

Coloring Comparability-ke Graphs

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1 Introduction

Many graph problems are NP-complete for general graphs. It is natural to consider that if the graph problem is tractable for a graph class $\mathcal{F}$, it is also tractable for a class of graphs which are close to graphs in $\mathcal{F}$. $\mathcal{F}+ke$ and $\mathcal{F}-ke$ graphs are classes of graphs close to $\mathcal{F}$. They are the classes of graphs obtained by adding or deleting k edges from graphs in $\mathcal{F}$. We can consider the complexity of several problems on such graph classes from parametric point of view. In general, problems become difficult as k increases. A problem with parameter k is called to be fixed parameter tractable if it can be solved in $f(k)|x|^c$ time, where f is an arbitrary function and $|x|$ is the size of input.

Vertex coloring problem is a very important graph problem, which is NP-complete for general graphs. Vertex coloring for parameterized graph classes are considered in [1]. It is shown in [1] that, when vertex coloring of $\mathcal{F}$ graphs is solved in polynomial time, vertex coloring of $\mathcal{F}+ke$ graphs is fixed parameter tractable if $\mathcal{F}$ is closed under identification of nonadjacent vertices, and that vertex coloring of $\mathcal{F}-ke$ graphs is fixed parameter tractable if $\mathcal{F}$ is closed under edge contraction.

In this paper, we consider vertex coloring of comparability-ke graphs. A comparability graph is an undirected graph which becomes a transitive graph if we give appropriate direction to each edge. As a comparability graph is a perfect graph, vertex coloring of comparability graphs can be solved in polynomial time [3]. In addition, comparability graphs are closed neither under identification of nonadjacent vertices nor under edge contraction. On comparability+ke graphs, vertex coloring is solved in polynomial time for $k = 1$ and NP-complete for $k \geq 2$ [4].

In this paper, we first show that vertex coloring of comparability-1e graphs is solved in polynomial time. Next, we show that vertex coloring of comparability-ke graphs can be polynomially reduced to a vertex coloring problem of comparability graphs with restrictions that given pairs of nodes should have the same color.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. Then vertex coloring problem is defined as follows.
VERTEX COLORING

Input : A graph $G = (V, E)$ and a positive integer $t \leq |V|$.

Question : Is G t -colorable? That is, is there a function $f : V \rightarrow \{1, 2, \dots, t\}$ that satisfies $f(u) \neq f(v)$ for all $(u, v) \in E$.

The chromatic number of G , denoted as $\chi(G)$, is the smallest t for which G is t -colorable. The clique number of a graph G , denoted as $\omega(G)$, is the degree of the maximum complete subgraph of G .

For a graph class $\mathcal{F}$, let $\mathcal{F}+ke$ be the class of graphs which can be obtained by adding at most k edges to an $\mathcal{F}$ graph. Similarly, let $\mathcal{F}-ke$ be the class of graphs which can be obtained by removing at most k edges from an $\mathcal{F}$ graph. The *modulator* of an $\mathcal{F}+ke$ graph $G = (V, E)$ is a set of edges $E_k (\subset E)$ s.t. $(V, E - E_k) \in \mathcal{F}$ and $|E_k| \leq k$. The modulator of an $\mathcal{F}-ke$ graph G is a set of edges E_k s.t. $(V, E \cup E_k) \in \mathcal{F}$ and $|E_k| \leq k$. In this paper, we assume that the modulator is given. For a fixed k , the modulator of $\mathcal{F}+ke$ or $\mathcal{F}-ke$ graphs can be found in polynomial time provided that it can be checked in polynomial time whether a graph is in class $\mathcal{F}$ or not.

A comparability graph is an undirected graph s.t. a transitive graph can be obtained by giving appropriate orientation to each edge. A directed graph is called transitive if $(u, v) \in E$ and $(v, w) \in E$, then $(u, w) \in E$ holds. It can be recognized whether a graph is a comparability graph or not and its transitive orientation can be found in $O(\gamma|E|)$ time, where γ is the maximum degree of a vertex [2]. If a comparability graph is given, its transitive orientation is obtained in linear time [5]. A comparability graph is a perfect graph, whose clique number equals the chromatic number. The clique number and the chromatic number of a comparability graph can be computed in polynomial time.

