthogonal complement of the Sp_{2n} -invariant space of pseudo-Eisenstein series with $\mathrm{Sp}_n \times \mathrm{Sp}_n$ -vanishing period. The results of [AGR93] imply that the $\mathrm{Sp}_n \times \mathrm{Sp}_n$ -distinguished automorphic spectrum contains no cuspidal automorphic functions. We study period integrals on the continuous spectrum. The technical heart of our work is a formula for the periods of pseudo-Eisenstein series that we explicitly describe below. Results that are mentioned below without reference are proved in [LO].

1. NOTATION

Let F be a number field and let $\mathbb{A}$ be the ring of adeles of F . We denote F -varieties in bold letters such as $\mathbf{X}$ and write $X = \mathbf{X}(F)$ for the corresponding set of F -points.

For an algebraic group $\mathbf{Q}$ defined over F we denote by $X^*(\mathbf{Q})$ the lattice of F -rational characters of $\mathbf{Q}$ and let $\mathfrak{a}_Q^* = X^*(\mathbf{Q}) \otimes \mathbb{R}$, $\mathfrak{a}_Q = \mathrm{Hom}(\mathfrak{a}_Q^*, \mathbb{R})$ the real vector space dual to $\mathfrak{a}_Q^*$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_Q$ the natural pairing between them. We view $\mathfrak{a}_Q^*$ and its dual as Euclidean spaces and denote the norm on either of them by $\| \cdot \|$. We denote by $\mathfrak{a}_{\mathbb{C}}$ the complexification of a real vector space $\mathfrak{a}$. We also set

$$\mathbf{Q}(\mathbb{A})^1 = \{q \in \mathbf{Q}(\mathbb{A}) : \forall \chi \in X^*(\mathbf{Q}), |\chi(q)| = 1\}.$$

There is an isomorphism

$$H_Q : \mathbf{Q}(\mathbb{A})^1 \backslash \mathbf{Q}(\mathbb{A}) \rightarrow \mathfrak{a}_Q$$

such that $e^{\langle \chi, H_Q(q) \rangle} = |\chi(q)|_{\mathbb{A}^*}$, $\chi \in X^*(\mathbf{Q})$, $q \in \mathbf{Q}(\mathbb{A})$.

Let δ_Q denote the modulus function of $\mathbf{Q}(\mathbb{A})$. It is a character of $\mathbf{Q}(\mathbb{A})^1 \backslash \mathbf{Q}(\mathbb{A})$ and therefore there exists $\rho_Q \in \mathfrak{a}_Q^*$ such that

$$\delta_Q(q) = e^{\langle 2\rho_Q, H_Q(q) \rangle}, \quad q \in \mathbf{Q}(\mathbb{A}).$$

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Let $\mathbf{G}$ be a reductive group and $\mathbf{P}_0$ a minimal parabolic subgroup of $\mathbf{G}$ both defined over F . In general we denote by $\mathfrak{S}_G$ a Siegel domain for $G \backslash \mathbf{G}(\mathbb{A})$ and by $\mathfrak{S}_G^1$ a Siegel domain for $G \backslash \mathbf{G}(\mathbb{A})^1$ (cf. [MW95, I.2.1]).

Let K be a maximal compact subgroup of $\mathbf{G}(\mathbb{A})$ in good position with respect to P_0 so that we have the Iwasawa decomposition $\mathbf{G}(\mathbb{A}) = \mathbf{P}_0(\mathbb{A})K$. The map $H_0 = H_{P_0} : \mathbf{P}_0(\mathbb{A}) \rightarrow \mathfrak{a}_{P_0}$ is extended to $\mathbf{G}(\mathbb{A})$ via the Iwasawa decomposition, i.e., $H_0(pk) = H_0(p)$, $p \in \mathbf{P}_0(\mathbb{A})$ and $k \in K$. Let $\mathbf{T}_G$ be the split part of the identity connected component of the center of $\mathbf{G}$. Applying the imbedding $x \mapsto 1 \otimes x : \mathbb{R} \rightarrow F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ we imbed $\mathbf{T}_G(\mathbb{R})$ in $\mathbf{T}_G(F_\infty) \hookrightarrow \mathbf{T}_G(\mathbb{A})$ and denote by A_G the image of the identity component $\mathbf{T}_G(\mathbb{R})^\circ$ in $\mathbf{T}_G(\mathbb{A})$. Then $H_G : A_G \rightarrow \mathfrak{a}_G$ is an isomorphism. Denote by $\nu \mapsto e^\nu$ its inverse.

