

ON LINEAR DIFFERENTIAL GAMES

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1. Statement of the Problems

This paper treats a certain class of linear differential games, the state of which is specified by a state vector in an n -dimensional Euclidean space R^n . It is assumed that the state vector $z(t)$ in R^n can be expressed as

$$z(t) = z_1(t) + \int_0^t F(t,s)u(s)ds + \int_0^t W(t,s)v(s)ds, \quad (1)$$

where $u(t)$ and $v(t)$ are r -vectors called strategies or controls of the first and the second players, respectively, $F(t,s)$ and $W(t,s)$ are $n \times r$ matrix functions which are assumed to be continuous, and $z_1(t)$ is a known n -dimensional vector function. It is assumed that the strategies $u(t)$ and $v(t)$ of the first and second players are measurable and constrained as

$$u(t) \in U, \quad v(t) \in V, \quad (2)$$

where U and V are certain sets in an r -dimensional Euclidean space. Such strategies $u(t)$ and $v(t)$ which satisfy above mentioned conditions are called admissible strategies. In this paper, we treat the case where the sets U and V are unit cube, respectively, i.e.,

$$U = V = \Omega = \left\{ u; |u_i| \leq 1 \quad (i = 1, \dots, r) \right\}. \quad (3)$$

We consider two kinds of differential game problems stated as follows:

Problem 1. Determine a saddle point for

$$J(u, v) = (z(T), z(T)) + \int_0^T \{ (u(t), Cu(t)) - (v(t), Dv(t)) \} dt, \quad (4)$$

subject to the constraints (1) and (2), where $(\cdot, \cdot)$ denotes the inner product in some finite-dimensional Euclidean space, C and D are $n \times n$ positive semi-definite diagonal matrices with nonnegative constant elements, and T is a fixed time. $J(u, v)$ is called a payoff. A saddle point is defined as the pair $u^0(t), v^0(t)$ satisfying the relation

$$J(u^0, v) \leq J(u^0, v^0) \leq J(u, v^0) \quad (5)$$

for arbitrary admissible strategies u and v . Namely, the first player is to select a strategy that minimizes the payoff and the second player is to select a strategy that maximizes the payoff. If (5) can be realized, u^0 and v^0 are called the optimal strategies and $J(u^0, v^0)$ is called the value of the game.

Problem 2. Let $T_{u,v}$ be a time corresponding to the strategies $u(t) \in U, v(t) \in V$ and satisfying $z(T_{u,v}) = 0$. Determine a time T_{u^0, v^0} and a pair of admissible strategies $u^0(t), v^0(t)$ such that

$$z(T_{u^0, v^0}) = 0, \quad T_{u^0, v^0} = \min_{u \in U} \max_{v \in V} T_{u, v}. \quad (6)$$

The second equation of (6) is equivalent to

$$T_{u^0, v} \leq T_{u^0, v^0} \leq T_{u, v^0}.$$

Such differential game problems have been treated by several authors [1] - [7]. In this paper, we consider the problems as an infinite-dimensional nonlinear programming problem [8]. By using the generalized Kuhn-Tucker theorem [9] in nonlinear programming, we derive a system of transcendental equations, the solution of which directly yields the optimal strategies.

We can mention two kinds of games describable in the form (1). One is a pursuit-evasion game governed by

$$\left. \begin{aligned} \dot{x}_p / dt &= A_p(t)x_p + B_p(t)u(t) + h_p(t), & x_p(0) &= x_{p0}, \\ \dot{x}_e / dt &= A_e(t)x_e + B_e(t)v(t) + h_e(t), & x_e(0) &= x_{e0}, \end{aligned} \right\} \quad (7)$$

where x_p is an n -vector representing the state of the pursuer, $u(t) \in U$ is an r -vector representing the control of the pursuer, $A_p(t)$, $B_p(t)$, and $h_p(t)$ are $n \times n$, $n \times r$, and $n \times 1$ known matrices, respectively, continuous in t , and identical statements apply to the evader and x_e , $v(t) \in V$, $A_e(t)$, $B_e(t)$, and $h_e(t)$. x_{p0} and x_{e0} are initial values. The state of the game is defined by

$$z(t) = Q \{ x_p(t) - x_e(t) \}, \quad (8)$$

where Q is an $n \times n$ positive semidefinite constant matrix. Then, the $z_1(t)$, $F(t, s)$, and $W(t, s)$ in (1) are given by

