

On some additive problems with primes and almost-primes

T. P. Peneva

§1. Introduction and statement of the result.

In 1937 I. M. Vinogradov [11] solved the ternary Goldbach problem proving that for every sufficiently large odd integer N the equation

$$p_1 + p_2 + p_3 = N \quad (1)$$

has solutions in prime numbers p_1, p_2, p_3 .

Two years later van der Corput [3] used the method of I. M. Vinogradov and established the existence of infinitely many arithmetic progressions of three different primes. A corresponding result for progressions of four or more primes has not been proved yet. In 1981, however, D. R. Heath-Brown [4] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is P_2 (as usual P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

Another famous and still unsolved number theory problem is the so-called *prime twins* conjecture, which asserts that there exist infinitely many primes p , such that $p + 2$ is also a prime. The most important achievement in studying this problem is due to Chen [2]. In 1973 he proved that there exist infinitely many primes p , such that $p + 2$ is P_2 .

In 1997 D. I. Tolev and the author [8] applied the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem as well as some arguments belonging to H. Mikawa, and proved that there exist infinitely many non-trivial arithmetic progressions of three primes, such that for two of them, p_1 and p_2 , say, both the numbers $p_1 + 2$, $p_2 + 2$ are almost-primes.

Later D. I. Tolev [9] obtained an extension of the above result by applying the vector sieve developed by Iwaniec [5] and used also by Brüdern and Fouvry [1]. He established that the equation

$$p_1 + p_2 = 2p_3$$

has infinitely many solutions in different primes p_1, p_2, p_3 , such that $p_1 + 2 = P_5$, $p_2 + 2 = P'_5$, $p_3 + 2 = P_8$.

Here we study the solvability of the equation (1) in primes p_1, p_2, p_3 , such that $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ are almost-primes. We follow the approach of [9] putting emphasis on the examining of the main term where we apply some arguments of [1] (for the other details the reader may refer to [9]).

Our main result is the following

Theorem. Suppose that $N \equiv 3 \pmod{6}$ is a sufficiently large integer. Then there exist infinitely many solutions of the equation (1) in primes p_1, p_2, p_3 , such that $p_1 + 2 = P_5$, $p_2 + 2 = P'_5$, $p_3 + 2 = P_8$.

In fact, the proof yields that for some constant $c_0 > 0$ there are at least $c_0 N^2 (\log N)^{-6}$ triplets of primes p_1, p_2, p_3 , satisfying (1) and such that for any prime factor p of $p_1 + 2$ or $p_2 + 2$ we have $p \geq N^{0.167}$ and for any prime factor p of $p_3 + 2$ we have $p \geq N^{0.116}$. Notice that if N is a sufficiently large odd integer, not satisfying the hypothesis of the Theorem, then for any solution of (1) we have $3 \mid p_1 p_2 p_3 (p_1 + 2) (p_2 + 2) (p_3 + 2)$. Therefore, by modifying slightly the given proof, we may obtain that for such N the equation (1) has infinitely many solutions in primes p_1, p_2, p_3 , such that $p_1 + 2 = P_6$, $p_2 + 2 = P'_6$, $p_3 + 2 = P_9$. Here the extra prime factor in P_r is 3.

Recently H. Mikawa (unpublished result) used the theory of "well-factorable" functions and showed that the power of N in the quantity D_3 (for the definition see formulas (2)) can be taken to be equal to $4/9$ instead of $1/3$. This enables us to prove the Theorem with $p_3 + 2 = P_6$.

We should also mention that by applying the method of [9], D. I. Tolev [10] proved that if N is a sufficiently large integer satisfying the congruent condition $N \equiv 5 \pmod{24}$ then the equation

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N$$

has infinitely many solutions in prime numbers p_1, p_2, p_3, p_4, p_5 such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ and $p_4 + 2$ has at most 6 prime factors and $p_5 + 2$ has at most 7 prime factors.

§2. Notations.

Let N be a sufficiently large integer, such that $N \equiv 3 \pmod{6}$ and $\alpha_1, \alpha_2, \alpha_3$ — constants satisfying $0 < \alpha_1, \alpha_2 < 1/4$, $0 < \alpha_3 < 1/6$, which we shall specify later.

