

Letter

Geometries from field theories

Sinya Aoki^{1,2,*}, Kengo Kikuchi¹, and Tetsuya Onogi³

¹*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

²*Center for Computational Sciences, University of Tsukuba, Ibaraki 305-8577, Japan*

³*Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan*

*E-mail: saoki@het.ph.tsukuba.ac.jp

Received June 8, 2015; Revised July 15, 2015; Accepted August 18, 2015; Published October 1, 2015

.....
We propose a method to define a $d + 1$ -dimensional geometry from a d -dimensional quantum field theory in the $1/N$ expansion. We first construct a $d + 1$ -dimensional field theory from the d -dimensional one via the gradient-flow equation, whose flow time t represents the energy scale of the system such that $t \rightarrow 0$ corresponds to the ultraviolet and $t \rightarrow \infty$ to the infrared. We then define the induced metric from $d + 1$ -dimensional field operators. We show that the metric defined in this way becomes classical in the large- N limit, in the sense that quantum fluctuations of the metric are suppressed as $1/N$ due to the large- N factorization property. As a concrete example, we apply our method to the $O(N)$ nonlinear σ model in two dimensions. We calculate the 3D induced metric, which is shown to describe an anti-de Sitter space in the massless limit. Finally, we discuss several open issues for future studies.
.....

Subject Index B30, B34, B35, B37, B39

1. Introduction One of the most surprising and significant findings in field theories and string theories is the anti-de Sitter/conformal field theory (AdS/CFT) (or, more generally, gravity/gauge theory) correspondence [1], which claims that a d -dimensional conformal field theory is equivalent to some $d + 1$ -dimensional (super-)gravity theory on the AdS background. After this proposal, there appeared a tremendous amount of evidence to support this correspondence. This equivalence is, however, still mysterious and needs to be understood, even though the open-string/closed-string duality may explain it.

In this paper, we consider such gravity/field theory correspondences from a different point of view, and propose an alternative method to define a geometry from a field theory. Explicitly, we consider a d -dimensional quantum field theory in the large- N expansion, and lift it to a $d + 1$ -dimensional one using the gradient flow [2–5], where the flow time t becomes an additional coordinate and represents the energy scale of the original d -dimensional theory. We then define the induced metric from the field in this $d + 1$ -dimensional theory. In this way, we define the metric from the original d -dimensional theory and its scale dependence; the method is quite generic and can be applied to all field theories in principle. As shown in the next section, however, the metric defined in this way becomes classical only in the large- N limit; thus, the use of the large- N expansion for field theories may be mandatory.

This paper is organized as follows. We first propose our method for general field theories in the large- N expansion to define the induced metric. We discuss some general properties of this construction. We next apply the method to the 2D $O(N)$ nonlinear σ model. We calculate the vacuum

expectation value (VEV) of the 3D induced metric in the particular definition, which is shown to describe (asymptotically) an AdS space in the massless (ultraviolet (UV)) limit. We finally discuss several remaining issues of the method for future studies.

2. Proposal We consider the generic large- N field $\varphi^{a,\alpha}(x)$ where x is d -dimensional space-time coordinate, $a = 1, 2, \dots$, is the large- N index, while α represents other indices such as spinor or vector indices, so that $h_{\alpha\beta}\varphi^{a,\alpha}(x)\varphi^{b,\beta}(x)$ can be made Lorentz invariant by a constant tensor $h_{\alpha\beta}$. We denote the action of this theory S .

We first extend the d -dimensional field $\varphi^{a,\alpha}(x)$ to $\phi^{a,\alpha}(t, x)$ in $d + 1$ dimensions, using the gradient-flow equation as [6]

$$\frac{d}{dt}\phi^{a,\alpha}(t, x) = -g^{ab}(\phi(t, x)) \frac{\delta S}{\delta \varphi^{b,\alpha}(x)} \Big|_{\varphi \rightarrow \phi}, \quad (1)$$

with an initial condition that $\phi^{a,\alpha}(0, x) = \varphi^{a,\alpha}(x)$, where g^{ab} is the metric of the space of the large- N index. Since the length dimension of t is 2 and $t \geq 0$, we introduce a new variable $\tau = 2\sqrt{t}$. (Here a factor 2 makes some later results simpler.) Then we denote the $d + 1$ -dimensional coordinate as $z = (\tau, x) \in \mathbb{R}^+ (= [0, \infty]) \times \mathbb{R}^d$ and the field as $\phi^{a,\alpha}(z)$.