$$\left. \begin{aligned} z_1(t) &= Q \left\{ X_p(t) \left(x_{p0} + \int_0^t X_p^{-1}(s) h_p(s) ds \right) - X_e(t) \left(x_{e0} + \int_0^t X_e^{-1}(s) h_e(s) ds \right) \right\}, \\ F(t, s) &= Q X_p(t) X_p^{-1}(s) B_p(s), \\ W(t, s) &= -Q X_e(t) X_e^{-1}(s) B_e(s), \end{aligned} \right\} \quad (9)$$

where $X_p(t)$ and $X_e(t)$ are $n \times n$ matrix functions satisfying

$$\left. \begin{aligned} dX_p(t) / dt &= A_p(t)X_p(t), & X_p(0) &= I \text{ (the identity)}, \\ dX_e(t) / dt &= A_e(t)X_e(t), & X_e(0) &= I. \end{aligned} \right\}$$

The other game describable in the form (1) is a control system subject to unpredictable disturbances. The state of the control system is governed by

$$dz/dt = A(t)z + B_1(t)u(t) + B_2(t)v(t) + h(t), \quad z(0) = z_0, \quad (10)$$

where $z(t)$ is an n -vector describing the state of the system, $u(t) \in U$ is an

r -vector representing the control, $v(t)$ is an r -vector representing unpredictable disturbance functions, $A(t)$, $B_1(t)$, $B_2(t)$, and $h(t)$ are $n \times n$, $n \times r$, $n \times r$, and $n \times 1$ known matrices, respectively, which are continuous in t . The known information concerning $v(t)$ is only the fact that $v(t) \in V$. In this case, the $z_1(t)$, $F(t, s)$, and $W(t, s)$ in (1) are given by

$$\left. \begin{aligned} z_1(t) &= X(t) \left(z_0 + \int_0^t \bar{X}^{-1}(s) h(s) ds \right), \\ F(t, s) &= X(t) \bar{X}^{-1}(s) B_1(s), \\ W(t, s) &= X(t) \bar{X}^{-1}(s) B_2(s), \end{aligned} \right\} \quad (11)$$

where z_0 is the initial value of $z(t)$, and $X(t)$ is an $n \times n$ matrix function satisfying

$$dX(t)/dt = A(t)X(t), \quad X(0) = I.$$

2. Fundamental Theorem and its Application to the Problem

Since the solution of the Problem 2 is obtained from the solution of the Problem 1, we consider the Problem 1 first. Let H be a real Hilbert space of r -dimensional functions square integrable over $[0, T]$. Then, the strategies $u(t)$ and $v(t)$ ($0 \leq t \leq T$) can be taken in H . Define the inner product of two vectors u^1 and u^2 in the Hilbert space H by

$$[u^1, u^2] = \int_0^T u^{1*}(t) u^2(t) dt = \int_0^T (u^1(t), u^2(t)) dt,$$

where $*$ denotes the transpose of a vector or a matrix. We define linear operators P and R , respectively, by

$$\left. \begin{aligned} Pu &= \int_0^T F(T, s) u(s) ds, \\ Rv &= \int_0^T W(T, s) v(s) ds, \end{aligned} \right\} \quad (12)$$

which map the Hilbert space H into the n -dimensional Euclidean space R^n . Then,

from (1) it follows that

$$z_T = z_1 + Pu + Rv, \quad (13)$$

where $z_T = z(T)$ and $z_1 = z_1(T)$. The payoff (4) is rewritten as

$$J(u, v) = (z_T, z_T) + (u, Cu) - (v, Dv). \quad (14)$$

By using (13), it follows that

$$\begin{aligned} (z_T, z_T) &= (Pu + Rv + z_1, Pu + Rv + z_1) \\ &= (Pu, Pu) + 2(Pu, Rv + z_1) + (Rv + z_1, Rv + z_1). \end{aligned} \quad (15)$$