For $n \in \mathbb{N}$ let $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n$, $J_n = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}$ and

$$\mathbf{Sp}_n = \{g \in \mathrm{GL}_{2n} : {}^t g J_n g = J_n\}.$$

Let $*$ be the automorphism of GL_n given by $g \mapsto g^* = w_n {}^t g^{-1} w_n$. The imbedding $g \mapsto \mathrm{diag}(g, g^*) : \mathrm{GL}_n \rightarrow \mathbf{Sp}_n$ identifies GL_n with the Siegel Levi subgroup of $\mathbf{Sp}_n$.

To every $\gamma = (n_1, \dots, n_k; r)$ where $k, r \geq 0$, $n_1, \dots, n_k > 0$, and $n_1 + \dots + n_k + r = n$ we associate the standard parabolic subgroup $\mathbf{P} = \mathbf{P}_\gamma = \mathbf{M} \ltimes \mathbf{U}$ consisting of block upper triangular matrices in $\mathbf{Sp}_n$ where

$$\mathbf{M} = \mathbf{M}_\gamma = \{\mathrm{diag}(g_1, \dots, g_k, h, g_k^*, \dots, g_1^*) : h \in \mathbf{Sp}_r, g_i \in \mathrm{GL}_{n_i}, i = 1, \dots, k\}.$$

(We call such $\mathbf{M}$'s standard Levi subgroups.) In particular, $\mathbf{P}_{(n;0)}$ is the Siegel parabolic subgroup of $\mathbf{Sp}_n$ and $\mathbf{P}_\gamma \subseteq \mathbf{P}_{(n;0)}$ if and only if $r = 0$.

Let $\delta_n = \mathrm{diag}(1, -1, 1, \dots, (-1)^{n-1}) \in \mathrm{GL}_n$ and $\epsilon_n = \mathrm{diag}(\delta_n, \delta_n^*) \in \mathbf{Sp}_n$.

For the rest of this note fix $n \in \mathbb{N}$ and let $\mathbf{G} = \mathbf{Sp}_{2n}$, $\epsilon = \epsilon_{2n}$ and $\mathbf{H} = \mathbf{C}_G(\epsilon)$, the centraliser of ϵ in $\mathbf{G}$.

Let $\mathbf{B} = \mathbf{P}_{(1, \dots, 1; 0)} = \mathbf{T} \ltimes \mathbf{N}$ be the standard Borel subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{N}$ and $\mathbf{T} = \mathbf{M}_{(1, \dots, 1; 0)}$. We call a Levi subgroup of $\mathbf{G}$ semi-standard if it contains $\mathbf{T}$.

For a standard Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we denote by $\mathcal{L}(\mathbf{M})$ the (finite) set of (semi-standard) Levi subgroups containing $\mathbf{M}$.

For a standard parabolic subgroup $P = M \ltimes U$ of G let $\Sigma_M = R(T_M, G) \subseteq \mathfrak{a}_M^*$ be the root system of G with respect to T_M and Σ_P^+ the subset of positive roots in Σ_M with respect to P .

We identify $\mathbf{G}/\mathbf{H}$ with the $\mathbf{G}$ -conjugacy class $\mathbf{X}$ of ϵ , a closed subvariety of $\mathbf{G}$, via $g\mathbf{H} \mapsto g\epsilon g^{-1}$. For $x \in X$ and a subgroup Q of G we denote by $Q \cdot x = \{qxq^{-1} : q \in Q\}$ the Q -conjugacy class of x and by $Q_x = \{q \in Q : qxq^{-1} = x\}$ the centraliser of x in Q .