A transitive graph can be represented by a Hasse diagram. If $(u, v) \in E$ and $(v, w) \in E$, (u, w) is omitted in a Hasse diagram. When there exists a path from u to v in a Hasse diagram, we call that u is an ancestor of v and v is a descendant of u , denoted as $u \prec v$. In this paper, for simplicity, we write a Hasse diagram as an undirected graph. All the edges are assumed to be downward edges.

For a transitive graph G , a function $f : V \rightarrow \{1, 2, \dots, \omega(G)\}$ is called a leveling function if $f(u) < f(v)$ is satisfied for all $(u, v) \in E$. We define *levmin* and *levmax* as follows.

$$\begin{aligned} \text{levmin}(v) &= \begin{cases} 1 & \text{if } v \text{ is a source} \\ \max_{(u,v) \in E} \text{levmin}(u) + 1 & \text{otherwise} \end{cases} \\ \text{levmax}(v) &= \begin{cases} \omega(G) & \text{if } v \text{ is a sink} \\ \max_{(v,u) \in E} \text{levmax}(u) - 1 & \text{otherwise} \end{cases} \end{aligned}$$

Then *levmin* and *levmax* are leveling functions. We call that *levmin*(v) is the level of vertex v , and that v is in level *levmin*(v). As a maximal path in the Hasse diagram corresponds to a maximal clique in a transitive graph, $\omega(G) = \max_{v \in V} \text{levmin}(v)$. If we color vertex v by $f(v)$ for some leveling function f , it is an optimal vertex coloring of G . In this paper, we call such coloring a levelwise coloring of G by f .

3 Coloring Comparability-ke Graphs

3.1 Coloring Comparability-1e Graphs

In this section, we consider vertex coloring of comparability-1e graphs. Let $G = (V, E)$ be a comparability-1e graph and $E_1 = \{(a, b)\}$ be the modulator of G . Let $G_c = (V, E \cup E_1)$.

First, we consider the relation between $\chi(G)$, $\omega(G)$ and $\omega(G_c)$. If $\omega(G) = \omega(G_c)$, then $\chi(G) = \omega(G_c)$ holds. If $\omega(G) = \omega(G_c) - 1$, then $\omega(G_c) - 1 \leq \chi(G) \leq \omega(G_c)$ holds. In the latter case, there may exist an $(\omega(G_c) - 1)$ -coloring of G . In the $(\omega(G_c) - 1)$ -coloring, a and b have the same color.

We first show that it is not difficult to check if $\omega(G) = \omega(G_c)$ or not.

Lemma 1 *The equality $\omega(G) = \omega(G_c) - 1$ holds iff there exists no vertex v ($v \neq a, b$) such that $\text{levmin}(v)$ is equal to $\text{levmin}(a)$ or $\text{levmin}(b)$ and $\text{levmin}(v) + \text{levmax}(v) - 1 = \omega(G_c)$.*

Proof First, we should note that, for a vertex v , the size of the maximum clique including v is $\text{levmin}(v) + \text{levmax}(v) - 1$.

($\rightarrow$) $\omega(G) = \omega(G_c) - 1$ implies that all the maximum cliques of G' includes the modulator (a, b) .

Assume that there exists a vertex v ($v \neq a$) such that $\text{levmin}(v) = \text{levmin}(a)$ and $\text{levmin}(v) + \text{levmax}(v) - 1 = \omega(G')$. As $\text{levmin}(v) + \text{levmax}(v) - 1 = \omega(G_c)$, v is included in a maximum clique of G_c . However, as $\text{levmin}(v) = \text{levmin}(a)$, v and a are not in the same clique. Therefore, G_c has a maximum clique which does not include (a, b) . It means that $\omega(G) = \omega(G_c)$.