We propose to define a $d + 1$ -dimensional metric as

$$\hat{g}_{\mu\nu}(z) := g_{ab}(\phi(z)) h_{\alpha\beta} \partial_\mu \phi^{a,\alpha}(z) \partial_\nu \phi^{b,\beta}(z), \quad (2)$$

where a mass dimension of the constant tensor $h_{\alpha\beta}$ ¹ must be $-2(1 + d_\varphi)$, with d_φ being that of φ , to make the metric dimensionless. This is an induced metric from a $d + 1$ -dimensional manifold $\mathbb{R}^+ \times \mathbb{R}^d$ on a curved space in $\mathbb{R}^N$ with the metric g_{ab} . Using the above definition, we then calculate the expectation values of $g_{\mu\nu}$ and its correlations as

$$\langle \hat{g}_{\mu\nu}(z) \rangle := \langle \hat{g}_{\mu\nu}(z) \rangle_S, \quad (3)$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle_S, \quad (4)$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle_S, \quad (5)$$

where $\langle \mathcal{O} \rangle_S$ is the expectation values of $\mathcal{O}(\varphi)$ in d dimensions with the action S as

$$\langle \mathcal{O} \rangle_S := \frac{1}{Z} \int \mathcal{D}\varphi \mathcal{O}(\varphi) e^{-S}, \quad Z := \int \mathcal{D}\varphi e^{-S} \quad (6)$$

in the large- N expansion. Even though the “composite” operator $\hat{g}_{\mu\nu}(z)$ contains a product of two local operators at the same point z , $\langle \hat{g}_{\mu\nu}(z) \rangle$ is finite as long as $\tau \neq 0$ [7]. This is the reason why we define the induced metric in $d + 1$ dimensions from ϕ , not the d -dimensional induced metric from φ , which diverges badly.

Thanks to the large- N factorization, quantum fluctuations of the metric $\hat{g}_{\mu\nu}$ are suppressed in the large- N limit. For example, the two-point correlation function of $\hat{g}_{\mu\nu}$ behaves as

$$\langle \hat{g}_{\mu\nu}(z_1) \hat{g}_{\alpha\beta}(z_2) \rangle = \langle \hat{g}_{\mu\nu}(z_1) \rangle \langle \hat{g}_{\alpha\beta}(z_2) \rangle + \mathcal{O}\left(\frac{1}{N}\right), \quad (7)$$

which shows that the induced metric $\hat{g}_{\mu\nu}$ is classical in the large- N limit, and quantum fluctuations are sub-leading and can be calculated in the $1/N$ expansion. A use of the $1/N$ expansion here seems

¹ In general, we may introduce a z -dependence tensor $h_{\alpha\beta}(z)$ here, but we consider the constant case only in this paper.

important for the geometrical interpretation of the metric $\hat{g}_{\mu\nu}$. For example, in the large- N limit, the VEV of the curvature tensor operator is directly obtained from the VEV of $\hat{g}_{\mu\nu}$ as in the classical theory.

3. An example: $O(N)$ nonlinear σ model in two dimensions As a concrete example of our proposal, we consider the $O(N)$ nonlinear σ model in two dimensions, whose action is given by

$$S = \frac{1}{2g^2} \int d^2x \sum_{a,b=1}^{N-1} g_{ab}(\varphi) \sum_{k=1}^2 \left(\partial_k \varphi^a(x) \partial^k \varphi^b(x) \right), \quad (8)$$

where

$$g_{ab}(\varphi) = \delta_{ab} + \frac{\varphi^a \varphi^b}{1 - \varphi \cdot \varphi}, \quad g^{ab}(\varphi) = \delta_{ab} - \varphi^a \varphi^b \quad (9)$$

with $\varphi \cdot \varphi = \sum_{a=1}^{N-1} \varphi^a \varphi^a$, and the N th component of φ is expressed in terms of other fields as $\varphi^N = \pm \sqrt{1 - \varphi \cdot \varphi}$, so that the metric g_{ab} appears in the action. The 3D metric $g_{\mu\nu}(z)$ will be extracted from this theory, according to our proposal.