We put

$$z_i = N^{\alpha_i}, \quad i = 1, 2, 3, \quad z_0 = (\log N)^{1000}, \quad D_0 = \exp((\log N)^{0.6}),$$

$$D_1 = D_2 = N^{1/2} \exp(-2(\log N)^{0.6}), \quad D_3 = N^{1/3} \exp(-2(\log N)^{0.6}), \quad (2)$$

$$P(z_0) = \prod_{2 < p < z_0} p, \quad P(z_0, z_i) = \prod_{z_0 \leq p < z_i} p, \quad i = 1, 2, 3.$$

Letters $m, n, d, l, k, h, \delta, \nu, t, \rho$ denote integers; $p, p_1, p_2, \dots$ — prime numbers. As usual $\mu(n)$, $\varphi(n)$ and $\tau(n)$ denote Möbius' function, Euler's function and the number of positive divisors of n , respectively; $(m_1, \dots, m_k)$ and $[m_1, \dots, m_k]$ denote the greatest common divisor and the least common multiple of $m_1, \dots, m_k$. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. The notation $p^\nu \parallel n$ means that $p^\nu \mid n$ and $p^{\nu+1} \nmid n$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$.

For squarefree odd integers k_1, k_2, k_3 and prime p we denote

$$I_{k_1, k_2, k_3}(N) = \sum_{\substack{p_1 + p_2 + p_3 = N \\ p_i + 2 \equiv 0(k_i), \ i=1,2,3}} \log p_1 \log p_2 \log p_3, \quad (3)$$

$$h_{k_1, k_2, k_3}(p) = \begin{cases} 1/(p-1)^3 & \text{if } p \nmid k_1 k_2 k_3, \ p \nmid N; \\ -1/(p-1)^2 & \text{if } p \nmid k_1 k_2 k_3, \ p \mid N; \\ -1/(p-1)^2 & \text{if } p \parallel k_1 k_2 k_3, \ p \nmid N+2; \\ 1/(p-1) & \text{if } p \parallel k_1 k_2 k_3, \ p \mid N+2; \\ 1/(p-1) & \text{if } p^2 \parallel k_1 k_2 k_3, \ p \nmid N+4; \\ -1 & \text{if } p^2 \parallel k_1 k_2 k_3, \ p \mid N+4; \\ -1 & \text{if } p^3 \parallel k_1 k_2 k_3, \ p \nmid N+6; \\ p-1 & \text{if } p^3 \parallel k_1 k_2 k_3, \ p \mid N+6; \end{cases} \quad (4)$$

$$\omega(k_1, k_2, k_3) = \prod_{p \mid k_1 k_2 k_3} \frac{1 + h_{k_1, k_2, k_3}(p)}{1 + h_{1,1,1}(p)}, \quad \Omega(k_1, k_2, k_3) = \frac{\omega(k_1, k_2, k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}, \quad (5)$$

$$\mathfrak{S}(N) = \prod_{p \nmid N} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \mid N} \left(1 - \frac{1}{(p-1)^2}\right). \quad (6)$$

§3. Outline of the proof.

Consider the sum

$$\Gamma = \sum_{\substack{p_1 + p_2 + p_3 = N \\ (p_i + 2, P(z_i)) = 1, \ i=1,2,3}} \log p_1 \log p_2 \log p_3.$$

Any non-trivial estimate from below of Γ implies the solvability of (1) in primes, such that $p_i + 2 = P_{h_i}$, $h_i = [\alpha_i^{-1}]$, $i = 1, 2, 3$. We see that

$$\Gamma = \sum_{p_1 + p_2 + p_3 = N} \log p_1 \log p_2 \log p_3 \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_6,$$

where

$$\Lambda_i = \begin{cases} \sum_{d \mid (p_i + 2, P(z_0, z_i))} \mu(d) & \text{for } i = 1, 2, 3, \\ \sum_{d \mid (p_{i-3} + 2, P(z_0))} \mu(d) & \text{for } i = 4, 5, 6. \end{cases}$$

Denote

$$\Lambda_i^\pm = \begin{cases} \sum_{d \mid (p_i + 2, P(z_0, z_i))} \lambda_i^\pm(d) & \text{for } i = 1, 2, 3, \\ \sum_{d \mid (p_{i-3} + 2, P(z_0))} \lambda_0^\pm(d) & \text{for } i = 4, 5, 6; \end{cases} \quad (7)$$

Define

$$\omega_1(p) = \omega(p, 1, 1), \quad \omega_2(p) = \omega(p, p, 1), \quad \omega_3(p) = \omega(p, p, p). \quad (15)$$

