

ON STARLIKENESS AND CONVEXITY OF FUNCTIONS AND THE SCHWARZIAN DERIVATIVE

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ABSTRACT. The purpose of this paper is to generalize Miller and Mocanu's result [2].

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, \text{ and } |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be in the class $\mathcal{S}^*$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathcal{U}$$

and is said to be in the class $\mathcal{C}$ if and only if

$$1 + \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in } \mathcal{U}.$$

We denote by $\{f, z\}$ the Schwarzian derivative, which is characterized by the equality

$$(1) \quad \{f, z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

In [1], Nunokawa et al. obtained the following result:

Theorem A. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$(2) \quad \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] \geq -\frac{1}{2} \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{S}^$.*

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. starlike, convex, Schwarzian derivative.

Remark. Theorem A is an extension of Miller and Mocanu [2], where the right hand side of (2) is improved from 0 to $-\frac{1}{2}$.

Further Miller and Mocanu [2] obtained the following results :

Theorem B. Let $f(z) \in \mathcal{A}$ satisfy

$$\operatorname{Re} \left[\left(1 + \frac{zf''(z)}{f'(z)} \right)^2 + z^2\{f, z\} \right] > 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Theorem C. Let $f(z) \in \mathcal{A}$ satisfy

$$\operatorname{Re} \left[\left(1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2\{f, z\}} \right] > 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Let us investigate improvements of these results in the next section.

2. Main Results

The following result will be required in our investigation :

Lemma. [3] Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and $\operatorname{Re}\{p(z_0)\} = 0$ ($p(z_0) \neq 0$). Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is a real number and

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{when } p(z_0) = ia, \ a > 0,$$

and

$$k \leq \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{when } p(z_0) = ia, \ a < 0.$$

Now we state our main result.

Theorem 1. Let $f(z) \in \mathcal{A}$ and satisfy one of the following inequalities :

$$\begin{aligned} (3) \quad & \operatorname{Re} \left[\left(\frac{zf'(z)}{f(z)} \right)^{4m-1} \left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\} \right) \right] \\ & < \frac{1}{2} \left| \frac{zf'(z)}{f(z)} \right|^{4m-2} \left(3 \left| \frac{zf'(z)}{f(z)} \right|^2 + 1 \right) \quad \text{in } \mathcal{U}, \end{aligned}$$

$$(4) \quad \operatorname{Re} \left[\left(\frac{zf'(z)}{f(z)} \right)^{4m-3} \left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\} \right) \right] \\ > -\frac{1}{2} \left| \frac{zf'(z)}{f(z)} \right|^{4m-4} \left(3 \left| \frac{zf'(z)}{f(z)} \right|^2 + 1 \right) \quad \text{in } \mathcal{U},$$

where m is a positive integer. Then $f(z) \in \mathcal{S}^*$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)},$$

then we easily have

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}$$

and from (1), by a simple calculation, we have

$$(5) \quad z^2\{f, z\} = z^2 \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{zf''(z)}{f'(z)} \right)^2 \\ = \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \frac{3}{2} \left(\frac{zp'(z)}{p(z)} \right)^2 + \frac{1}{2} \{1 - p(z)^2\}.$$

To prove $\operatorname{Re} \{zf'(z)/f(z)\} > 0$ in $\mathcal{U}$, we show $\operatorname{Re} \{p(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re} \{p(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} \{p(z_0)\} = 0 \quad (p(z_0) \neq 0),$$

then from Lemma we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik,$$

and (3), (4) and (5) imply

$$(6) \quad \left(\frac{z_0f'(z_0)}{f(z_0)} \right)^l \left(1 + \frac{z_0f''(z_0)}{f'(z_0)} + z_0^2\{f, z_0\} \right) \\ = (ia)^l \left[ia + ik + ik + \frac{z_0^2p''(z_0)}{p(z_0)} - \frac{3}{2}(ik)^2 + \frac{1}{2}\{1 - (ia)^2\} \right] \\ = (ia)^l \left[i \left\{ a + k + k \left(1 + \frac{z_0p''(z_0)}{p'(z_0)} \right) \right\} + \frac{3}{2}k^2 + \frac{1}{2}(1 + a^2) \right],$$

where l is a positive integer. Let J be the right hand side of (6). For the case $l = 2n - 1$,

$$J = (ia)^{2n-1} \left[i \left\{ a + k + k \left(1 + \frac{z_0p''(z_0)}{p'(z_0)} \right) \right\} + \frac{3}{2}k^2 + \frac{1}{2}(1 + a^2) \right].$$

Therefore we have

$$\operatorname{Re}\{J\} = (-1)^n a^{2n-1} \left[a + k + k \left(1 + \operatorname{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right].$$

Considering the geometrical property, we notice that the tangential vector of the curve $p(z) = p(z_0)$ moves to positive direction near the point $p(z_0)$. In short, $p(z)$ is convex in the neighborhood of the point $p(z_0)$, or

$$1 + \operatorname{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \geq 0.$$

