

A survey on unitary representation of braid groups

By

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Abstract

The Long-Moody construction is a method for constructing representations of the braid groups B_n from representations of the semidirect product of free group F_n and braid group, $F_n \rtimes B_n$ [33]. It is well-established that significant representations of braid groups, such as the Burau representation [9], can be obtained by the method. In our previous work [19], we extended the method by analytical techniques. The Katz-Long-Moody construction is defined as an endofunctor on the category of representations of $F_n \rtimes B_n$. This construction facilitates the acquisition of an infinite sequence of representations of braid groups by combining with the Long-Moody construction.

Here we focus on the following open problem regarding the Long-Moody construction: Can every unitary representation of a braid group be constructed via the Long-Moody construction? To shed light on the problem, this article gives an overview of the (unitary) representation of B_n , followed by an outline of the Katz-Long-Moody construction.

§ 1. Representation of Braid Group

Braid group representations play a central role in modern mathematics, with far-reaching applications in topology, representation theory, and mathematical physics. This profound significance arises from the fact that braid groups can be defined from multiple perspectives, notably through both algebraic and topological frameworks.

In 1925, Artin defined the braid group B_n as the group of isotopy classes of structures formed by the intertwining of n strings [1].

Definition 1.1 (The Artin braid group B_n). The Artin braid group B_n is the group generated by $n - 1$ generators $\sigma_1, \dots, \sigma_{n-1}$ and two braid relations.

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1. $\sigma_i \sigma_j = \sigma_j \sigma_i \quad (|j - i| > 1)$
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (i = 1, \dots, n-2)$

The braid relations represent the correspondences between the structures of geometric braids that are transformed into each other through isotopy deformations.

Birman provided a unified perspective on braid groups, mapping class groups, and knot theory [5]. The braid group can also be defined as the mapping class group of an n -punctured closed disk. The generators of this group are half-twists, which are self-homeomorphisms that exchange two points.

Definition 1.2 (Mapping class group). Let D_n denote a closed disk with n punctures, i.e., the closed disk D with n interior points, $p_1, \dots, p_n$, removed. The braid group is defined as the mapping class group of D_n that fixes the boundary ∂D :

$$B_n \cong \text{MCG}(D_n) = \left\{ f : D_n \rightarrow D_n \mid f \text{ is a homeomorphism and } f|_{\partial D} = \text{id} \right\} / \sim,$$

where $\sim$ denotes the equivalence relation by isotopy.

Definition 1.3 (half-twist). The standard generators of $\text{MCG}(D_n)$ are given by the *half-twists*, which exchange adjacent punctures. More precisely, for each $i = 1, \dots, n-1$, define the half-twist $h_i : D_n \rightarrow D_n$ as follows:

- In a small neighborhood containing the adjacent punctures p_i and p_{i+1} , the map h_i rotates the neighborhood by an angle of π , thereby exchanging p_i and p_{i+1} .
- This map is extended continuously to the entire D_n while fixing the boundary ∂D .

The half-twists $\{h_1, h_2, \dots, h_{n-1}\}$ generate $\text{MCG}(D_n)$ and satisfy the braid relations.

The pure braid group P_n is a particularly important subgroup which was defined as the kernel of the canonical projection from B_n to symmetric group S_n .

Definition 1.4 (Pure braid group). Let S_n be a symmetric group of rank n . The pure braid group P_n is defined as the kernel of the surjective homomorphism

$$\Pi : B_n \rightarrow S_n, \quad \sigma_i \mapsto (i, i+1),$$

which sends each standard Artin generator σ_i of the braid group B_n to the adjacent transposition $(i, i+1) \in S_n$. A set of generators for P_n is given by

$$\sigma_{ij} := \sigma_i \cdots \sigma_{j-2} (\sigma_{j-1})^2 (\sigma_i \cdots \sigma_{j-2})^{-1}, \quad \text{for } 1 \leq i < j \leq n.$$

For detail, see [24].

In this section, we will introduce previous studies on representations of the braid group that are of importance, focusing mainly on the following three properties: **faithfulness**, **irreducibility**, and **unitarity**. Here we define the three properties as follows.

Let G be a group and let $\rho : G \rightarrow \text{GL}(V)$ be a linear representation of G .

Definition 1.5 (Faithful Representation). A representation ρ is called **faithful** if $\ker(\rho) = \{e\}$, where e is the identity element of G . Equivalently, ρ is injective.