($\leftarrow$) If there exists no vertex v ($v \neq a, b$) such that $\text{levmin}(v)$ is equal to $\text{levmin}(a)$ or $\text{levmin}(b)$ and $\text{levmin}(v) + \text{levmax}(v) - 1 = \omega(G_c)$, all the vertices which is in some maximum clique of G_c must be connected with a and b . Thus, the modulator (a, b) is included in all the maximum cliques of G_c . $\square$

The condition of this lemma is checked easily using levmin and levmax of each node. Thus, it can be checked in polynomial time if the condition is satisfied. In the following, we consider the graphs satisfying the condition of Lemma 1.

Even though $\omega(G) = \omega(G_c) - 1$ holds, it is not always possible to color G with $\omega(G_c) - 1$ colors. We consider how to compute if G is $(\omega(G_c) - 1)$ -colorable. To consider the coloring of G , we first obtain a transitive orientation of G_c . Let G_t be the obtained transitive graph. G_t is represented as a Hasse diagram $H = (V, E_H)$. In the following, we assume w.l.o.g. that $a \prec b$ in H . In a Hasse diagram, all the vertices in a path must have different colors. However, in this case, as we consider the coloring of G , we admit that a and b have the same color in H .

We consider to modify the graph without changing its chromatic number. Let w, x, y, z be the vertices satisfying the following conditions: $(w, x), (y, x), (w, z) \in E_H$ and w and x are in the same $(\omega(G_c) - 1)$ -clique of G_c (see Fig.1). Let $H' = (V, E_H \cup \{(y, z)\})$ and let G' be the comparability-1e graph represented by H' when the modulator is added.

Lemma 2 $\chi(G') = \chi(G)$.

Proof As G' is obtained by adding edges to G , a coloring of G' is also a coloring of G . That is, $\chi(G') \geq \chi(G)$ holds. We show that if G is $(\omega(G_c) - 1)$ -colorable, then G' is also $(\omega(G_c) - 1)$ -colorable.

Consider a $(\omega(G_c) - 1)$ coloring of G . Let $U(v)$ be the set of colors used in v and its ancestors, and $L(v)$ be the set of colors used in v and its descendants. As $(w, x) \in E_H$, it holds that $U(w) \cap L(x) = \emptyset$. Similarly, $U(y) \cap L(x) = \emptyset$ and $U(w) \cap L(z) = \emptyset$ also

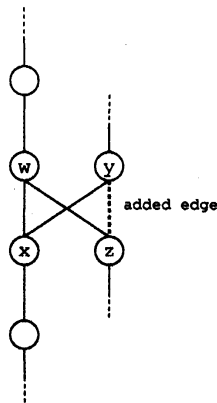


Figure 1: Add an edge to generate H' .

hold. In addition, as w and x are in the same $(\omega(G_c) - 1)$ -clique of G_c , $U(w) \cup L(x) = \{1, \dots, \omega(G_c) - 1\}$. Therefore, we can see that $U(y) \subseteq U(w)$ and $L(z) \subseteq L(x)$ hold. It follows that $U(y) \cap L(z) = \emptyset$. It means that even when (y, z) is added to H , no path of the Hasse diagram contains the same color (except two endpoints of the modulator). $\square$

Add edges satisfying the above condition as far as possible. Let the resulting Hasse diagram be H_+ and the corresponding comparability-1e graph be G_+ . Lemma2 shows that $\chi(G_+) = \chi(G)$. Let $V_a = \{v \mid a \prec v \text{ in } H_+\}$ and $V_b = \{v \mid v \prec b \text{ in } H_+\}$.

Lemma 3 *There exists an $(\omega(G_c) - 1)$ -coloring of G_+ iff there exists no $(\omega(G_c) - 1)$ -clique that does not include an endpoint of the modulator and whose all vertices are in $V_a \cup V_b$.*

Proof The vertices in V_a or V_b cannot be colored with the same color as a and b because the vertices in $V_a(V_b)$ and $a(b)$ are on the same path in H_+ . If there exists an $(\omega(G_c) - 1)$ -clique that does not include an endpoint of the modulator and whose all vertices are in $V_a \cup V_b$, there exists no $(\omega(G_c) - 1)$ -coloring of G_+ because only $\omega(G_c) - 2$ colors can be used to color the clique.