Let P^* now be the adjoint operator of P , then P^* maps R^n into H and satisfies the relation

$$(x, Pu) = (P^*x, u),$$

where $x \in R^n$ and $u \in H$. From (12), it can be easily seen that [10]

$$P^*x = F^*(T, t)x, \quad (16)$$

where F^* denotes the transposed matrix of F . Then, (15) can be rewritten as

$$(z_T, z_T) = (u, P^*Pu) + 2(u, P^*(Rv + z_1)) + (Rv + z_1, Rv + z_1).$$

Therefore, the payoff (14) can be written as

$$\begin{aligned} J(u, v) &= (u, (P^*P + C)u) + 2(u, P^*(Rv + z_1)) + (Rv + z_1, Rv + z_1) - (v, Dv) \\ &= (v, (R^*R - D)v) + 2(v, R^*(Pu + z_1)) + (Pu + z_1, Pu + z_1) + (u, Cu). \end{aligned} \quad (17)$$

Since $(u, (P^*P + C)u) = (Pu, Pu) + (u, Cu) \geq 0$, it can be seen that

$J(u, v)$ is convex with respect to $u \in H$. But the convexity or concavity of $J(u, v)$ with respect to $v \in H$ cannot be asserted. Define a mapping g , which maps H_1 ($H_1 \subset H$) into H , by

$$g(u) = \begin{pmatrix} 1 - u_1^2(t) \\ 1 - u_2^2(t) \\ \vdots \\ 1 - u_n^2(t) \end{pmatrix}. \quad (18)$$

Then, the constraints $u(t), v(t) \in \Omega$ can be expressed as

$$g(u) \geq 0, \quad g(v) \geq 0. \quad (19)$$

Define a closed bounded convex subset X of H by

$$X = \{u \in H : g(u) \geq 0\}.$$

Since $J(u, v)$ is continuous and convex with respect to u on X , from [11, Theorem 2.1] there exists an element u^0 in X such that

$$\inf_{u \in X} J(u, v) = J(u^0, v).$$

Concerning an element v^0 in X which maximizes $J(u, v)$, the existence cannot be asserted. Therefore, we assume the existence of v^0 . In the following, necessary conditions for the optimal strategies will be derived.

First, we show a theorem which corresponds to a special case of the Theorem V.3.3.2 in [9] and is available for our problem.

Theorem 1. Let f be a real-valued differentiable functional on the Hilbert space H . Let $x^0 \in X$ maximize $f(x)$ subject to the constraint $g(x) \geq 0$. Then, there exists a $\lambda^0 \in H$ such that

$$\lambda^0 \geq 0, \quad (20)$$

and that the Lagrangian expression

$$\Phi(x, \lambda) = f(x) + [\lambda, g(x)] \quad (21)$$

satisfies the following relations:

$$\delta_x \Phi((x^0, \lambda^0); \xi) = 0 \quad \text{for all } \xi \in H, \quad (22)$$

$$[\lambda^0, g(x^0)] = 0, \quad (23)$$

where $\delta_x \Phi((x^0, \lambda^0); \xi)$ represents the partial Fréchet differential of Φ with respect to x at (x^0, λ^0) with increment ξ , which is defined by

$$\delta_x \bar{\Phi}((x^0, \lambda^0); \xi) = \lim_{\varepsilon \rightarrow 0} \frac{\Phi(x^0 + \varepsilon \xi, \lambda^0) - \Phi(x^0, \lambda^0)}{\varepsilon}. \quad (24)$$

Proof. Since the function g defined by (18) is regular [9] at every point of X , it follows from [9, Theorem V.3.3.1] that the relation

$$\left. \begin{aligned} \delta g(x^0; \xi) + g(x^0) &\geq 0 \\ -\delta f(x^0; \xi) &\geq 0. \end{aligned} \right\} \quad \text{implies} \quad (25)$$

Define a linear continuous transformation $A(\xi)$ on H into H by

$$A(\xi) = \delta g(x^0; \xi), \quad (26)$$

and a linear functional $\phi(\xi)$ on H by

$$\phi(\xi) = -\delta f(x^0; \xi). \quad (27)$$

Since A is bounded, it follows that a subset Y_A in H defined by

$$Y_A = \{A^*(\lambda) \in H: \lambda \in H, \lambda \geq 0\} \quad (28)$$

is regularly convex [9], where A^* is the adjoint transformation of A . Further,

$$A(\xi) + g(x^0) \geq 0 \quad \text{implies} \quad \phi(\xi) \geq 0.$$

Therefore, from [9, Corollary IV. 3], there exists a $\lambda^0 \geq 0$ ($\lambda^0 \in H$) such that

$$[\lambda^0, A(\xi)] = \phi(\xi) \quad \text{for all} \quad \xi \in H.$$

Furthermore,

$$[\lambda^0, g(x^0)] = 0.$$

Thus, the Theorem 1 is proved.