Definition 1.6 (Irreducible Representation). The representation ρ is said to be *irreducible* if V has no proper nontrivial G -invariant subspaces. That is, if $W \subseteq V$ is a subspace such that

$$\rho(g)W \subseteq W \quad \text{for all } g \in G,$$

then W must be either $W = \{0\}$ or $W = V$. Equivalently, ρ is irreducible if there is no nontrivial decomposition of V into a direct sum of G -invariant subspaces.

Definition 1.7 (Unitary Representation relative to H). Let V be a Hilbert space and the Hermitian transpose of a matrix A is denoted by $A^\dagger$. A representation ρ is said to be unitary relative to H if there exists a positive definite Hermitian matrix H such that

$$\rho(g)^\dagger H \rho(g) = H \quad \text{for any } g \in C^N.$$

More generally, for (p, q) -Hermitian matrix $H_{p,q}$, we also define the unitarity relative to $H_{p,q}$ in the similar way.

In particular, the focus is on unitary representations, which plays a crucial role in knot invariants, quantum computing, and other related fields. Unitary representations serve as a bridge between algebra and physics, finding applications in exchange statistics of particles in two-dimensional conformal field theory or knot invariants. For instance, they yield powerful knot invariants, such as the Jones polynomial [22, 23], which have significant applications in 3-manifold theory and TQFTs. A key question proposed in [6] is whether all unitary braid group representations can be obtained via the Long–Moody construction [33, 4], which we will explain in the next section 2.

§ 1.1. Burau Representation

One of the earliest and most studied representations is the Burau representation. Burau constructed one of the first linear representations of these groups in 1936 [9], a representation based on the L-polynomial [43] (or the Alexander polynomial), which is an invariant of the links. The Burau representation is a $(n - 1)$ -dimensional representation of B_n over the ring $\mathbb{Z}[t, t^{-1}]$, often given in “reduced” form of dimension $n - 1$ (or an unreduced form of dimension n).

Definition 1.8 (Braid representation). Let $n > 1$ and let B_n be the braid group generated by the elementary braids $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, satisfying the standard braid relations. The *Braid representation* is a $(n-1)$ -dimensional representation of B_n over the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$, defined by:

$$\rho : B_n \rightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$$

where the generators σ_i are mapped to the following $(n-1) \times (n-1)$ matrices:

$$\rho(\sigma_i) = I_{i-1} \oplus \begin{bmatrix} 0 & t \\ 1 & 1-t \end{bmatrix} \oplus I_{n-i-1},$$

where I_k denotes the $k \times k$ identity matrix.

In considering B_n as $\mathrm{MCG}(D_n)$, we obtain a homological representation, a braid action on the homology of topological spaces obtained from punctured disks. Indeed, the Braid representation is shown to be equivalent to a specific homological representation and its faithfulness is demonstrated through this method.

For small n , the Braid representation is faithful – it is known to be faithful for $n = 2, 3$. It was shown that the Braid representation is not faithful for $n \geq 5$ (see [36, 34, 2]). Unitarity of the Braid representation was proved by Squier [41].

§ 1.2. Gassner Representation

The Gassner representation is a multi-variable generalization of Braid's representation, defined for the pure braid group P_n . Introduced by Gassner in 1961 [16], it is given on a free module of rank $n-1$ with one independent Laurent polynomial variable for each of the $n-1$ gaps between braid strands.

Definition 1.9 (Gassner representation). The *Gassner representation* is a $(n-1)$ -dimensional representation of P_n over the ring $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$, defined by:

$$\rho : P_n \rightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}])$$

where the generators $\sigma_{i,j}$ of P_n are mapped to matrices with entries in the Laurent polynomial ring. The standard form of the Gassner representation is given by:

$$\rho(\sigma_{i,j}) = I_{j-2} \oplus \begin{bmatrix} 0 & t_j \\ 1 & 1-t_j \end{bmatrix} \oplus I_{n-j-1}$$

The question “Is the Gassner representation of P_n faithful for $n \geq 4$?” still remains open. In preprint [11], Cosgrove showed that the Gassner matrices for the generators of

the free subgroup are pairwise free. Unitarity of the Gassner representation was proved by Long [32].

§ 1.3. Lawrence–Krammer–Bigelow Representation

Jones introduced new representations related to knot invariants [21]. This period saw the emergence of representations via the Hecke algebra, which not only led to the discovery of the Jones polynomial, but also provided new insights into the structure of braid groups.

Birman–Murakami–Wenzl algebra is a two-parameter generalization of the Hecke algebra introduced by Birman, Murakami, and Wenzl in [7, 37] that corresponds to the Kauffman polynomial for knots. This yields the Birman–Murakami–Wenzl representation of B_n .

