

INCLUSION PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. The object of the present paper is to give sharp forms of inclusion properties of the class $P(p, \alpha, \beta)$ under operators $J_{p,c}$, F_m and $J_{p,1}^\lambda$.

1. INTRODUCTION

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be a member of the class $P(p, \alpha)$ if it satisfies the inequality $\operatorname{Re}\{f'(z)/z^{p-1}\} > \alpha$ ($z \in U$) for some α ($0 \leq \alpha < p$). The classes $P(1, 0)$ and $P(p, 0)$ were investigated by MacGregor [5] and Umezawa [6], respectively.

For a function $f(z)$ belonging to $A(p)$, we define the generalized Bernardi integral operator $J_{p,c}$ by

$$\begin{aligned} J_{p,c}(f) &= \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \\ &= z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} a_{p+n} z^{p+n} \quad (p \in \mathbb{N}; c > -p). \end{aligned} \quad (1.2)$$

The operator $J_{1,c}$ for $c \in \mathbb{N}$ was introduced by Bernardi [1]. Clearly, from (1.2) we see that

$$f(z) \in A(p) \implies J_{p,c}(f) \in A(p) \quad (c > -p). \quad (1.3)$$

Thus, by applying the operator $J_{p,c}$ successively, we can obtain

$$J_{p,c}^n(f) = \begin{cases} J_{p,c}(J_{p,c}^{n-1}(f)) & (n \in \mathbb{N}) \\ f(z) & (n = 0). \end{cases} \quad (1.4)$$

Suppose also that

$$\begin{aligned} F_m(f) &= J_{p,c_m}(J_{p,c_{m-1}} \dots (J_{p,c_1}(f))) \\ &= z^p + \sum_{n=1}^{\infty} \left[\prod_{j=1}^m \frac{p+c_j}{p+n+c_j} \right] a_{p+n} z^{p+n} \quad (c_j > -p; m \in \mathbb{N}). \end{aligned} \quad (1.5)$$

For an analytic function $g(z)$ given by $g(z) = \sum_{n=0}^{\infty} b_{p+n} z^{p+n}$ in U , and for a real number λ , Flett [3] define the multiplier transformation $I^\lambda g(z)$ by

$$I^\lambda g(z) = \sum_{n=0}^{\infty} (p+n+1)^{-\lambda} b_{p+n} z^{p+n} \quad (z \in U). \quad (1.6)$$

The function $I^\lambda g(z)$ is clearly analytic in U . It may be regarded as a fractional integral (for $\lambda > 0$) or a fractional derivative (for $\lambda < 0$) of $g(z)$. Furthermore, in terms of the Gamma function, we have

$$I^\lambda g(z) = \frac{1}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} g(tz) dt \quad (\lambda > 0). \quad (1.7)$$

Denote by $D^\lambda g(z)$ the multiplier transformation $I^{-\lambda} g(z)$ for $\lambda \geq 0$, i.e.,

$$D^\lambda g(z) = I^{-\lambda} g(z) = \sum_{n=0}^{\infty} (p+n+1)^\lambda b_{p+n} z^{p+n} \quad (\lambda \geq 0; z \in U). \quad (1.8)$$

From (1.4) and (1.6),

$$J_{p,1}^m(f) = (p+1)^m I^m(f) \quad (m \in \mathbb{N}; f \in A(p)). \quad (1.9)$$

Thus, one can define the operator $J_{p,1}^\lambda$ (depending on a continuous parameter $\lambda > 0$) by

$$J_{p,1}^\lambda(f) = (p+1)^\lambda I^\lambda(f) \quad (\lambda > 0; f \in A(p)). \quad (1.10)$$

Making use of the fractional derivative operator and operators $J_{p,c}$, F_m and $J_{p,1}^\lambda$ as mentioned above, Cho [2] introduced and studied the class $P(p,\alpha,\beta)$ defined by

$$P(p,\alpha,\beta) = \{f \in A(p) : (p+1)^{-\beta} D^\beta f \in P(p,\alpha)\},$$

where $0 \leq \alpha < p$ and $\beta \geq 0$. Observe that $P(p,\alpha,0) = P(p,\alpha)$. If $\beta \geq 0$ and $0 \leq \alpha_1 \leq \alpha_2 < p$, then $P(p,\alpha_2,\beta) \subset P(p,\alpha_1,\beta)$. The class $P(1,\alpha,\beta)$ was introduced and studied by Kim, Lee and Srivastava [4]. In [2], Cho showed that

(i) if $f(z) \in P(p,\alpha,\beta)$, then $J_{p,c}(f)$, $F_m(f)$ and $J_{p,1}^\lambda(f)$ are also in the class $P(p,\alpha,\beta)$, where $c \in \mathbb{N}$ and $c_j \in \mathbb{N}$,

(ii) if $0 \leq \alpha < p$ and $\beta \geq 0$, then $P(p,\alpha,\beta+1) \subset P(p,\mu,\beta)$, where $\mu = (2\alpha(p+1)+p)/(2(p+1)+1)$.

