

Deformations of Cones

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1. Deformations of isolated singularities.

Let X be an affine scheme of finite type over a field k , A a ring with an augmentation $A \rightarrow k$. By a deformation X_A of X over A , we mean a product diagram

$$\begin{array}{ccc} X_A & \longleftarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } A & \longleftarrow & \text{Spec } k \end{array} \quad X \simeq X_A \otimes_A k,$$

where X_A is flat over $\text{Spec } A$. If X'_A is another deformation of X over A , then X_A and X'_A are isomorphic if there exists an isomorphism $f: X_A \rightarrow X'_A$ over A which induces the identity over $\text{Spec } k$.

Throughout this lecture, we assume the following:

- i) X has only isolated singularities.
- ii) A is an artinian local k -algebra with residue field k (e.g., $A = k[t]/t^n$ etc.).

We write $X = \text{Spec } B$ with $B = k[x]/(f_1, \dots, f_q)$ where $x = (x_1, \dots, x_N)$ is a set of variables and $f_i \in k[x]$. For simplicity of notation, we set $P = k[x]$ and

$F = (f_1, \dots, f_q)P$. For some integer p , we have a resolution

$$(1) \quad P^p \xrightarrow{R} P^q \xrightarrow{f} P \longrightarrow B \longrightarrow 0$$

of B .

We note that any deformation X_A of X over A is an affine scheme ([EGA I.5.1.9]). So we can write $X_A = \text{Spec } B_A$ with an A -algebra B_A such that $B_A \otimes_A k = B$. Moreover, X_A can be embedded in $\mathbb{A}_A^N = \text{Spec } P_A$ where $P_A = A[x]$. By an embedded deformation of X (with respect to $\mathbb{A}_A^N = \text{Spec } P$) over A , we mean a closed subscheme X_A of $\mathbb{A}_A^N$ flat over $\text{Spec } A$ which induces the subscheme X of $\mathbb{A}^N$.

Proposition 1. The resolution (1) lifts to a resolution

$$(2) \quad P_A^p \xrightarrow{R_A} P_A^q \xrightarrow{f_A} P_A \longrightarrow B_A \longrightarrow 0.$$

In fact, the assertion is equivalent to the flatness of X_A over $\text{Spec } A$

We consider the following three deformation functors:

$\text{Defs}(X) : A \longrightarrow$ the set of isomorphic classes of
deformations of X over A .

$\text{Emb.Defs}(X) : A \longrightarrow$ the set of embedded deformations of
 X over A .

$\text{Defs of Res} : A \longrightarrow$ the set of isomorphic classes of
liftings (2) of (1).

Then we have three natural morphisms of functors:

$$\begin{array}{ccc}
 & \text{Emb.Defs}(X) & \\
 \nearrow & & \searrow \\
 \text{Defs of Res} & \xrightarrow{\quad} & \text{Defs}(X) .
 \end{array}$$

Corollary. These morphisms are smooth (see [F], Definition 2.2).

2. The tangent space of $\text{Defs}(X)$.

Let $A = k[t]/t^2$ where t is a variable. The set of isomorphic classes of deformations of X over A is called the tangent space of the functor $\text{Defs}(X)$. A deformation X_A of X over A is defined by $f_{A,i} \in P_A$ ($1 \leq i \leq q$). We write

$$f_{A,i} = f_i + g_i t \quad \text{with } g_i \in P.$$

The flatness of X_A over $\text{Spec } A$ is equivalent to the existence of an A -valued $p \times q$ -matrix R_A such that $R_A \equiv R \pmod{(t)}$ and $f_A R_A = 0$, where f_A denote the vector $(f_{A,1}, \dots, f_{A,q})$. If we write $R_A = R + St$ with a k -valued $p \times q$ -matrix S , then the above condition is equivalent to $gR + fS = 0$. Hence, we have

$$f_A = f + gt \text{ defines a deformation of } X.$$

$$\iff gR \equiv 0 \pmod{F}.$$

$$\iff g \text{ defines a } P\text{-homomorphism}$$

$$P^q/RP^p = F \longrightarrow B.$$

$$\iff g \text{ defines a } B\text{-homomorphism } F/F^2 \longrightarrow B.$$

We define the normal sheaf by

$$N_B = N_X = \text{Hom}_B(F/F^2, B).$$

Then the tangent space of $\text{Emb. Defs}(X)$ is isomorphic to N_B .