Now, we apply Theorem 1 to our problem. Let $u^0 \in X$ minimize $J(u, v)$ subject to the constraint $g(u) \geq 0$, then there exists a $\lambda^0 \in H$ such that

$$\lambda^0 \geq 0, \quad (29)$$

and that the Lagrangian expression

$$K(u, v, \lambda) = J(u, v) - [\lambda, g(u)] \quad (30)$$

satisfies the following relations:

$$\delta_u K((u^0, v, \lambda^0); \xi) = 0 \text{ for all } \xi \in H, \quad (31)$$

$$[\lambda^0, g(u^0)] = 0. \quad (32)$$

Analogously, let $v^0 \in X$ maximize $J(u, v)$ subject to the constraint $g(v) \geq 0$, then there exists a $\mu^0 \in H$ such that

$$\mu^0 \geq 0, \quad (33)$$

and that the Lagrangian expression

$$L(u, v, \mu) = J(u, v) + [\mu, g(v)] \quad (34)$$

satisfies the following relations:

$$\delta_v L((u, v^0, \mu^0); \xi) = 0 \text{ for all } \xi \in H, \quad (35)$$

$$[\mu^0, g(v^0)] = 0. \quad (36)$$

Henceforth, u^0 , v^0 , λ^0 , and μ^0 which satisfy these relations are simply written as u , v , λ , and μ , since no confusion may occur. In view of (17) and (24), the partial Frechet differential of $K(u, v, \lambda)$ with respect to u at (u, v, λ) with increment ξ can be evaluated as

$$\delta_u K((u, v, \lambda); \xi) = 2 [(P^*P + C)u, \xi] + 2 [P^*(Rv + z_1), \xi] - \left[\frac{\partial g(u)}{\partial u} \lambda, \xi \right],$$

where $\partial g(u)/\partial u$ denotes an $r \times r$ matrix defined by

$$\frac{\partial g(u)}{\partial u} = \left[\frac{\partial g_1(u)}{\partial u_j} \right] = -2 \begin{bmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & & \\ \vdots & & \ddots & \\ 0 & & & u_r \end{bmatrix}.$$

Hence, from (31) it follows that

$$(P^*P + C)u + P^*Rv + P^*z_1 - \frac{1}{2} \frac{\partial g(u)}{\partial u} \lambda = 0. \quad (37)$$

Define a new vector x in H by

$$x = Cu - \frac{1}{2} \frac{\partial g(u)}{\partial u} \lambda = \begin{bmatrix} c_1 u_1 + \lambda_1 u_1 \\ \vdots \\ c_r u_r + \lambda_r u_r \end{bmatrix}, \quad (38)$$

where c_i ($i=1, \dots, r$) are the elements of the diagonal matrix C and non-negative. Then, (37) is rewritten as

$$P^*(Pu + Rv + z_1) + x = 0. \quad (39)$$

In the same way, since

$$\delta_v L((u, v, \mu); \xi) = 2 \left[(R^*R - D)v, \xi \right] + 2 \left[R^*(Pu + z_1), \xi \right] + \left[\frac{\partial g(v)}{\partial v} \mu, \xi \right],$$

it follows from (35) that

$$(R^*R - D)v + R^*Pu + R^*z_1 + \frac{1}{2} \frac{\partial g(v)}{\partial v} \mu = 0. \quad (40)$$

Define a new vector y in H by

$$y = Dv - \frac{1}{2} \frac{\partial g(v)}{\partial v} \mu = \begin{bmatrix} d_1 v_1 + \mu_1 v_1 \\ \vdots \\ d_r v_r + \mu_r v_r \end{bmatrix}, \quad (41)$$

where d_i ($i=1, \dots, r$) are the elements of the diagonal matrix D and nonnegative.