3.1 Solution to the gradient-flow equation in large N In the previous study [8], the solution of the gradient-flow equation was obtained in the momentum space as

$$\phi^a(t, p) = f(t) e^{-p^2 t} \sum_{n=0}^{\infty} : X_{2n+1}(\varphi, p, t) : \quad (10)$$

where X_{2n+1} only contains φ^{2n+1} terms and is $O(1/N^{2n+1})$. The leading order term X_1 is given by $X_1^a(\varphi, p, t) = \varphi^a(p)$ with

$$f(t) = \frac{1}{\sqrt{1 - 2\lambda J(t)}}, \quad J(t) = \int_0^t ds I(s), \quad I(t) = \int \frac{d^2q}{(2\pi)^2} \frac{q^2 e^{-2q^2 t}}{q^2 + m^2}, \quad (11)$$

where $\lambda = g^2 N$ is the 't Hooft coupling constant, and m is the dynamically generated mass, which satisfies

$$1 = \lambda \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2}. \quad (12)$$

Introducing the momentum cut-off Λ , we have

$$f(t) = e^{-m^2 t} \sqrt{\frac{\log(1 + \Lambda^2/m^2)}{\text{Ei}(-2t(\Lambda^2 + m^2)) - \text{Ei}(-2tm^2)}}, \quad (13)$$

where $\text{Ei}(x)$ is the exponential integral function defined by $\text{Ei}(-x) = \int dx e^{-x}/x$. The two-point function, which dominates in the large- N limit, is calculated as

$$\langle \phi^a(t, x) \phi^b(s, y) \rangle_S = \int \frac{d^2q}{(2\pi)^2} \frac{e^{-q^2(t+s)} e^{iq(x-y)}}{q^2 + m^2} \delta_{ab} \frac{\lambda}{N} f(t) f(s) + O(N^{-2}). \quad (14)$$

3.2 Induced metric An induced metric for this model is given by

$$\hat{g}_{\mu\nu}(z) := h g_{ab}(\phi(z)) \partial_\mu \phi^a(z) \partial_\nu \phi^b(z), \quad (15)$$

where $z = (2\sqrt{t}, x)$, and the constant h is introduced so that the mass dimension of the metric operator $\hat{g}_{\mu\nu}(z)$ is zero. This is the induced metric from a 3D manifold $\mathbb{R}^+ \times \mathbb{R}^2$ on the $N - 1$ -dimensional sphere defined by ϕ^a .

The VEV of the metric, $g_{\mu\nu}$, which does not depend on x due to the translational invariance of the 2D $O(N)$ nonlinear σ model, can easily be calculated in the large- N limit as $g_{i\tau}(\tau) = g_{\tau i}(\tau) = 0$ for $i = 1, 2$, while

$$g_{ij}(z) := \langle \hat{g}_{ij}(z) \rangle = h \langle g_{ab}(\phi) \partial_i \phi^a(t, x) \partial_j \phi^b(t, x) \rangle \simeq \frac{h}{2} \delta_{ij} \lambda f^2(t) I(t) = \frac{h}{2} \delta_{ij} \frac{\dot{f}(t)}{f(t)} \quad (16)$$

for $i, j = 1, 2$, where we use an equality that $\dot{f}(t) := df(t)/dt = \lambda f(t)^3 I(t)$. Furthermore,

$$g_{\tau\tau}(\tau) = \frac{\tau^2 h}{4} \langle \dot{\phi}^a(t, x) g_{ab}(\phi(t, x)) \dot{\phi}^b(t, x) \rangle, \quad (17)$$

which, after using the gradient-flow equation, is evaluated as

$$g_{\tau\tau}(\tau) \simeq \frac{\tau^2 h}{4} \left[\langle \nabla^2 \phi \cdot \nabla^2 \phi \rangle - \langle \phi \cdot \nabla^2 \phi \rangle^2 \right] = -\frac{\tau h}{4} \frac{d}{d\tau} \left(\frac{\dot{f}}{f} \right). \quad (18)$$