2. PSEUDO-EISENSTEINS SERIES

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a standard parabolic subgroup of $\mathbf{G}$. For any $R > 0$ let $C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ be the space of continuous cuspidal functions ϕ on $\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A})$ such that for all $N > 1$ we have

$$\sup_{m \in \mathfrak{S}_M^1, a \in A_M, k \in K} |\phi(amk)| \|m\|^N e^{R\|H_P(a)\|} < \infty.$$

For any $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ define the pseudo-Eisenstein series $\theta_{P,\phi}$ on $G \backslash \mathbf{G}(\mathbb{A})$ by the absolutely convergent series

$$\theta_{P,\phi}(g) = \sum_{\gamma \in P \backslash G} \phi(\gamma g).$$

For $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ with $\|\mathrm{Re} \lambda + \rho_P\| < R$ we write

$$\phi[\lambda](g) = e^{-\langle \lambda, H_P(g) \rangle} \int_{A_M} e^{-\langle \lambda + \rho_P, H_P(a) \rangle} \phi(ag) da.$$

Let $C_R^\infty(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ be the smooth part of $C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$. For $\phi \in C_R^\infty(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ we have the inversion formula

$$\phi(g) = \int_{\lambda_0 + i\mathfrak{a}_M^*} \phi[\lambda]_\lambda(g) d\lambda$$

for any $\lambda_0 \in \mathfrak{a}_M^*$ with $\|\lambda_0 + \rho_P\| < R$. Moreover, for any $R' < R$ and $N > 0$ we have

$$(1) \quad \sup_{m \in \mathfrak{S}_M^1, k \in K, \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*, \|\lambda + \rho_P\| \leq R'} |\phi[\lambda](mk)| (\|m\| + \|\lambda\|)^N < \infty.$$

We wish to study the period integrals

$$\mathcal{P}_H(\theta_{P,\phi}) := \int_{H \backslash \mathbf{H}(\mathbb{A})} \theta_{P,\phi}(h) dh.$$

It is easy to see that for R large enough $g \mapsto \sum_{\gamma \in P \backslash G} |\phi(\gamma g)|$ is bounded on $\mathbf{G}(\mathbb{A})^1$ for every $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ and therefore $\mathcal{P}_H(\theta_{P,\phi})$ is defined by an absolutely convergent integral for all $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$.

We express $\mathcal{P}_H(\theta_{P,\phi})$ as a sum parameterised by certain double cosets in $P \backslash G / H$ of certain $\mathbf{H}(\mathbb{A})$ -invariant linear forms, called intertwining periods, on representations of $\mathbf{G}(\mathbb{A})$ induced from $\mathbf{P}(\mathbb{A})$. In order to formulate our results we explain in §3 the relevant results concerning the double coset decomposition and in §4 we define and state the convergence of the intertwining periods.

3. DOUBLE COSETS

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a standard parabolic subgroup of $\mathbf{G}$. The map $g \mapsto g\epsilon g^{-1}$ defines a bijection

$$P \backslash G / H \simeq P \backslash X.$$

Instead of double cosets we study P -conjugacy classes in X .

3.1. Admissible orbits. An element $x \in X$ (or $P \cdot x$) is called M -admissible if $N_G(M) \cap P \cdot x$ is not empty. We denote by $[P \backslash X]_{\mathrm{adm}}$ the set of M -admissible P -conjugacy classes in X .

If $x \in X$ is M -admissible and $y \in N_G(M) \cap P \cdot x$ then $N_G(M) \cap P \cdot x = M \cdot y$ is a unique M -orbit. The correspondence $P \cdot x \mapsto M \cdot y$ is a bijection

$$[P \backslash X]_{\mathrm{adm}} \simeq M \backslash (X \cap N_G(M))$$

between M -admissible P -conjugacy classes in X and M -conjugacy classes in $N_G(M) \cap X$.

The group $N_G(M)$ acts on $\mathfrak{a}_M^*$ (M acts trivially) and in particular, $x \in N_G(M) \cap X$ acts as an involution on $\mathfrak{a}_M^*$ and decomposes it into a direct sum of the ± 1 -eigenspaces which we denote by $(\mathfrak{a}_M^*)_x^\pm$ respectively. (A similar decomposition applies to the dual space $\mathfrak{a}_M = (\mathfrak{a}_M)_x^+ \oplus (\mathfrak{a}_M)_x^-$.) Any such x defines

$$\mathbf{L} = \mathbf{L}(x) = \cap_{\mathbf{L}' \in \mathcal{L}(\mathbf{M}), x \in \mathbf{L}' \mathbf{L}' \in \mathcal{L}(\mathbf{M})}$$

so that $(\mathfrak{a}_M^*)_x^+ = \mathfrak{a}_L^*$ (cf. [Art82, p. 1299]).