The Lawrence–Krammer–Bigelow representation is a landmark representation of B_n as a homological representations of the two-parameter Hecke algebra. Zinno constructed a bridge parametrized by Young diagrams between the Lawrence–Krammer–Bigelow representation and the Birman–Murakami–Wenzl algebra. As the two representations are thus the same, we can prove the irreducibility of one and the faithfulness of the other [46].

Here we show the definition by Paoluzzi and Paris [3].

Definition 1.10 (Lawrence–Krammer–Bigelow Representation).

Consider the ring $R = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of Laurent polynomials in two variables and the (abstract) free R -module:

$$V = \bigoplus_{1 \leq i < j \leq n} R e_{ij}$$

For $k \in \{1, \dots, n-1\}$ define the R -homomorphism $\rho_k: V \rightarrow V$ by

$$\rho_k(e_{ij}) = \begin{cases} x e_{i-1j} + (1-x)e_{ij} & \text{if } k = i-1 \\ e_{i+1j} - xy(x-1)e_{kk+1} & \text{if } k = i < j-1 \\ -x^2 y e_{kk+1} & \text{if } k = i = j-1 \\ e_{ij} - y(x-1)^2 e_{kk+1} & \text{if } i < k < j-1 \\ e_{ij-1} - xy(x-1)e_{kk+1} & \text{if } i < j-1 = k \\ x e_{ij+1} + (1-x)e_{ij} & \text{if } k = j \\ e_{ij} & \text{otherwise.} \end{cases}$$

The Lawrence–Krammer representation is faithful for all n [2, 30] and is unitary for suitable parameters q and t [8].

One of the significance of Lawrence’s result, as stated in [31], lies in its relation to monodromy representation of a vector bundle with a flat connection. A notable finding

is the establishment of a connection between these representations and the Hecke algebra representations from the field of conformal field theory, particularly the contributions of Tsuchiya and Kanie [44] or Kohno [27].

In [35], Marin defined a generalized the Lawrence-Krammer-Bigelow representation for any finite type Artin group and any braid group associated with a complex reflection group, as monodromy of certain flat connection.

§ 1.4. Monodromy Representation

Monodromy representation is a (anti-)representation of the fundamental group of the domain of a differential equation. A linear transformation of the fundamental solution matrix is determined by analytic continuation along the paths corresponding to the generators of the fundamental group of the domain. The seminal research on the monodromy representation of the Knizhnik-Zamolodchikov(KZ) equation [26], a representation of the (pure) braid group, is exemplified by [15, 28, 44]. It is also known that the KZ equation can be used to obtain the Kontsevich invariant of links [29]. However, in this particular investigation, a comprehensive exploration of these concepts will not be undertaken. Instead, an explanation will be conducted to establish a relationship with the Long-Moody construction, as outlined below, with a focus on the KZ-type equation, wherein the coefficient matrices of the differential equation are complex constant matrices. This is because, in our next paper [38], we will show the correspondence between the Katz-Long-Moody construction, which is an extension of the Long-Moody construction, and the middle convolution of KZ-type equations.

Definition 1.11 (KZ-type equation).

Let $n > 1$, and N be positive integers and assume that $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

KZ type equation is a linear partial differential equations

$$\frac{\partial u}{\partial z_i} = \sum_{i \neq j} \frac{A_{ij}}{z_i - z_j} u, \quad i = 1, \dots, n$$

where $A_{i,j}$ are $N \times N$ complex constant matrices.

Besides, we assume the following integrability condition;

$$\begin{aligned} A_{ij} &= A_{ji} & 1 \leq i < j \leq n \\ [A_{i,j}, A_{k,l}] &= O & \{i, j\} \cap \{k, l\} = \emptyset \\ [A_{i,j}, A_{i,k} + A_{j,k}] &= O & i \neq j, i \neq k, j \neq k \end{aligned}$$

As the fundamental group of the domain is isomorphic to P_n , monodromy representation of the n -variable KZ-type equation is a (anti-)representation of P_n . Constantinescu gave representations of the braid group as monodromy representations of a

hypergeometric function, defined monodromy invariant Hermitian matrices, and showed the unitarity of the braid representations [10]. The result also includes the case of the Burau and Gassner representations and includes applications to two-dimensional conformal field theory, specifically, applications to the Coulomb gas representation. In the next section, this article focuses on middle convolution as an important research area in this field. Middle convolution constitutes a method of applying the Riemann-Liouville transformation and gauge transformation to a solution of Fuchsian system, thereby yielding another Fuchsian system [13, 14, 25]. Haraoka extended the result to obtain n -variable KZ-type equations [17, 18], which is called multiplicative middle convolution. Haraoka's multiplicative middle convolution is the method to construct irreducible representation of P_n .