Thus the expectation values of the induced metric turns out to be diagonal as

$$g_{\mu\nu} = \begin{pmatrix} B(\tau) & 0 & 0 \\ 0 & A(\tau) & 0 \\ 0 & 0 & A(\tau) \end{pmatrix}, \quad (19)$$

where we define

$$A(\tau) = \frac{h}{2} \frac{\dot{f}(t)}{f(t)} \Big|_{t=\tau^2/4}, \quad B(\tau) = -\frac{\tau}{2} A_{,\tau},$$

and $f_{,\tau}$ means the derivative of f with respect to τ . This $A(\tau)$, and hence also $B(\tau)$, is finite in the $\Lambda \rightarrow \infty$ limit as

$$A(\tau) = -\frac{m^2 h}{2} \left[1 + \frac{e^{-\tau^2 m^2/2}}{\text{Ei}(-\tau^2 m^2/2) m^2 \tau^2/2} \right]. \quad (20)$$

From the metric, we can calculate the VEV of composite operators such as the Einstein tensor $G_{\mu\nu}(\hat{g})$ as $\langle G_{\mu\nu}(\hat{g}) \rangle = G_{\mu\nu}(\langle \hat{g} \rangle)$, thanks to the factorization in the large- N limit.

After a little algebra, we obtain

$$G_{\tau\tau} = \frac{A_{,\tau}^2}{4A^2}, \quad G_{ij} = \delta_{ij} \left[\frac{A_{,\tau\tau}}{2B} - \frac{A_{,\tau} B_{,\tau}}{4B^2} - \frac{A_{,\tau}^2}{4AB} \right], \quad (21)$$

and $G_{i\tau} = G_{\tau i} = 0$.

3.3 Massless limit and AdS space We consider the massless limit ($m \rightarrow 0$), where A and its derivatives are given by

$$A \simeq -\frac{1}{\tau^2} \frac{h}{\log(m^2)} \left[1 + O\left(\frac{1}{\log(m^2)}\right) \right]; \quad (22)$$

$A_{,\tau} \simeq -2A/\tau$ and $A_{,\tau\tau} \simeq 6A/\tau^2$. We here use the expansion $\text{Ei}(-x) = \log x + \gamma + \sum_{n=1}^{\infty} (-x)^n / (n \cdot n!)$. In order to have positive and finite $g_{\mu\nu}$ in the massless limit, we take $h = -R_0^2 \log(m^2 R_0^2)$,

where a mass dimension of the constant R_0 is -1 . We thus obtain $g_{\tau\tau} = \frac{R_0^2}{\tau^2}$, $g_{ij} = \delta_{ij} \frac{R_0^2}{\tau^2}$, so that

$$ds^2 = \frac{R_0^2}{\tau^2} \left[d\tau^2 + (d\vec{x})^2 \right], \quad (23)$$

which describes the Euclidean AdS space. Indeed, the Einstein tensor reads

$$G_{\mu\nu} = -\Lambda_0 g_{\mu\nu}, \quad \Lambda_0 = -\frac{1}{R_0^2}, \quad (24)$$

which give the negative cosmological constant Λ_0 . It is interesting to see that the AdS geometry is realized for the conformal field theory defined in the massless limit, which corresponds to the UV fixed point of the theory.

3.4 Metric in UV and IR limits In the short-distance (UV) limit ($m\tau \rightarrow 0$), we have $A \simeq -h/[\tau^2 \log(m^2\tau^2)]$, so that $g_{\tau\tau} \simeq -h/[\tau^2 \log(m^2\tau^2)]$ and $g_{ij} \simeq -\delta_{ij}h/[\tau^2 \log(m^2\tau^2)]$.