Then, (40) is rewritten as

$$R^*(Rv + Pu + z_1) - y = 0. \quad (42)$$

Since $\lambda(t) \geq 0$ and $g(u(t)) \geq 0$ on the interval $[0, T]$, it follows from (32) that the equation

$$(\lambda(t), g(u(t))) = \lambda^*(t)g(u(t)) = 0$$

holds for almost all $t \in [0, T]$. Therefore,

$$\left. \begin{aligned} \lambda_1(t) &= 0 & \text{if } -1 < u_1(t) < 1, \\ \lambda_1(t) &\geq 0 & \text{if } |u_1(t)| = 1. \end{aligned} \right\}$$

Thus, the relation between $u_1(t)$ and $\lambda_1(t)$ can be shown as Fig. 1 (a). The relation between $u_1(t)$ and $\lambda_1(t)u_1(t)$ and then the relation between $u_1(t)$ and $c_1(t)u_1(t) + \lambda_1(t)u_1(t)$ can also be obtained successively from Fig. 1 (a) as shown in Fig. 1 (b) and (c), respectively. Hence, the relation between $x \in H$ and $u \in X$, which is defined by (38), can be expressed as

$$u(t) = \varphi(x(t)), \text{ or } u_i(t) = \varphi_i(x_i(t)) \quad (i=1, \dots, r), \quad (43)$$

where the nonlinear function φ_i is shown in Fig. 2 (a), which can be obtained from Fig. 1 (c) directly.

Analogously, the relation between $y \in H$ and $v \in X$, which is defined by (41), can be expressed as

$$v(t) = \psi(y(t)), \text{ or } v_i(t) = \psi_i(y_i(t)) \quad (i=1, \dots, r), \quad (44)$$

where the nonlinear function ψ_i is shown in Fig. 2 (b). If we use a notation such that

$$\left. \begin{aligned} \text{sat } \alpha &= \alpha & \text{if } |\alpha| \leq 1, \\ \text{sat } \alpha &= \text{sgn } \alpha & \text{if } |\alpha| \geq 1, \end{aligned} \right\}$$

then,

$$\varphi(x) = \begin{bmatrix} \text{sat}(x_1 / c_1) \\ \vdots \\ \text{sat}(x_r / c_r) \end{bmatrix}, \quad \psi(y) = \begin{bmatrix} \text{sat}(y_1 / d_1) \\ \vdots \\ \text{sat}(y_r / d_r) \end{bmatrix}.$$

Substituting (43) and (44) into (39) and (42) yields

$$\left. \begin{aligned} P^*(P\varphi(x) + R\psi(y) + z_1) + x &= 0, \\ R^*(P\varphi(x) + R\psi(y) + z_1) - y &= 0. \end{aligned} \right\} \quad (45)$$

Equation (45) is a system of nonlinear integral equations from which $x \in H$ and $y \in H$ can be determined. This system of nonlinear integral equations can be reduced to a system of transcendental equations as follows.

By setting

$$P\varphi(x) + R\psi(y) + z_1 = \alpha, \quad (46)$$

we obtain

$$x = -P^*\alpha, \quad y = R^*\alpha, \quad (47)$$

where α is a vector in R^n . In view of (13), it is clear that the α defined by (46) represents $z_T = z(T)$. Substituting (47) into (46) yields a transcendental equation:

$$-P\varphi(P^*\alpha) + R\psi(R^*\alpha) + z_1 = \alpha. \quad (48)$$

Let $f_i(T, t)$ and $w_i(T, t)$ ($i=1, \dots, r$) be n -dimensional column vectors of the matrices $F(T, t)$ and $W(T, t)$, respectively, i.e.,

$$\left. \begin{aligned} F(T, t) &= \begin{bmatrix} f_1(T, t) & f_2(T, t) & \dots & f_r(T, t) \end{bmatrix}, \\ W(T, t) &= \begin{bmatrix} w_1(T, t) & w_2(T, t) & \dots & w_r(T, t) \end{bmatrix}. \end{aligned} \right\}$$

Then, since

$$P^*\alpha = F^*(T, t)\alpha = \begin{bmatrix} f_1^*(T, t)\alpha \\ \vdots \\ f_r^*(T, t)\alpha \end{bmatrix}, \quad R^*\alpha = W^*(T, t)\alpha = \begin{bmatrix} w_1^*(T, t)\alpha \\ \vdots \\ w_r^*(T, t)\alpha \end{bmatrix},$$

(48) can be rewritten as

$$\alpha = \sum_{i=1}^r \int_0^T \left[w_i(T, t) \psi_i(w_i^*(T, t)\alpha) - f_i(T, t) \varphi_i(f_i^*(T, t)\alpha) \right] dt + z_1. \quad (49)$$

