

On a sufficient condition for starlikeness

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Abstract

We give a sufficient condition for a normalized analytic function to be starlike.

Keywords: α -convex functions, starlike functions, convex functions, univalent functions

2000 Mathematics Subject Classification. Primary 30C45.

1. Introduction and Result

P. T. Mocanu [1] defined α -convex functions as follows: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\Delta = \{z : |z| < 1\}$, with $f(z)f'(z)/z \neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α -convex in Δ if and only if the inequality

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0$$

holds in Δ . For the functions above, S. S. Miller, P. T. Mocanu and M. O. Read [2] proved the following theorem.

Theorem A. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is α -convex in the unit disc Δ , then $f(z)$ is starlike and univalent in Δ . Moreover, if $\alpha \geq 1$, then $f(z)$ is convex for $|z| < 1$, and if $\alpha \leq -1$, then $1/f(1/z)$ is convex for $|z| > 1$.*

In this paper, we partly improve Theorem A and to do so, we need the following lemma [3], [4].

Lemma A. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in the unit disc Δ and supposed that there exists a point $z_0 \in \Delta$ such that*

$$\operatorname{Re}\{p(z)\} > 0 \text{ for } |z| < |z_0|,$$

$$\operatorname{Re}\{p(z_0)\} = 0 \text{ and } p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg\{p(z_0)\} = \frac{\pi}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg\{p(z_0)\} = -\frac{\pi}{2}$$

where $p(z_0) = \pm ia$ and $a > 0$.

Theorem. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ and let us put

$$F(z) = \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right].$$

Then, for the case $\alpha \geq 0$ or $\alpha < -2$, if $F(z)$ does not take pure imaginary value l where $|l| \geq \sqrt{(2 + \alpha)\alpha}$, then $f(z)$ is starlike in Δ and for the case $-2 \leq \alpha < 0$, if

$$\operatorname{Re}\{F(z)\} > 0 \text{ in } \Delta,$$

then $f(z)$ is starlike in Δ .

Proof. For the case $\alpha > 0$, from the hypothesis of the theorem we obtain $f'(z) \neq 0$ in Δ , because if there exists a point $z_0 \in \Delta$ such that

$$f'(z_0) = 0,$$

this contradicts the hypothesis of the theorem.

Let us put

$$p(z) = \frac{zf'(z)}{f(z)} \text{ and } p(z) \neq 0 \text{ in } \Delta.$$

Then it follows that

$$F(z) = p(z) + \alpha \frac{zp'(z)}{p(z)}.$$

If there exists a point $z_0 \in \Delta$ such that

$$\operatorname{Re}\{p(z)\} > 0 \text{ for } |z| < |z_0|,$$

$$\operatorname{Re}\{p(z_0)\} = 0 \text{ and } p(z_0) \neq 0,$$

then from Lemma A, for the case $\arg\{p(z_0)\} = \pi/2$, $p(z_0) = ia$ and $a > 0$, we have

$$F(z_0) = p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = ia + i\alpha k$$

and so, $F(z_0)$ is a pure imaginary value.

Then it follows that

$$\begin{aligned}\operatorname{Im}\{F(z_0)\} &= a + \alpha k \\ &\geq \frac{1}{2} \left\{ (2 + \alpha)a + \frac{\alpha}{a} \right\} \\ &\geq \sqrt{(2 + \alpha)\alpha}.\end{aligned}$$

This contradicts the hypothesis of the theorem.

For the case $\arg\{p(z_0)\} = -\pi/2$, $p(z_0) = -ia$ and $a > 0$, we have also

$$F(z_0) = -ia + i\alpha k$$

and

$$\begin{aligned}\operatorname{Im}\{F(z_0)\} &= -a + \alpha k \\ &\leq -a - \frac{1}{2}\alpha \left(a + \frac{1}{a} \right) \\ &= - \left\{ (2 + \alpha)a + \frac{\alpha}{a} \right\} \\ &\leq -\sqrt{(2 + \alpha)\alpha}.\end{aligned}$$

This is also contradicts the hypothesis of the theorem and it completes the proof of the case $\alpha > 0$.

For the case $\alpha < -2$, applying the same method and Lemma A, we can obtain the proof of the theorem.

Finally, for the case $-2 \leq \alpha < 0$, it depends on [2].

References

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