Next we shall kill the effect of automorphisms of $\mathbb{A}_A^N$. Let h be an automorphism of $\mathbb{A}_A^N$ over A which induces the identity on $\mathbb{A}_k^N$. Then h corresponds to $\tilde{h}: P_A \rightarrow P_A$ given by $\tilde{h}(x_i) = x_i + y_i t$ with $y_i \in P$. Hence $h^{-1}(X_A)$ is defined by $f + g't = f_A(x+yt)$. If we let J denote the matrix $(\frac{\partial f_i}{\partial x_j})$, then we have $g' = g + y^t J$.

Letting θ_X and $\theta_{\mathbb{A}}$ denote the sheaf of k -derivations of X and $\mathbb{A}^N$, respectively, we have the exact sequence

$$0 \longrightarrow \theta_X \xrightarrow{J} \theta_{\mathbb{A}|X} \longrightarrow N_X.$$

We set $T_X^1 = \text{Coker}(\theta_{\mathbb{A}|X} \longrightarrow N_X)$. Then the tangent space of $\text{Defs}(X)$ is isomorphic to T_X^1 . We note that the support of T_X^1 is concentrated at the singular locus of X .

For example, assume that X is normal and that $\dim X \geq 2$. We set $U = X - (\text{singular locus})$. Since $T_X^1 = 0$ on U , we have an exact sequence

$$\begin{aligned} (*) \quad H^0(U, \theta_X) &\longrightarrow H^0(U, \theta_{\mathbb{A}|U}) \longrightarrow H^0(U, N_X) \\ &\longrightarrow H^1(U, \theta_X). \end{aligned}$$

Since $\text{depth } \mathcal{O}_{X,x} \geq 2$ for any $x \in X$, the restriction maps $H^0(X, \mathcal{O}_X) \longrightarrow H^0(U, \mathcal{O}_U)$ and $H^0(X, \mathcal{O}_X|_X) \longrightarrow H^0(U, \mathcal{O}_U|_U)$ are bijective. Hence we can identify T_X^1 with a subspace of $\text{Coker}(H^0(U, \mathcal{O}_U|_U) \longrightarrow H^0(U, N_X))$.

Theorem 1. (Schlessinger) If X is an affine scheme over a field k with only isolated singularities, then there exists a formal versal deformation of X parametrized by a complete local ring $\hat{R}$.

For the proof, see [F], Proposition 3.10. A formal versal deformation of X means a hull of the functor $\text{Defs}(X)$. By the definition of a hull, if we let $\mathfrak{m}$ denote the maximal ideal of $\hat{R}$, we have

$$\mathfrak{m}/\mathfrak{m}^2 \simeq \text{tangent space of } \text{Defs}(X) \simeq T_X^1.$$

Problem. Compute $\hat{R}$.

We say that X is unobstructed, if the functor $\text{Defs}(X)$ is smooth, or equivalently, if $\hat{R}$ is a formal power series ring.

In the following cases, X is unobstructed.

- (i) X is a complete intersection.
- (ii) (Schaps) X is a Cohen-Macaulay subscheme of codimension 2 in an affine space.