3. Solution of the Transcendental Equation

Let us define a mapping A by

$$A\alpha = \sum_{i=1}^r \int_0^T \left[w_i(T, t) \psi_i(w_i^*(T, t)\alpha) - f_i(T, t) \varphi_i(f_i^*(T, t)\alpha) \right] dt + z_1, \quad (50)$$

which maps R^n into R^n . Let $(A\alpha)_j$, f_{ij} , w_{ij} , and z_{ij} denote the j th component of n -vectors $A\alpha$, f_i , w_i , and z_1 , respectively. Then,

$$(A\alpha)_j = \sum_{i=1}^r \int_0^T \left[w_{ij}(T, t) \psi_i(w_i^*(T, t)\alpha) - f_{ij}(T, t) \varphi_i(f_i^*(T, t)\alpha) \right] dt + z_{ij}.$$

Since $|\varphi_i| \leq 1$, $|\psi_i| \leq 1$ ($i=1, \dots, r$), it follows that

$$|(A\alpha)_j| \leq \sum_{i=1}^r \int_0^T \left[|w_{ij}(T, t)| + |f_{ij}(T, t)| \right] dt + |z_{ij}| \quad (j=1, \dots, n). \quad (51)$$

Since the functions f_{ij} , w_{ij} ($i=1, \dots, r$; $j=1, \dots, n$) are assumed to be continuous on the closed interval $0 \leq t \leq T$, (51) shows that the mapping A maps a closed bounded convex subset of R^n into itself. Furthermore, the mapping is continuous. Therefore, by Brouwer's fixed-point theorem [12], we can conclude that there exists a point α such that $A\alpha = \alpha$. Namely,

Theorem 2. Let the vector functions f_i and w_i ($i=1, \dots, r$) be continuous on the closed interval $[0, T]$. Then, there exists a solution of (49).

The solution of (49) may be computed by the method of successive approximations:

$$\alpha_k = A \alpha_{k-1} \quad (k = 1, 2, \dots). \quad (52)$$

As to the convergence of the successive approximations (52), we can propose:

Theorem 3. Let the nonnegative constants c_i, d_i ($i=1, \dots, r$) be all positive. Further, let us assume that

$$\sum_{i=1}^r \int_0^T \left[\frac{1}{c} \|f_i(T, t)\|^2 + \frac{1}{d} \|w_i(T, t)\|^2 \right] dt < 1, \quad (53)$$

where $c = \min(c_1, c_2, \dots, c_r)$, $d = \min(d_1, d_2, \dots, d_r)$, and $\| \cdot \|$ denotes the Euclidean norm in R^n . Then, the successive approximations (52), starting with an arbitrary α_0 , converge to a unique solution of (49).

Proof. Let α and β be arbitrary points in R^n . By using the Schwarz inequality and the relations:

$$\left. \begin{aligned} |\varphi_i(f_i^*(T, t)\alpha) - \varphi_i(f_i^*(T, t)\beta)| &\leq \frac{1}{c} |f_i^*(T, t)(\alpha - \beta)| \\ &\leq \frac{1}{c} \|f_i(T, t)\| \|\alpha - \beta\|, \\ |\psi_i(w_i^*(T, t)\alpha) - \psi_i(w_i^*(T, t)\beta)| &\leq \frac{1}{d} \|w_i(T, t)\| \|\alpha - \beta\|, \\ (i = 1, \dots, r) \end{aligned} \right\} \quad (54)$$

it follows that

$$\begin{aligned} |(A\alpha)_j - (A\beta)_j| &\leq \|\alpha - \beta\| \sum_{i=1}^r \int_0^T \left[\frac{1}{c} |f_{ij}(T, t)| \|f_i(T, t)\| \right. \\ &\quad \left. + \frac{1}{d} |w_{ij}(T, t)| \|w_i(T, t)\| \right] dt \end{aligned}$$

$$\leq \|\alpha - \beta\| \left\{ \frac{1}{c} \left\{ \int_0^T \sum_{i=1}^r f_{ij}^2(\tau, t) dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^r \|f_i(\tau, t)\|^2 dt \right\}^{\frac{1}{2}} \right. \\ \left. + \frac{1}{d} \left\{ \int_0^T \sum_{i=1}^r w_{ij}^2(\tau, t) dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^r \|w_i(\tau, t)\|^2 dt \right\}^{\frac{1}{2}} \right\}.$$