§ 2. Long-Moody Construction

The Long-Moody construction is a method for constructing representations of the braid groups B_n from representations of the semidirect product of free group F_n and braid group, $F_n \rtimes B_n$ [33]. $F_n \rtimes B_n$ is defined by the Artin representation. The Artin representation is a homomorphism $\theta : B_n \rightarrow \text{Aut}(F_n)$ induced by the left action of B_n on the fundamental group $\pi_1(D_n) = F_n$, corresponding to half-twists on D_n .

Definition 2.1 (Artin representation). Let x_j ($j = 1, \dots, n$) be the generators of F_n , and σ_i ($j = 1, \dots, n$) be the generators of B_n . The Artin representation θ is defined by:

$$\begin{array}{ccc} \theta: B_n & \longrightarrow & \text{Aut}(F_n) \\ \Psi & & \Psi \\ \sigma_i & \longmapsto & \theta_{\sigma_i} \end{array}$$

$$\theta_{\sigma_i}(x_j) := \begin{cases} x_{i+1} & j = i \\ x_{i+1}^{-1} x_i x_{i+1} & j = i + 1 \\ x_j & j \neq i, i + 1 \end{cases}.$$

Definition 2.2 (Semidirect product $F_n \rtimes_{\theta} B_n$). Let B_n and F_n be two groups, and let $\theta : B_n \rightarrow \text{Aut}(F_n)$ be a homomorphism, where $\text{Aut}(F_n)$ is the group of automorphisms of F_n . The (outer) semidirect product of B_n and F_n with respect to θ is the group denoted by $F_n \rtimes_{\theta} B_n$, defined as follows:

1. The underlying set of $F_n \rtimes_{\theta} B_n$ is the Cartesian product $F_n \times B_n$.
2. The product is given by:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot \theta_{g_1}(h_2), g_1 \cdot g_2),$$

where $h_1, h_2 \in F_n$, $g_1, g_2 \in B_n$, and $\theta_{g_1}(h_2)$ denotes the action of g_1 on h_2 via θ .

Wada further generalized the Artin representation, thereby yielding group invariants of knots [45]. Moreover, the generalized representation, Wada representation, was classified by Ito [20].

The Long-Moody construction is defined as follows. Hereafter, we denote the field by $\mathbf{k}$. We write $F_n \rtimes_{\theta} B_n$ as $F_n \rtimes B_n$ for short.

Definition 2.3 (Long-Moody construction). Let V be a finite dimensional $\mathbf{k}$ -vector space. Let x_j ($j = 1, \dots, n$) be the generators of F_n , and σ_i ($j = 1, \dots, n$) be the generators of B_n . For any representation $\rho: F_n \rtimes B_n \longrightarrow \mathrm{GL}(V)$, we have a representation

$$\rho^{\mathrm{LM}}: B_n \longrightarrow \mathrm{GL}(V^{\oplus n})$$

given by

$$\rho^{\mathrm{LM}}(\sigma_i) := \begin{pmatrix} s_i & & \\ & \ddots & \\ & & s_i \end{pmatrix} \cdot \begin{pmatrix} I_{N(i-1)} & & \\ & R_i & \\ & & I_{N(n-i-1)} \end{pmatrix},$$

$$R_i := \begin{pmatrix} 0 & g_i \\ I_N & I_N - g_{i+1} \end{pmatrix},$$

where $g_i := \rho(x_i)$ and $s_i := \rho(\sigma_i)$.

Bigelow generalized the method and proved that the method is also applicable to any subgroup of B_n in [4].

Theorem 2.4 (generalized Long-Moody construction).

Let V be a finite-dimensional $\mathbf{k}$ -vector space, and let B be any subgroup of B_n . Let $\sigma_1, \dots, \sigma_{n-1}$ be the generators of B_n . For

$$\rho: F_n \rtimes B \longrightarrow \mathrm{GL}(V),$$

we obtain a homomorphism of B

$$\rho^{\mathrm{LM}}|_B: B \longrightarrow \mathrm{GL}(V^{\oplus n}).$$

The homomorphism is called Long-Moody construction of a subgroup B .

The Long-Moody construction has many applications to knot invariants and homological stability [39, 40]. Takano described the Long-Moody construction using Fox derivatives and constructed the twisted Alexander invariants of knots [42]. Soulé extended the Long-Moody construction to the other groups or Wada representation [39, 40]. It is observed that the unitary representations such that Burau representation, Gassner representation, Jones representations, etc., are obtained through the

Long-Moody construction. Long proved that the Long-Moody construction preserves unitarity [33] by the result obtained by Deligne-Mostow [12] .