As briefly discussed for generic cases, had we defined the d -dimensional metric directly from the d -dimensional field theory as $\hat{g}_{ij}^d(x) := g_{ab}(\varphi(x))\partial_i\varphi^a(x)\partial_j\varphi^b(x)$, the VEV of $\hat{g}_{ij}^d(x)$ would become UV divergent due to the short distance singularity of $\varphi^a(x)\varphi^b(y)$ at $x \rightarrow y$. In contrast, the $d+1$ -dimensional metric $\hat{g}_{\mu\nu}$ defined from the d -dimensional field theory via Eq. (2) is free from UV divergence, since the flowed field $\phi(t, x)$ and any local composite operators constructed from it are expected to be finite as long as $t \sim \tau^2$ is nonzero [7–10]. This is the reason why we employ the flowed field and why the induced metric is defined on $d+1$ dimensions, not on d dimensions. Consequently, the classical metric, $g_{\mu\nu}$, is UV finite in our proposal, as it should be. Note that the UV divergence presented in the 2D $O(N)$ nonlinear σ model appears, e.g., in the $m\tau \rightarrow 0$ limit as $g_{\mu\nu} \sim 1/(\tau^2 \log m^2\tau^2)$.

In this limit, the “effective” cosmological constant is given as

$$\Lambda_0^{\text{eff}} = -\frac{\log(m^2\tau^2)}{R_0^2 \log(m^2R_0^2)}, \quad (25)$$

where the non-conformal natures of the original 2D asymptotic-free field theory appear in its $\log \tau^2$ dependence.

In the $m\tau \rightarrow \infty$ (infrared (IR)) limit, on the other hand, $A \simeq h/\tau^2$, which gives $g_{\tau\tau} \simeq h/\tau^2$ and $g_{ij} \simeq \delta_{ij}h/\tau^2$. Since $\Lambda_0^{\text{eff}} = [R_0^2 \log(m^2R_0^2)]^{-1}$ in this limit, the theory becomes asymptotically AdS ($\Lambda_0^{\text{eff}} < 0$) if $\log(m^2R_0^2) < 0$. This result looks rather nontrivial, since the massive theory is expected naively to become trivially conformal due to decouplings of all massive modes in the IR limit.

Assuming the Einstein equation, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we can define the energy momentum tensor $T_{\mu\nu}$. In both the UV and IR limits, we have $T_{\mu\nu} = \delta_{\mu\nu}/(8\pi G\tau^2)$, which does not depend on m and therefore holds even at $m = 0$. As a consequence, $T_{\mu\nu}^{\text{matter}}$ vanishes in both the UV and IR limits, where we define $T_{\mu\nu}^{\text{matter}} := T_{\mu\nu} + g_{\mu\nu}\Lambda_0/(8\pi G)$.

4. Discussions In our proposal, a correspondence between a geometry and a field theory is explicit by construction, and the method proposed here can be applied to an arbitrary quantum field theory, as long as the large- N expansion is employed. If the theory is solvable in the large- N limit, the VEV of $\hat{g}_{\mu\nu}$ can be calculated exactly. Let us discuss the remaining issues that should be investigated in future studies.

First, we do not yet have a full dictionary to interpret a quantum field theory in terms of the corresponding metric operator and vice versa. For the 2D $O(N)$ nonlinear σ model, the dynamically generated mass m can be extracted from the asymptotic behavior of the metric that $g_{\mu\nu} \sim 1/[\tau^2 \log(m^2 \tau^2)]$ as $m\tau \rightarrow 0$. However, this gives only a partial knowledge of the field theory. It would be nice if we could determine the whole structure of the matter content based on a gravity action that gives the same Einstein equation that we have assumed.

Secondly, since our method can be applied to a large class of quantum field theories, one should investigate what kind of geometry emerges from various large- N models other than the 2D $O(N)$ nonlinear σ model, including conformal theories. One possible direction is to introduce a source field in the original theory to break the translational invariance to generate geometries with nontrivial z dependence. In particular, it would be a challenge to find the field theory set-up that induces the black hole geometry.