In the case of $p = 1$, these results correspond to the results by Kim, Lee and Srivastava [4]. In the present paper, we give the sharp forms of these results simply.

2. INCLUSION PROPERTIES

Our first result for the class $P(p,\alpha,\beta)$ is contained in

THEOREM I. If $f(z)$ is in the class $P(p,\alpha,\beta)$, then $J_{p,c}(f)$ belongs to the class $P(p,\mu,\beta)$, where

$$\mu = p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c}.$$

The result is sharp.

PROOF. It follows from the definitions (1.2) and (1.8) that

$$\begin{aligned} (p+1)^{-\beta} D^\beta (J_{p,c}(f)) &= J_{p,c}((p+1)^{-\beta} D^\beta f) \\ &= (p+c) \int_0^1 t^{c-1} \{(p+1)^{-\beta} D^\beta f(tz)\} dt. \end{aligned} \quad (2.1)$$

Therefore, setting

$$H(z) = (p+1)^{-\beta} D^{\beta}(J_{p,c}(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta} D^{\beta}f(z), \quad (2.2)$$

we must show that

$$\operatorname{Re} \left[\frac{H'(z)}{z^{p-1}} \right] > \mu \quad (0 \leq \alpha < p; c > -p; z \in \mathbb{U}) \quad (2.3)$$

whenever $h(z) \in P(p, \alpha)$. Note that (2.1) gives

$$\operatorname{Re} \left[\frac{H'(z)}{z^{p-1}} \right] = (p+c) \int_0^1 t^{p+c-1} \operatorname{Re} \left[\frac{h'(tz)}{(tz)^{p-1}} \right] dt. \quad (2.4)$$

Since $h(z) \in P(p, \alpha)$, we have

$$\operatorname{Re} \left[\frac{h'(tz)}{(tz)^{p-1}} \right] > \frac{p - (p-2\alpha)t}{1+t} \quad (0 < t \leq 1; z \in \mathbb{U}) \quad (2.5)$$

and hence (2.4) yields

$$\begin{aligned} \operatorname{Re} \left[\frac{H'(z)}{z^{p-1}} \right] &> (p+c) \int_0^1 t^{p+c-1} \frac{p - (p-2\alpha)t}{1+t} dt \\ &= p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c}. \end{aligned} \quad (2.6)$$

Further, to show that the result is sharp, we consider the function

$$f_0(z) = z^p + \sum_{n=1}^{\infty} \frac{2(p-\alpha)(p+1)^{\beta}}{(p+n)(p+n+1)^{\beta}} (-1)^n z^{p+n}, \quad (2.7)$$

which belongs to the class $P(p, \alpha, \beta)$. Since

$$\begin{aligned} H_0(z) &= (p+1)^{\beta} D^{\beta}(J_{p,c}(f_0(z))) \\ &= z^p + 2(p-\alpha)(p+c) \sum_{n=1}^{\infty} \frac{(-1)^n}{(p+n)(p+n+c)} z^{p+n} \quad (c > -p), \end{aligned}$$

one can easily show that $J_{p,c}(f_0(z)) \in P(p, \mu, \beta)$, but $J_{p,c}(f_0(z)) \notin P(p, \mu', \beta)$ if $\mu' > \mu$. This completes the proof of Theorem 1.

REMARK I. For $0 \leq \alpha < p$ and $c > -p$, we have from the right-hand side of (2.6) that $\alpha \leq \mu < p$, and hence $P(p, \mu, \beta) \subset P(p, \alpha, \beta)$ ($\beta \geq 0$).

COROLLARY I. If $f(z) \in P(p, \alpha, \beta)$, then $F_m(f(z))$ defined by (1.5) belongs to the class $P(p, \mu_m, \beta)$, where

$$\mu_j = p + 2(p - \mu_{j-1})(p + c_j) \sum_{n=1}^{\infty} \frac{(-1)^n}{p+n+c_j} \quad (j = 1, 2, 3, \dots, m)$$

and $\mu_0 = \alpha$. The result is sharp.

Next, we derive

THEOREM 2. If $f(z)$ is in the class $P(p, \alpha, \beta)$, then $J_{p,1}^\lambda(f(z))$ defined by (1.10) belongs to the class $P(p, \gamma, \beta)$, where

$$\gamma = p + 2(p - \alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+1}{p+n+1} \right)^\lambda.$$

The result is sharp.