We call $x \in N_G(M) \cap X$ (or $M \cdot x$) M -standard relevant if M has the form

$$M = M_{(r_1, r_1, \dots, r_k, r_k, s_1, \dots, s_l, t_1, \dots, t_m; u)}$$

and

$$L(x) = M_{(2r_1, \dots, 2r_k, s_1, \dots, s_l; v)}$$

(with k, l, m, u or v possibly zero) where $t_1, \dots, t_m$ are even and $v = u + t_1 + \dots + t_m$.

The following lemma reduces the study of $M \backslash (N_G(M) \cap X)$ to M -standard relevant M -conjugacy classes.

Lemma 3.1. *Let M be a standard Levi subgroup of G and $x \in N_G(M) \cap X$. Then there exists $n \in N_G(T)$ such that nMn^{-1} is a standard Levi subgroup of G , $n\mathbf{x}n^{-1}$ is nMn^{-1} -standard relevant and $L(n\mathbf{x}n^{-1}) = nL(x)n^{-1}$.*

3.2. Stabilizers and exponents. Let $x \in N_G(M) \cap X$. Then $\mathbf{P}_x = \mathbf{M}_x \ltimes \mathbf{U}_x$. Set $\mathbf{M}_x(\mathbb{A})^{(1)} = \mathbf{M}_x(\mathbb{A}) \cap \mathbf{M}(\mathbb{A})^1$ and note that $\mathbf{M}_x(\mathbb{A})^{(1)}$ contains (possibly strictly) $\mathbf{M}_x(\mathbb{A})^1$. The map H_M defines an isomorphism

$$\mathbf{M}_x(\mathbb{A})^{(1)} \backslash \mathbf{M}_x(\mathbb{A}) \simeq (\mathfrak{a}_M)_x^+.$$

Furthermore, $\mathbf{M}_x(\mathbb{A}) = (\mathbf{M}_x(\mathbb{A}) \cap A_M) \cdot \mathbf{M}_x(\mathbb{A})^{(1)}$ and $\mathbf{M}_x(\mathbb{A}) \cap A_M = (A_M)_x^+$ where $(A_M)_x^+ = e^{(\mathfrak{a}_M)_x^+}$.

Consequently, there exists a unique $\rho_x \in (\mathfrak{a}_M^*)_x^+$ such that

$$(2) \quad e^{\langle \rho_x, H(a) \rangle} = \delta_{P_x}(a) \delta_P(a)^{-\frac{1}{2}} \quad \text{or equivalently} \quad \delta_{P_x}(a) = e^{\langle \rho_x + \rho_P, H(a) \rangle}, \quad a \in (A_M)_x^+.$$

Remark 3.2. *The vector ρ_x (with a slightly different convention) was encountered in the setup of [Off06]. It does not show up in the cases considered in [LR03] by [ibid., Proposition 4.3.2]. Note that in our case δ_{P_x} is non-trivial on $\mathbf{M}_x(\mathbb{A})^{(1)}$ in general. This is in contrast with the cases considered in [LR03] and [Off06] where $\mathbf{M}_x(\mathbb{A})^{(1)} = \mathbf{M}_x(\mathbb{A})^1$.*

3.3. Cuspidal orbits. Let $x \in N_G(M) \cap X$ be M -standard relevant and assume further that $M = M_{(r_1, r_1, \dots, r_k, r_k, s_1, \dots, s_l; 0)}$ (i.e., $m = u = 0$) and $L(x) = M_{(2r_1, \dots, 2r_k, s_1, \dots, s_l; 0)}$. Thus,

$$M \simeq \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_1} \times \dots \times \mathrm{GL}_{r_k} \times \mathrm{GL}_{r_k} \times \mathrm{GL}_{s_1} \times \dots \times \mathrm{GL}_{s_l}.$$

The stabiliser M_x can be described as follows. The element x (in fact its M -conjugacy class) defines a decomposition $s_i = p_i + q_i$, $i = 1, \dots, l$ so that

$$M_x \simeq \mathrm{GL}_{r_1} \times \dots \times \mathrm{GL}_{r_k} \times (\mathrm{GL}_{p_1} \times \mathrm{GL}_{q_1}) \times \dots \times (\mathrm{GL}_{p_l} \times \mathrm{GL}_{q_l})$$

where GL_{r_i} is imbedded (twisted) diagonally in $\mathrm{GL}_{r_i} \times \mathrm{GL}_{r_i}$, $i = 1, \dots, k$ and $(\mathrm{GL}_{p_i} \times \mathrm{GL}_{q_i})$ is imbedded as the group of fixed points of an involution with signature (p_i, q_i) in GL_{s_i} , $i = 1, \dots, l$.