3. Deformations of cones.

Let Y be a smooth subscheme in $\mathbb{P}^m$, C the cone

over Y in $\mathbb{A}^{m+1}$, and $\bar{C} = C \cup Y_\infty$ the cone over Y in $\mathbb{P}^{m+1}$. We ask to relate deformations of C , Y , and $\bar{C}$. We assume that Y is arithmetically normal (i.e., the systems of hypersurfaces of any degree are complete), or equivalently, C is normal. Let v be the vertex of C and $\bar{C}$, $U = C - v$, and $\bar{U} = \bar{C} - v$. We set

$$L = \mathbb{P}^{m+1} - v = \text{line bundle } \mathcal{O}_{\mathbb{P}^{m+1}}(1),$$

$$V = L - (0\text{-section}).$$

Then $L = \underline{\text{Spec}} S(\mathcal{O}(-1)) = \underline{\text{Spec}} \bigoplus_{n=-\infty}^0 \mathcal{O}(n)$ and

$V = \underline{\text{Spec}} \bigoplus_{n=-\infty}^{\infty} \mathcal{O}(n)$ is the $\mathbb{G}_m$ -bundle associated to L .

For any sheaf M on $\mathbb{P}^m$, we have

$$\begin{aligned} H^q(L, \pi^*M) &= \bigoplus_{n=-\infty}^0 H^q(\mathbb{P}^m, M(n)), \\ H^q(V, \pi^*M) &= \bigoplus_{n=-\infty}^{\infty} H^q(\mathbb{P}^m, M(n)), \end{aligned}$$

where π denote the natural projections onto $\mathbb{P}^m$.

Letting $\theta_{X/Y}$ denote the sheaf of derivations of X over Y , we have the standard exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \theta_{V/\mathbb{P}^m} & \longrightarrow & \theta_V & \longrightarrow & \pi^*\theta_{\mathbb{P}^m} \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \parallel \\ 0 & \longrightarrow & \pi^*\mathcal{O}_{\mathbb{P}^m} & \longrightarrow & \pi^*\mathcal{O}(1)^{m+1} & \longrightarrow & \pi^*\theta_{\mathbb{P}^m} \longrightarrow 0 \end{array}$$

Moreover, letting N_U and N_Y denote the normal sheaves of U and Y in V and $\mathbb{P}^m$, respectively, we have a commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^* \mathcal{O}_Y & \longrightarrow & \mathcal{O}_U & \longrightarrow & \pi^* \mathcal{O}_Y \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
(**) \quad 0 & \longrightarrow & \pi^* \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{V|U} & \longrightarrow & \pi^* \mathcal{O}_{\mathbb{P}^m|Y} \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & N_U & \xrightarrow{\sim} & \pi^* N_Y \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

where all vertical and horizontal lines are exact. We note that the sequence (*) is derived from the second vertical line in (**). Thus we get a commutative diagram

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
H^0(C, \mathcal{O}_{V|C}) & \longrightarrow & H^0(C, N_C) & \longrightarrow & H^0(C, T_C^1) & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
H^0(U, \mathcal{O}_{V|U}) & \longrightarrow & H^0(U, N_U) & \longrightarrow & H^1(U, \mathcal{O}_U) & & \\
\downarrow \wr & & \downarrow \wr & & & & \\
\bigoplus_{n=-\infty}^{\infty} H^0(Y, \mathcal{O}(n+1))^{m+1} & \longrightarrow & \bigoplus_{n=-\infty}^{\infty} H^0(Y, N_Y(n)) & & & &
\end{array}$$

This shows that T_C^1 is graded as $\bigoplus_{n=-\infty}^{\infty} T_C^1(n)$.

Let $\text{Hilb}(\bar{C})$ and $\text{Hilb}(Y)$ denote the Hilbert functors of $\bar{C}$ and Y in $\mathbb{P}^{m+1}$ and $\mathbb{P}^m$, respectively.

Theorem 2. (1) (Pinkham) If $T_C^1(n) = 0$ for $n > 0$, then the natural morphism

$$\text{Hilb}(\bar{C}) \longrightarrow \text{Defs}(C)$$

is smooth.

(2) (Schlessinger) If $T_C^1(n) = 0$ for $n \neq 0$, then every deformation of C is a cone, namely the natural morphism

$$\text{Hilb}(Y) \longrightarrow \text{Defs}(C)$$

is smooth.

Proof. (1) It suffices to prove the following:

(i) The tangent map is surjective.