PROOF. Making use of (1.7) and (1.8), the definition (1.10) yields

$$\begin{aligned} (p+1)^{-\beta} D^\beta (J_{p,1}^\lambda(f(z))) &= J_{p,1}^\lambda((p+1)^{-\beta} D^\beta(f(z))) \\ &= \frac{(p+1)^\lambda}{\Gamma(\lambda)} \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} (p+1)^{-\beta} D^\beta(f(tz)) dt \quad (\lambda > 0; \beta \geq 0) \end{aligned} \quad (2.8)$$

Therefore, setting

$$G(z) = (p+1)^{-\beta} D^\beta (J_{p,1}^\lambda(f(z))) \quad \text{and} \quad h(z) = (p+1)^{-\beta} D^\beta(f(z)), \quad (2.9)$$

we have to show that

$$\operatorname{Re} \left(\frac{G'(z)}{z^{p-1}} \right) > \gamma \quad (z \in U) \quad (2.10)$$

whenever $h(z) \in P(p, \alpha)$. Applying (2.5), we obtain

$$\begin{aligned} \operatorname{Re} \left(\frac{G'(z)}{z^{p-1}} \right) &= \frac{(p+1)^\lambda}{\Gamma(\lambda)} \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} t^p \operatorname{Re} \left(\frac{h'(tz)}{(tz)^{p-1}} \right) dt \\ &> \frac{(p+1)^\lambda}{\Gamma(\lambda)} \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} t^p \frac{p - (p-2\alpha)t}{1+t} dt \end{aligned}$$

$$= p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+1}{p+n+1} \right)^{\lambda}. \quad (2.11)$$

To show that the result is sharp, we take the function $f_0(z)$ given by (2.7). Since

$$\begin{aligned} G_0(z) &= (p+1)^{-\beta} D^{\beta} (J_{p,1}^{\lambda} (f_0(z))) \\ &= z^p + 2(p-\alpha) \sum_{n=1}^{\infty} \left(\frac{p+1}{p+n+1} \right)^{\lambda} \frac{(-1)^n}{p+n} z^{p+n} \end{aligned} \quad (2.12)$$

with $0 \leq \alpha < p$, $\lambda > 0$ and $\beta \geq 0$, we see that $J_{p,1}^{\lambda} (f_0(z)) \in P(p, \gamma, \beta)$, but $J_{p,1}^{\lambda} (f_0(z)) \notin P(p, \gamma', \beta)$ if $\gamma' > \gamma$. Thus we complete the proof of the theorem.

REMARK 2. For $0 \leq \alpha < p$ and $\lambda > 0$, we have from the right-hand side of (2.11) that $\alpha \leq \gamma < p$, and hence $P(p, \gamma, \beta) \subset P(p, \alpha, \beta)$ ($\beta \geq 0$).

COROLLARY 2. If $0 \leq \alpha < p$ and $0 \leq \beta < \rho$, then $P(p, \alpha, \rho) \subset P(p, \gamma_0, \beta)$, where

$$\gamma_0 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \left(\frac{p+n}{p+n+1} \right)^{\rho-\beta}.$$

The result is sharp.

PROOF. Setting $\lambda = \rho - \beta > 0$ in Theorem 2, we observe that

$$\begin{aligned} f(z) \in P(p, \alpha, \rho) &\implies J_{p,1}^{\rho-\beta} (f(z)) \in P(p, \gamma_0, \rho) \\ &\iff (p+1)^{-\rho} D^{\rho} (J_{p,1}^{\rho-\beta} (f(z))) \in P(p, \gamma_0) \\ &\iff (p+1)^{-\beta} D^{\beta} (f(z)) \in P(p, \gamma_0) \\ &\iff f(z) \in P(p, \gamma_0, \beta). \end{aligned} \quad (2.13)$$

COROLLARY 3. If $0 \leq \alpha < p$ and $\beta \geq 0$, then $P(p, \alpha, \beta+1) \subset P(p, \gamma_1, \beta)$, where

$$\gamma_1 = p + 2(p-\alpha) \sum_{n=1}^{\infty} (-1)^n \frac{p+1}{p+n+1}.$$

The result is sharp.

PROOF, Putting $\lambda = 1$ in Theorem 2, we have

$$\begin{aligned} f(z) \in P(p, \alpha, \beta+1) &\implies J_{p,1}^1(f(z)) \in P(p, \gamma_1, \beta+1) \\ &\iff (p+1)^{-\beta-1} D^{\beta+1}(J_{p,1}^1(f(z))) \in P(p, \gamma_1) \\ &\iff (p+1)^{-\beta} D^{\beta}(f(z)) \in P(p, \gamma_1) \\ &\iff f(z) \in P(p, \gamma_1, \beta). \end{aligned}$$

REMARK 3, We note that the several cases of Theorem 1 and Theorem 2, for the special values of p , α , β , c and λ , will improve some known results.

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