We call x as above M -standard cuspidal if there exists $0 \leq l_1 \leq l$ such that $p_i = q_i$ for $i = 1, \dots, l_1$ (in particular, $s_1, \dots, s_{l_1}$ are even) and $s_i = 1$, $l_1 + 1 \leq i \leq l$.

More generally, we say that $x \in N_G(M) \cap X$ is M -cuspidal if there exists $n \in N_G(T)$ such that nMn^{-1} is a standard Levi subgroup of G and nxn^{-1} is nMn^{-1} -standard cuspidal.

Let $[X]_{M, \text{cusp}}$ be the set of M -cuspidal M -conjugacy classes in $N_G(M) \cap X$.

4. INTERTWINING PERIODS

Our formula for the period integral of a pseudo-Eisenstein series is in terms of certain $\mathbf{H}(\mathbb{A})$ -invariant linear forms on induced representations of $\mathbf{G}(\mathbb{A})$ that we call intertwining periods. In this section we recall their definition for the pair (G, H) . They were introduced and studied in the Galois case in [JLR99] and [LR03].

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a parabolic subgroup of $\mathbf{G}$. Let $\mathcal{A}_P(G)$ be the space of continuous functions φ on $\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A})$ such that $\varphi(ag) = e^{\langle \rho_P, H_0(a) \rangle} \varphi(g)$ for all $a \in A_M$, $g \in G(\mathbb{A})$ and $\varphi(g) \ll \|g\|^N$ for some N .

Note that $\phi[\lambda] \in \mathcal{A}_P(G)$ for every $R > 0$, $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ and $\lambda \in \mathfrak{a}_{M, \mathbb{C}}^*$ with $\|\mathrm{Re} \lambda + \rho_P\| < R$.

Denote by $\mathcal{A}_P^{rd}(G)$ the subspace of $\mathcal{A}_P(G)$ consisting of φ such that for all $N > 0$

$$\sup_{m \in \mathfrak{S}_M^1, k \in K} |\varphi(mk)| \|m\|^N < \infty.$$

For instance, it follows from [MW95, Lemma I.2.10] that $\mathcal{A}_P^{rd}(G)$ contains the space of smooth functions $\varphi \in \mathcal{A}_P(G)$ of uniform moderate growth such that $m \mapsto \varphi(mg)$ is a cuspidal function on $\mathbf{M}(\mathbb{A})$ for all $g \in \mathbf{G}(\mathbb{A})$.

For $\varphi \in \mathcal{A}_P(G)$ and $\lambda \in \mathfrak{a}_{M, \mathbb{C}}^*$ let $\varphi_\lambda(g) = e^{\langle \lambda, H(g) \rangle} \varphi(g)$, $g \in \mathbf{G}(\mathbb{A})$.

For $\varphi \in \mathcal{A}_P(G)$ and $\lambda \in \rho_x + (\mathfrak{a}_{M, \mathbb{C}}^*)_x^-$, whenever convergent, we define

$$J(\varphi, x, \lambda) = \int_{\mathbf{P}_x(\mathbb{A}) \backslash \mathbf{G}_x(\mathbb{A})} \int_{M_x \backslash \mathbf{M}_x(\mathbb{A})^{(1)}} \delta_{P_x}^{-1}(m) \varphi_\lambda(mh\eta) \, dm \, dh$$

where $\eta \in G$ is such that $x = \eta\eta^{-1}$. (Recall that $\mathbf{M}_x(\mathbb{A})^{(1)} = \mathbf{M}_x(\mathbb{A}) \cap \mathbf{M}(\mathbb{A})^1$ and ρ_x is defined by (2).) Note that the expression does not depend on η , since $G_x\eta$ is determined by x . Furthermore, $J(\varphi, x, \lambda)$ only depends on the M -conjugacy class of x .