It is clear that if $p > 2$ then

$$\omega_1(p) = \begin{cases} 1 & \text{if } p \mid N; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } p \mid N+2; \\ \frac{(p-1)(p-2)}{p^2-3p+3} & \text{if } p \nmid N(N+2); \end{cases} \quad \omega_2(p) = \begin{cases} \frac{p-1}{p-2} & \text{if } p \mid N; \\ 0 & \text{if } p \mid N+4; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } p \nmid N(N+4); \end{cases} \quad (16)$$

$$\omega_3(p) = \begin{cases} 4 & \text{if } p=3; \\ \frac{(p-1)^3}{p^2-3p+3} & \text{if } p \mid N+6, \quad p > 3; \\ 0 & \text{if } p \nmid N+6. \end{cases}$$

By (4), (5), (14), (15) we get

$$\omega(k_1, k_2, k_3) = \prod_{\substack{p^\nu \parallel k_1 k_2 k_3 \\ 1 \leq \nu \leq 3}} \omega_\nu(p). \quad (17)$$

The next statement is the analogue of Lemma 12 of [1]. The Lemma follows easily from (16), (17).

Lemma 1. *For squarefree odd k , let*

$$\omega^*(k) = \prod_{p \mid k} \omega_1(p).$$

If k_1, k_2, k_3 is a triplet of integers, we put $k_{1,2} = (k_1, k_2)$, $k_{1,3} = (k_1, k_3)$, $k_{2,3} = (k_2, k_3)$. Then

(i) there exists a function g of the three variables $k_{i,j}$, such that for any squarefree odd k_1, k_2, k_3 we have

$$\omega(k_1, k_2, k_3) = \omega^*(k_1) \omega^*(k_2) \omega^*(k_3) g(k_{1,2}, k_{1,3}, k_{2,3})$$

and

$$g(k_{1,2}, k_{1,3}, k_{2,3}) \leq 10 (\max k_{i,j})^{10};$$

(ii) for any squarefree odd k_1, k_2, k_3 we have the inequality

$$\omega(k_1, k_2, k_3) \leq 10 \tilde{\omega}(k_1) \tilde{\omega}(k_2) \tilde{\omega}(k_3),$$

where $\tilde{\omega}(m)$ is the multiplicative function defined on squarefree odd m by

$$\tilde{\omega}(p) = \begin{cases} 2 & \text{if } p \nmid N+6; \\ 2p^{1/3} & \text{if } p \mid N+6. \end{cases}$$

Suppose that the integers $d_1, d_2, d_3, \delta_1, \delta_2, \delta_3$ satisfy the conditions imposed in (13). Using (5) and (17) we easily get

$$\Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3) = \Omega(d_1, d_2, d_3) \Omega(\delta_1, \delta_2, \delta_3).$$

Note that $\Omega(\delta_1, \delta_2, \delta_3)$ is a symmetrical function with respect to $\delta_1, \delta_2, \delta_3$. Hence, we obtain by (11), (13)

$$W = \sum_{i=1}^6 L_i H_i - 5L_7 H_7, \quad (18)$$

where

$$\begin{aligned} L_1 &= \sum_{d_i | P(z_0, z_i), i=1,2,3} \lambda_1^-(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3), \\ L_2 &= \sum_{d_i | P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^-(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3), \\ L_3 &= \sum_{d_i | P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^-(d_3) \Omega(d_1, d_2, d_3), \\ L_4 = L_5 = L_6 = L_7 &= \sum_{d_i | P(z_0, z_i), i=1,2,3} \lambda_1^+(d_1) \lambda_2^+(d_2) \lambda_3^+(d_3) \Omega(d_1, d_2, d_3), \\ H_1 = H_2 = H_3 = H_7 &= \sum_{\delta_i | P(z_0), i=1,2,3} \lambda_0^+(\delta_1) \lambda_0^+(\delta_2) \lambda_0^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3), \\ H_4 = H_5 = H_6 &= \sum_{\delta_i | P(z_0), i=1,2,3} \lambda_0^-(\delta_1) \lambda_0^+(\delta_2) \lambda_0^+(\delta_3) \Omega(\delta_1, \delta_2, \delta_3). \end{aligned}$$

It is easy to prove the following

Lemma 2. Suppose that $\phi(n_1, n_2, n_3)$ is a function defined on the set of integers and such that for any two triplets n_1, n_2, n_3 and l_1, l_2, l_3 , satisfying $(n_1 n_2 n_3, l_1 l_2 l_3) = 1$, we have $\phi(n_1 l_1, n_2 l_2, n_3 l_3) = \phi(n_1, n_2, n_3) \phi(l_1, l_2, l_3)$. Then the function

$$\Phi(n) = \sum_{d_1, d_2, d_3 | n} \phi(d_1, d_2, d_3)$$

is multiplicative.