Definition 2.5. If ρ is unitary and $\rho(\sigma_i)$ has no eigenvalue ± 1 , then so is ρ^{LM} .

§ 3. Katz-Long-Moody Construction

In our previous work [19], we extended the Long-Moody construction by unifying the analytical technique based on Katz's theory on local systems [25]. It is imperative to note that middle convolution is a method employed to derive specific differential equations from a given differential equation. Moreover, given that monodromy representations can be obtained from differential equations, middle convolution can be regarded as a technique for constructing monodromy representations from monodromy representations. To address this, we propose a method for obtaining new representations while preserving the information of F_n by using the convolution approach of Dettweiler and Reiter [13].

Definition 3.1 (Dettweiler-Reiter's convolution). Let V be an N -dimensional linear space over $\mathbf{k}$. Let $\lambda \in \mathbf{k}^\times$, and let x_j be the generators of F_n . Let $\lambda \in \mathbf{k}^\times$, and let x_i be the generators of F_n . For any $\rho: F_n \longrightarrow \mathrm{GL}(V)$,

$$\rho_\lambda^{DR}: F_n \longrightarrow \mathrm{GL}(V)$$

$$\rho_\lambda^{DR}(x_i) := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \lambda(\rho(x_1) - 1) \cdots \lambda(\rho(x_{i-1}) - 1) \lambda \rho(x_i) \rho(x_{i+1}) - 1 \cdots \rho(x_n) - 1 & & & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Here, we denote 1 as I_N .

Using Dettweiler-Reiter's convolution, the twisted Long-Moody construction is defined as follows.

Definition 3.2 (Twisted Long-Moody construction). Let V be an N -dimensional vector space over the field $\mathbf{k}$. Let the generators of F_n and B_n be $x_j, \sigma_i, j = 1, \dots, n, i = 1, \dots, n-1$. Besides, let B be any subgroup of B_n , and $\lambda \in \mathbf{k}^\times$. Here, for

$$\rho: F_n \rtimes B \longrightarrow \mathrm{GL}(V),$$

we can construct a representation

$$\rho_\lambda^{LM} : F_n \rtimes B \longrightarrow \mathrm{GL}(V^{\oplus n})$$

such that $\rho_\lambda^{LM}(x_j) := \rho_\lambda^{DR}(x_j)$ and $\rho_\lambda^{LM}(\sigma_i) := \rho^{LM}(\sigma_i)$, for the generators of F_n and B_n , as x_j and σ_i , $j = 1, \dots, n$, $i = 1, \dots, n-1$.

To confirm that ρ_λ^{LM} becomes a representation, it is sufficient to check that $\rho_\lambda^{LM}(x_j)$ and $\rho_\lambda^{LM}(\sigma_i)$ satisfies the Artin representation-determined relations and braid relations.

Finally, we define the Katz-Long-Moody construction as the action on the following quotient space. Define subspaces of V , K and L , as follows. For $2 \leq j \leq n$,

$$K := \left\{ \begin{pmatrix} w_2 \\ \vdots \\ w_n \end{pmatrix} ; w_j \in \mathrm{Ker}(\rho(x_j) - I_N) \right\},$$

$$L := \bigcap_{j=2}^n \mathrm{Ker}(\rho_\lambda^{LM}(x_j) - I_{N(n-1)}).$$

Then, $K + L$ is ρ_λ^{LM} -invariant. For the proof, see [19].

Definition 3.3 (the Katz-Long-Moody construction).

We define the Katz-Long-Moody construction ρ_λ^{KLM} as the action of ρ_λ^{LM} on $V/(K + L)$.

In our forthcoming paper [38], we established a correspondence between the Katz-Long

-Moody construction and the multiplicative middle convolution for n -variable KZ-type equations and proved that the Katz-Long-Moody construction preserves unitarity:

If ρ is unitary relative to $H_{p,q}$, then ρ_λ^{KLM} also is unitary relative to a Hermitian matrix $\tilde{H}_{\tilde{p},\tilde{q}}$.

In this paper, we focused on the Long-Moody construction, a method for constructing various well-known representations of braid groups, and give an overview of the Katz-Long-Moody construction, which is an extension of the Long-Moody construction. We also give a brief survey of braid group representations to provide a clue to the open problem of the unitarity of the Long-Moody construction and braid group representations. As previous research on the expression of braided groups spans a wide range of fields, some of it could not be fully introduced here. Kindly refer to the following documents as appropriate: [6, 24].

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