Let $\Sigma_{P, x} = \{\alpha \in \Sigma_P^+ : -x\alpha \in \Sigma_P^+\}$. For $\gamma > 0$ let

$$\mathfrak{D}_x(\gamma) = \{\lambda \in \rho_x + (\mathfrak{a}_{M, \mathbb{C}}^*)_x^- : \mathrm{Re} \langle \lambda, \alpha^\vee \rangle > \gamma, \forall \alpha \in \Sigma_{P, x}\}.$$

Theorem 4.1. *There exists $\gamma > 0$ such that for any M -cuspidal $x = \eta\eta^{-1}$ and $\varphi \in \mathcal{A}_P^{rd}(G)$ the integral defining $J(\varphi, x, \lambda)$ is absolutely convergent for $\lambda \in \mathfrak{D}_x(\gamma)$.*

5. THE PERIOD OF A PSEUDO-EISENSTEIN SERIES

Fix $R > 0$ large enough so that $\mathcal{P}_H(\theta_{P,\phi})$ is defined by an absolutely convergent integral for all $\phi \in C_R(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$.

Theorem 5.1. *There exists $\gamma > 0$ such that for any $\phi \in C_R^\infty(\mathbf{U}(\mathbb{A})M \backslash \mathbf{G}(\mathbb{A}))$ we have*

$$\int_{H \backslash \mathbf{H}(\mathbb{A})} \theta_{P,\phi}(h) dh = \sum_{x \in [X]_{M,\text{cusp}}} \int_{\lambda_x + i(\mathfrak{a}_M^*)_x^-} J(\phi[\lambda], x, \lambda) d\lambda$$

where the integrals are absolutely convergent and for any $x \in [X]_{M,\text{cusp}}$ we fix $\lambda_x \in \mathfrak{D}_x(\gamma)$ such that $\|\text{Re } \lambda_x + \rho_P\| < R$. In particular, $\mathcal{P}_H(\theta_{P,\phi}) = 0$ if $[X]_{M,\text{cusp}}$ is empty (and in particular, unless $M \subseteq M_{(n;0)}$).

We briefly explain the main steps of the proof. All integrals involved are absolutely convergent. Expanding $\theta_{P,\phi}$ as a sum over $P \backslash G$ and unfolding, we get that

$$\int_{H \backslash \mathbf{H}(\mathbb{A})} \theta_\phi(h) dh = \sum_{x \in P \backslash X} I_x(\phi)$$

where

$$I_x(\phi) = \int_{P_x \backslash \mathbf{H}(\mathbb{A})} \phi(h\eta) dh$$

and $\eta \in G$ is such that $x = \eta\epsilon\eta^{-1}$. Unless x is M -admissible, the integral I_x factors through a constant term in M and the cuspidality of ϕ implies that $I_x(\phi) = 0$. Our analysis of M -admissible orbits implies that

$$\int_{H \backslash \mathbf{H}(\mathbb{A})} \theta_\phi(h) dh = \sum_{x \in M \backslash (N_G(M) \cap X)} I_x(\phi).$$

Well-known vanishing results of periods of cuspidal functions (cf. [AGR93] and [JR92]) together with our study of the stabiliser M_x imply that $I_x(\phi) = 0$ unless x is M -cuspidal. The sum on the right hand side is therefore only over $[X]_{M,\text{cusp}}$. For M -cuspidal x , a partial Fourier inversion formula with respect to the decomposition $\mathfrak{a}_M = (\mathfrak{a}_M)_x^+ \oplus (\mathfrak{a}_M)_x^-$ implies that

$$I_x(\phi) = \int_{\mathbf{P}_x(\mathbb{A}) \backslash \mathbf{H}(\mathbb{A})} \int_{M_x \backslash \mathbf{M}_x(\mathbb{A})^{(1)}} \left(\int_{\lambda_x + i(\mathfrak{a}_M^*)_x^-} \phi[\lambda]_\lambda(mh\eta_x) d\lambda \right) \delta_{P_x}^{-1}(m) dm dh$$

for any $\lambda_x \in \rho_x + (\mathfrak{a}_{M,\mathbb{C}}^*)_x^-$ such that $\|\rho_P + \text{Re } \lambda_x\| < R$. By Theorem 4.1 and (1) the triple integral converges provided that $\lambda_x \in \mathfrak{D}_x(\gamma)$ for suitable γ . Changing the order of integration we obtain

$$I_x(\phi) = \int_{\lambda_x + i(\mathfrak{a}_M^*)_x^-} J(\phi[\lambda], x, \lambda) d\lambda.$$

The theorem follows.

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