Hence,

$$\|A\alpha - A\beta\| \leq \left[\frac{1}{c} \int_0^T \sum_{i=1}^r \|f_i(\tau, t)\|^2 dt \right. \\ \left. + \frac{1}{d} \int_0^T \sum_{i=1}^r \|w_i(\tau, t)\|^2 dt \right] \|\alpha - \beta\|. \quad (55)$$

Inequality (55) shows that the mapping A defined by (50) is a contraction mapping under the condition (53) [13]. Thus, the theorem is proved.

4. Solution to the Problem 2

If $C=D=0$ in (4), the payoff becomes

$$\hat{J}(u, v) = (z(T), z(T)). \quad (56)$$

In this case, equation (49) which determines the vector α becomes

$$\alpha = \sum_{i=1}^r \int_0^T \left[w_i(\tau, t) \operatorname{sgn}(w_i^*(\tau, t)\alpha) \right. \\ \left. - f_i(\tau, t) \operatorname{sgn}(f_i^*(\tau, t)\alpha) \right] dt + z_1. \quad (57)$$

Let us introduce a real number ε and a vector β in R^n such that

$$\alpha = \varepsilon \beta, \quad \varepsilon > 0, \quad \|\beta\| = 1.$$

Then, (57) can be rewritten as

$$\varepsilon \beta = \sum_{i=1}^r \int_0^T \left[w_i(T, t) \operatorname{sgn}(w_i^*(T, t) \beta) - f_i(T, t) \operatorname{sgn}(f_i^*(T, t) \beta) \right] dt + z_1. \quad (58)$$

As mentioned before, the vector α defined by (46) represents $z_T = z(T)$. Hence, if there exists a time T such that the value $\hat{J}(u^0, v^0)$ of the game vanishes, then there exists a solution of the Problem 2. Letting $\varepsilon \rightarrow 0$ in (58) yields a equation

$$\sum_{i=1}^r \int_0^T \left[w_i(T, t) \operatorname{sgn}(w_i^*(T, t) \beta) - f_i(T, t) \operatorname{sgn}(f_i^*(T, t) \beta) \right] dt + z_1 = 0. \quad (59)$$

The time T and the n -vector β ($\|\beta\|=1$) which satisfy (59) give a solution to the Problem 2, i.e.,

$$T_{u^0, v^0} = T. \quad (60)$$

Furthermore, from (43), (44), and (47) it follows that

$$u^0(t) = - \begin{bmatrix} \operatorname{sgn}(f_1^*(T, t) \beta) \\ \vdots \\ \operatorname{sgn}(f_r^*(T, t) \beta) \end{bmatrix}, \quad v^0(t) = \begin{bmatrix} \operatorname{sgn}(w_1^*(T, t) \beta) \\ \vdots \\ \operatorname{sgn}(w_r^*(T, t) \beta) \end{bmatrix}. \quad (61)$$