(ii) Let $A' \rightarrow A$ be a small extension of artinian local k -algebras with residue field k , $C_A \in \text{Defs}(C)/_A$ ($= A$ -valued point of $\text{Defs}(C)$), $C_{A'} \in \text{Defs}(C)/_{A'}$, and $\bar{C}_A \in \text{Hilb}(\bar{C})/_A$ such that $C_{A'}$ induces C_A over $\text{Spec } A$ and $\bar{C}_A$ induces C_A on C . Then we can find $\bar{C}_{A'} \in \text{Hilb}(\bar{C})/_A$, which induces $\bar{C}_A$ over $\text{Spec } A$. (see [F], Definition 2.2).

In order to prove (i), we note that

$$H^0(\bar{C}, N_{\bar{C}}) \simeq H^0(\bar{U}, N_{\bar{U}}) = \bigoplus_{n=-\infty}^0 H^0(Y, N_Y(n)) ,$$

and that

$$\bigoplus_{n=-\infty}^{\infty} H^0(Y, N_Y(n)) = H^0(U, N_U) \longrightarrow T_C^1$$

is surjective. Hence the hypothesis implies that the tangent map $H^0(\bar{C}, N_{\bar{C}}) \longrightarrow T_C^1$ is surjective.

(ii) In general, the obstruction ξ for extending $\bar{C}_A$ to $\bar{C}_A$, lies in $H^1(\bar{C}, N_{\bar{C}})$. Since we have $\text{depth } \mathcal{O}_{C,z} \geq 2$ for any point $z \in C$, we have an injection

$$H^1(\bar{C}, N_{\bar{C}}) \hookrightarrow H^1(\bar{U}, N_{\bar{U}}) = \bigoplus_{n=-\infty}^0 H^1(Y, N_Y(n))$$

$$H^1(U, N_U) = \bigoplus_{n=-\infty}^{\infty} H^1(Y, N_Y(n))$$

$\downarrow$

The existence of C_A , implies that the image of ξ in $H^1(U, N_U)$ is zero. This proves the existence of $\bar{C}_A$.

The proof of (2) is similar, so we omit it. q.e.d.

Corollary. In the above situation, assume that $\dim Y \geq 2$ and that $\mathcal{O}_Y(1)$ is sufficiently ample in the sense that

$$H^1(Y, \mathcal{O}_Y(n)) = 0 \quad \text{for } n \neq 0,$$

$$H^1(Y, \Theta_Y(n)) = 0 \quad \text{for } n \neq 0.$$

Then every deformation of C is a cone.

Finally we study the special case: Y is a rational curve in $\mathbb{P}^m$ of degree m with a generic point $(1, t, t^2, \dots, t^m)$.

Y is defined by

$$\text{rank} \begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \\ x_1 & x_2 & \dots & x_m \end{pmatrix} < 2.$$

Theorem 3. If Y is a rational curve in $\mathbb{P}^m$ of degree m , then the natural morphism

$$\text{Hilb}(\overline{C}) \longrightarrow \text{Defs}(C)$$

is smooth.

This follows from $H^1(Y, \mathcal{O}_Y(n)) = H^1(Y, \mathcal{O}_Y(n)) = 0$ for $n > 0$.

We refer to the result of Nagata [N]: Let X be a surface in $\mathbb{P}^{m+1}$ of degree m .

(1) If X is singular, X is a cone over the singular point.

(2) If X is non-singular, X is a scroll, i.e., a rational ruled surface embedded linearly on fibres, unless $m = 4$ and X is the Veronese embedding of $\mathbb{P}^2$.

Using this fact, Pinkham proved

Corollary. Let $M = \text{Spf}(\hat{R})$ be the parameter space of a versal deformation of C .

- i) For $m = 2$ or 3 , M is smooth.
- ii) For $m = 4$, M has two components of dimension 3 and 1, which correspond to scrolls and Veronese embeddings of $\mathbb{P}^2$, respectively.
- iii) For $m > 4$, M_{red} is smooth of dimension $m - 1$, but $\hat{R}$ has non-zero nilpotent elements.

(Notes by E. Horikawa)

References

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