Applying Lemma 1 and Lemma 2 we find asymptotic formulas for the sums H_i . Define

$$H^{(\mu)} = \sum_{\delta_i | P(z_0), i=1,2,3} \mu(\delta_1) \mu(\delta_2) \mu(\delta_3) \Omega(\delta_1, \delta_2, \delta_3). \quad (19)$$

Lemma 3. *We have*

$$H_i = H^{(\mu)} + \mathcal{O}((\log N)^{-10}), \quad 1 \leq i \leq 7,$$

and

$$H^{(\mu)} \asymp (\log z_0)^{-3}. \quad (20)$$

Now we are able to estimate from below the quantity W , defined by (18). We put

$$\mathcal{F}(z_0, z_i) = \prod_{z_0 \leq p < z_i} \left(1 - \frac{\omega_1(p)}{p-1}\right), \quad s_i = \frac{\log D_i}{\log z_i}, \quad i = 1, 2, 3, \quad (21)$$

where $\omega_1(p)$ is defined by (16). Suppose that $c^* > 0$ is an absolute constant and let θ_i, s_i , $i = 1, 2, 3$ satisfy

$$\theta_1 + \theta_2 + \theta_3 = 1, \quad \theta_i > 0, \quad f(s_i) - 2\theta_i F(s_i) > c^*, \quad i = 1, 2, 3,$$

where f and F are the functions of the linear sieve. Following the arguments in the proof of Lemma 15 of [9] it is easy to establish that

$$W \geq H^{(\mu)} \prod_{j=1}^3 \mathcal{F}(z_0, z_j) \left(\sum_{i=1}^3 (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}((\log N)^{-1/3}) \right). \quad (22)$$

Finally, we choose

$$\alpha_1 = \alpha_2 = 0.167, \quad \alpha_3 = 0.116, \quad \theta_1 = \theta_2 = 0.345, \quad \theta_3 = 0.31$$

and compute that for sufficiently large N we have

$$f(s_i) - 2\theta_i F(s_i) > 10^{-5}, \quad i = 1, 2, 3. \quad (23)$$

Therefore, using (2), (20)–(23) we get

$$W \gg \frac{1}{\log^3 N}.$$

The last estimate and (9), (12) imply

$$\Gamma \gg \frac{N^2}{\log^3 N},$$

which suffices to complete the proof of the Theorem.

References

- [1] J. Brüdern, E. Fouvry, *Lagrange's Four Squares Theorem with almost prime variables*, J. Reine Angew. Math. **454** (1994), 59–96.
- [2] J. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica **16** (1973), 157–176.
- [3] J. G. van der Corput, *Über Summen von Primzahlen und Primzahlquadraten*, Math. Ann. **116** (1939), 1–50.
- [4] D. R. Heath-Brown, *Three primes and an almost-prime in arithmetic progression*, J. London Math. Soc. (2), **23** (1981), 396–414.
- [5] H. Iwaniec, *On sums of two norms from cubic fields*, Journées de théorie additive des nombres, Université de Bordeaux I, (1977), 71–89.
- [6] H. Iwaniec, *Rosser's sieve*, Acta Arith. **36** (1980), 171–202.
- [7] H. Iwaniec, *A new form of the error term in the linear sieve*, Acta Arith. **37** (1980), 307–320.
- [8] T. P. Peneva, D. I. Tolev, *An additive problem with primes and almost-primes*, Acta Arith. **83** (1998), 155–169.
- [9] D. I. Tolev, *Arithmetic progressions of prime-almost-prime twins*, to appear in Acta Arithmetica.
- [10] D. I. Tolev, *On the representation of an integer as a sum of five squares of prime numbers of a special type*, preprint of Plovdiv University "Paisii Hilendarski", 1998.
- [11] I. M. Vinogradov, *Representation of an odd number as a sum of three primes*, Dokl. Akad. Nauk SSSR **15** (1937), 169–172 (in Russian).

Institute of Mathematics
 University of Tsukuba
 Tsukuba-shi
 Ibaraki 305
 JAPAN

E-mail: tpeneva@hotmail.com