Thirdly, the fluctuations around the background geometry should be studied. In principle, we can calculate an arbitrary correlation function for the metric $\hat{g}_{\mu\nu}$ including the quantum fluctuation of the metric in the $1/N$ expansion. In practice, however, calculations in the next leading order become much more complicated than those in the large- N limit [8]. Although no action is explicitly given for $\hat{g}_{\mu\nu}$ in our approach, one may effectively define the quantum theory of the metric in this way. It would be interesting to investigate whether this quantum theory is renormalizable (or even UV finite) or not, in contrast to the unrenormalizable quantum theory of the Einstein gravity. Furthermore, it is also interesting to calculate an effective action for the composite operator of $\hat{g}_{\mu\nu}$.

Since Eq. (1) is not a unique way to define the flow equation, and thus the $d+1$ -dimensional field ϕ from d -dimensional φ , a dependence of the induced metric on the flow equation should be investigated.

Finally, in the case of gauge theories, simple choices for the induced metric may be

$$\hat{g}_{\mu\nu}(z) := h \sum_{i,j=1}^d \text{Tr} D_\mu F_{ij}(z) D_\nu F^{ij}(z), \quad (26)$$

$$\hat{g}_{\mu\nu}(z) := h \sum_{\alpha=0}^d \text{Tr} F_{\mu\alpha}(z) F_\nu{}^\alpha(z), \quad (27)$$

both of which are invariant under the τ -independent gauge transformation [3–5]. Here D_i ($i = 1, \dots, d$) is the covariant derivative in d dimensions while $D_\tau := \partial_\tau$, and then the field strength is given as $F_{\mu\nu} := [D_\mu, D_\nu]$. It will be interesting to calculate the induced metric from the large- N gauge theory in two dimensions ('t Hooft model) [11] with our method.

Acknowledgements

S.A. would like to thank Drs N. Ishibashi, S. Sugimoto, T. Takayanagi, and Y. Yokokura for their useful comments. We also thank Dr S. Yamaguchi for his comment on the AdS metric. This work was supported by a Grant-in-Aid for JSPS Fellows Grant Number 25-1336, by a Grant-in-Aid of the Japanese Ministry of Education (Nos. 25287046, 26400248), and by MEXT SPIRE and JICFuS.

Funding

Open Access funding: SCOAP³.

References

- [1] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [[arXiv:hep-th/9711200](#)] [[Search INSPIRE](#)].
- [2] R. Narayanan and H. Neuberger, J. High Energy Phys. **0603**, 064 (2006) [[arXiv:hep-th/0601210](#)] [[Search INSPIRE](#)].
- [3] M. Lüscher, J. High Energy Phys. **1008**, 071 (2010); **1403**, 092 (2014) [erratum] [[arXiv:1006.4518](#) [hep-lat]] [[Search INSPIRE](#)].
- [4] M. Lüscher, Commun. Math. Phys. **293**, 899 (2010) [[arXiv:0907.5491](#) [hep-lat]] [[Search INSPIRE](#)].
- [5] M. Lüscher, PoS LATTICE **2013**, 016 (2014) [[arXiv:1308.5598](#) [hep-lat]] [[Search INSPIRE](#)].
- [6] K. Kikuchi and T. Onogi, J. High Energy Phys. **1411**, 094 (2014) [[arXiv:1408.2185](#) [hep-th]] [[Search INSPIRE](#)].
- [7] M. Lüscher and P. Weisz, J. High Energy Phys. **1102**, 051 (2011) [[arXiv:1101.0963](#) [hep-th]] [[Search INSPIRE](#)].
- [8] S. Aoki, K. Kikuchi, and T. Onogi, J. High Energy Phys. **1504**, 156 (2015) [[arXiv:1412.8249](#) [hep-th]] [[Search INSPIRE](#)].
- [9] H. Makino and H. Suzuki, Prog. Theor. Exp. Phys. **2015**, 033B08 (2015) [[arXiv:1410.7538](#) [hep-lat]] [[Search INSPIRE](#)].
- [10] H. Makino, F. Sugino, and H. Suzuki, [[arXiv:1412.8218](#) [hep-lat]] [[Search INSPIRE](#)].
- [11] G. 't Hooft, Nucl. Phys. B **75**, 461 (1974).