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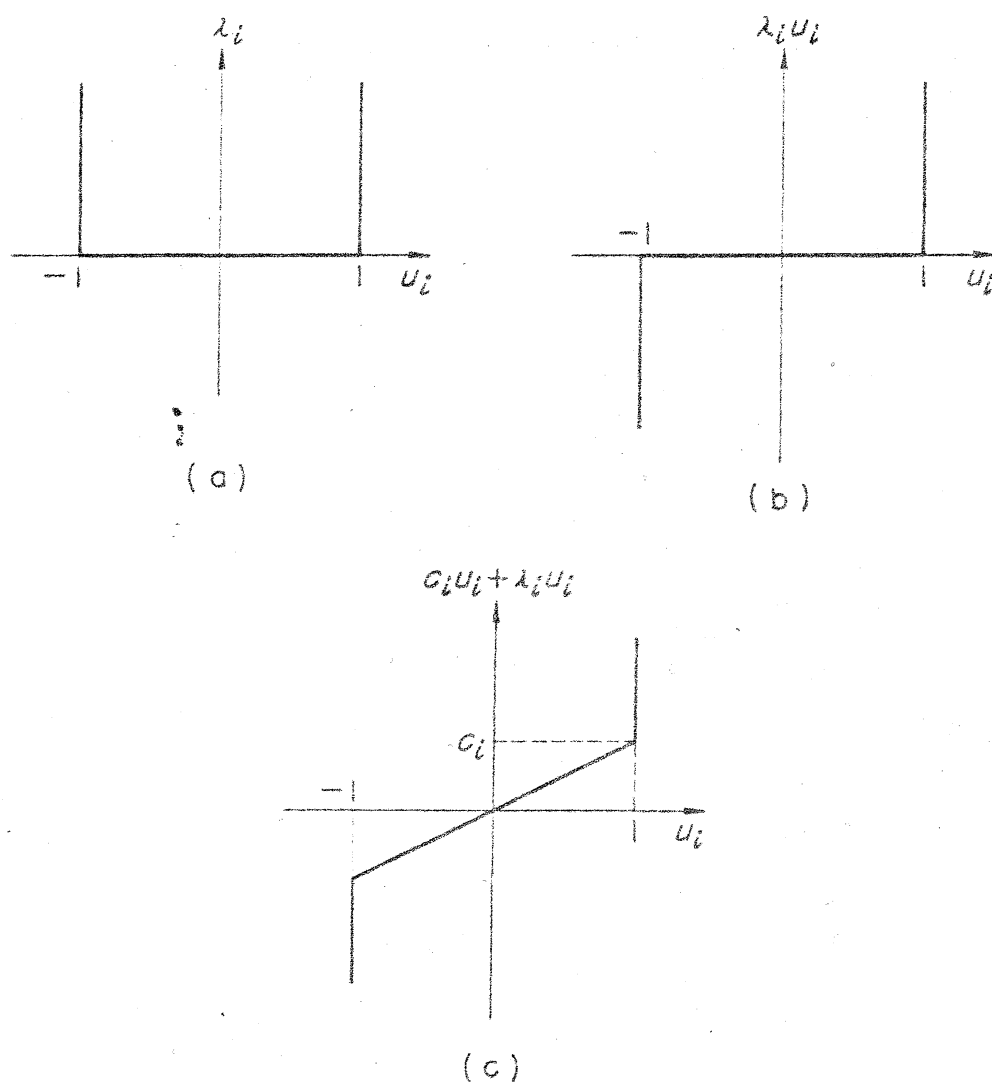


Fig. 1 Relations between the variables.

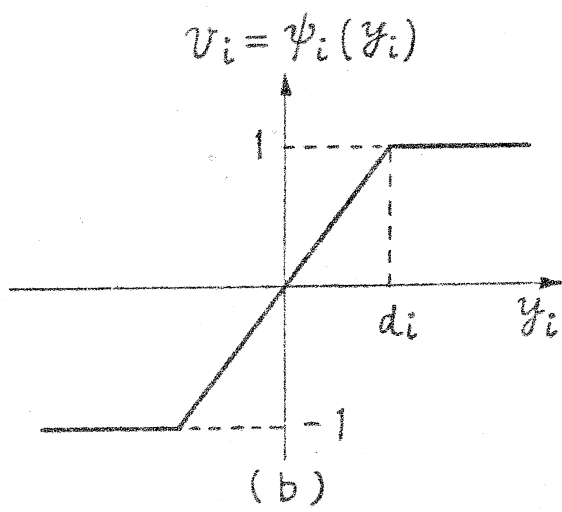
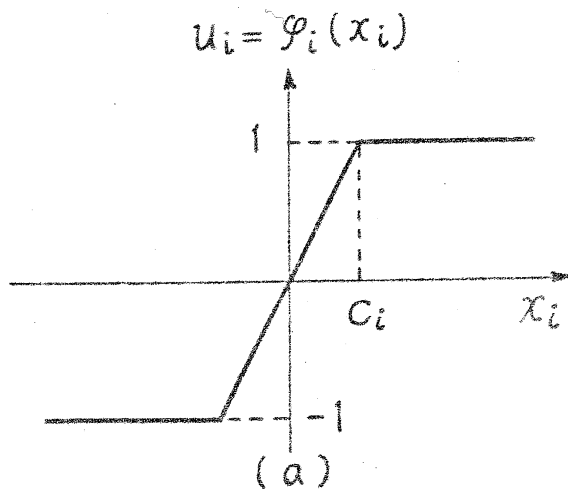


Fig. 2 Nonlinear characteristics.