(i) Case $n = 2m$:

$$\begin{aligned} \operatorname{Re}\{J\} &= (-1)^{2m} a^{4m-1} \left[a + k + k \left(1 + \operatorname{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right] \\ &\geq a^{4m-1} (a + k) \\ &= -a^{4m-2} (-a^2 - ak) \\ &\geq -a^{4m-2} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\} \\ &= -a^{4m-2} \left(-\frac{3}{2} a^2 - \frac{1}{2} \right) \\ &= \frac{1}{2} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^{4m-2} \left(3 \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^2 + 1 \right). \end{aligned}$$

(ii) Case $n = 2m - 1$:

$$\begin{aligned} \operatorname{Re}\{J\} &= (-1)^{2m-1} a^{4m-3} \left[a + k + k \left(1 + \operatorname{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right] \\ &\leq -a^{4m-3} (a + k) \\ &= a^{4m-4} (-a^2 - ak) \\ &\leq a^{4m-4} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\} \\ &= a^{4m-4} \left(-\frac{3}{2} a^2 - \frac{1}{2} \right) \\ &= -\frac{1}{2} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^{4m-4} \left(3 \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^2 + 1 \right). \end{aligned}$$

These contradict (3) and (4), respectively. Hence we must have

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{in } \mathcal{U}$$

or

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathcal{U},$$

which means $f(z) \in \mathcal{S}^*$. This completes our proof.

Setting $\alpha = 1$ in Theorem 1, we obtain

Corollary 1. Let $f(z) \in \mathcal{A}$ and suppose that

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\} \right) \right] > -\frac{1}{2} \left(1 + 3 \left| \frac{zf'(z)}{f(z)} \right|^2 \right) \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{S}^*$.

Corollary 1 is better than Theorem A.

Theorem 2. Let $f(z) \in \mathcal{A}$ and suppose that

$$(7) \quad \operatorname{Re} \left[\left(1 + \frac{zf''(z)}{f'(z)} \right)^{2n} + z^2\{f, z\} \right] + (-1)^{n+1} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^{2n} > 0 \quad \text{in } \mathcal{U}$$

for positive integer n . Then $f(z) \in \mathcal{C}$.

Proof. Let us put

$$q(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

Note that $q(0) = 1$. Then from (1), we easily have

$$(8) \quad z^2\{f, z\} = zq'(z) - \frac{1}{2}q(z)^2 + \frac{1}{2}.$$

To prove $1 + \operatorname{Re} \{zf''(z)/f'(z)\} > 0$ in $\mathcal{U}$, we show $\operatorname{Re} \{q(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re} \{q(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} \{q(z_0)\} = 0 \quad (q(z_0) \neq 0),$$

then from Lemma a real number k ($k \neq 0$) exists such that

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik.$$

From (7) and (8), we have

$$\begin{aligned} & \operatorname{Re} \left[\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right)^{2n} + z_0^2 \{f, z_0\} \right] + (-1)^{n+1} \left| 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right|^{2n} \\ &= \operatorname{Re} \left\{ (q(z_0))^{2n} + z_0 q'(z_0) - \frac{1}{2} q(z_0)^2 + \frac{1}{2} \right\} + (-1)^{n+1} |q(z_0)|^{2n} \\ &= \operatorname{Re} \left\{ (ia)^{2n} - ak - \frac{1}{2} (ia)^2 + \frac{1}{2} \right\} + (-1)^{n+1} |ia|^{2n} \\ &\leq (-1)^n a^{2n} - \frac{1}{2} (a^2 + 1) + \frac{1}{2} (a^2 + 1) + (-1)^{n+1} |a|^{2n} \\ &= 0. \end{aligned}$$

This is in contradiction to (7). Hence we must have

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{in } \mathcal{U}$$

or

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in } \mathcal{U}.$$

Therefore $f(z) \in \mathcal{C}$ and our result is established.

Taking $n = 1$ in Theorem 2, we have

Corollary 2. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f, z\}\right] + \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 > 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Corollary 2 is better than Theorem B.

Theorem 3. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^{2n-1} e^{z^2\{f, z\}}\right] \neq 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Proof. Let us take the same function $q(z)$ as in the proof of Theorem 2. Then from the assumption of theorem and (8), we find

$$\begin{aligned} & \operatorname{Re}\left[\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)^{2n-1} e^{z_0^2\{f, z_0\}}\right] \\ &= \operatorname{Re}\left[(q(z_0))^{2n-1} e^{z_0 q'(z_0) - \frac{1}{2} q(z_0)^2 + \frac{1}{2}}\right] \\ &= \operatorname{Re}\left[(ia)^{2n-1} e^{-ak + \frac{1}{2} a^2 + \frac{1}{2}}\right] \\ &= \operatorname{Re}\left[i(-1)^{n+1} a^{2n-1} e^{-ak + \frac{1}{2} a^2 + \frac{1}{2}}\right] \\ &= 0. \end{aligned}$$

This is a contradiction to the assumption. Hence we must have

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{in } \mathcal{U}$$

or

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in } \mathcal{U},$$

which yields our result.

Putting $n = 1$ in Theorem 3, we have

Corollary 3. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re} \left[\left(1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2\{f,z\}} \right] \neq 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Corollary 3 is a revision of Theorem C.

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