Otherwise, we can color all the vertices using colors $\{1, 2, \dots, \omega(G_c) - 1\}$ in the following manner. Color a and b with 1. In each $(\omega(G_c) - 1)$ -clique, color the vertex of the smallest level not in $V_a \cup V_b$ with 1. From the assumption, each $(\omega(G_c) - 1)$ -clique includes at least one vertex colored with 1. In addition, we can show that no clique includes more than one vertices colored with 1.

Assume that a vertex d is colored with 1. Then, there exists a vertex e satisfying $(e, d) \in H_+$, $e \in V_b$ and d, e are in the same $(\omega(G_c) - 1)$ -clique, and a vertex f satisfying $(e, f) \in H_+$ and $f \in V_b$. From the construction of H_+ , for each vertex g s.t. $(g, d) \in H_+$, an edge (g, f) must exist in H_+ . It means that any predecessor of d is included in V_b . Therefore, no two vertices in a $(\omega(G_c) - 1)$ -clique can be colored with 1.

Color the other vertices with the following rule: if v is a source, color v with 2, and otherwise color v with the minimum number which is not used in the predecessors of v in H_+ . We can observe that the coloring rule approves that no two vertices in a clique has the same color. In addition, we can show by induction on the level that the color of node

v with $\text{levmin}(v) = i$ is at most i . Therefore, the color of a sink is at most $\omega(G_c) - 1$. $\square$

[Algorithm COLOR-1E]

Input: comparability-1e graph $G = (V, E)$, modulator $E_1 = \{(a, b)\}$

Output: chromatic number of G

1. Compute a transitive orientation of $G_c = (V, E \cup E_1)$ (represented by a Hasse diagram $H = (V, E_H)$) and compute $\chi(G_c)$.
2. Repeat the following until no more edge is added.
Find the vertices w, x, y, z satisfying the following conditions: w and x are in the same $(\omega(G_c) - 1)$ -clique of G_c , $(w, x) \in E_H$, $(y, x) \in E_H$ and $(w, z) \in E_H$. If there exist such vertices, add an edge (y, z) to E_H .
Let G_+ be the obtained graph.
3. Compute $V_a = \{v \mid a \prec v \text{ in } H_+\}$ and $V_b = \{v \mid v \prec b \text{ in } H_+\}$.
4. Compute a subgraph of H_+ induced by $V_a \cup V_b - \{a, b\}$.
5. If the induced subgraph includes a path of length $\chi(G_c) - 1$, then output $\chi(G_c)$. Otherwise, output $\chi(G_c) - 1$.

Theorem 4 *Vertex coloring problem of comparability-1e graphs can be solved in polynomial time.*

3.2 Reduction to a Restricted Coloring of Comparability Graphs

In this section, we show that vertex coloring of comparability- k e graphs can be reduced to a kind of vertex coloring problem of comparability graphs. We define pair coloring problem as follows.

PAIR COLORING

Input : A graph $G = (V, E)$, a set of pairs of vertices $P \subseteq V^2$ and a positive integer k .

Output : If there exists a vertex coloring of G with k colors which colors u and v with the same color for all $(u, v) \in P$.

If there is no restriction on G , pair coloring is equivalent to vertex coloring of the graph obtained from G by identifying all the pairs of vertices in P . However, many graph classes including comparability graphs are not closed under identification of nonadjacent vertices.

Theorem 5 *Vertex coloring of comparability- k e graphs can be reduced to pair coloring of comparability graphs.*

Proof For an instance $\langle G = (V, E), k \rangle$ of vertex coloring of comparability- k e graphs, we construct an instance of pair coloring of comparability graphs $\langle G' = (V', E'), P, k' \rangle$ as follows.

Let M be the modulator of G and $G_t = (V, E_t)$ be a transitive graph obtained from $(V, E \cup M)$. Instead of G' , we define a transitive graph $G'_t = (V', E'_t)$ obtained from G' . Let $B = \{w \mid (u, w) \in E_t \text{ and } (w, v) \in E_t \text{ for some } (u, v) \in M\}$. For each vertex $w \in B$, we add a vertex w' . That is, $V' = V \cup \{w' \mid w \in B\}$. Let $E'_t = \{(r, w) \mid (r, w) \in E_t, w \in$

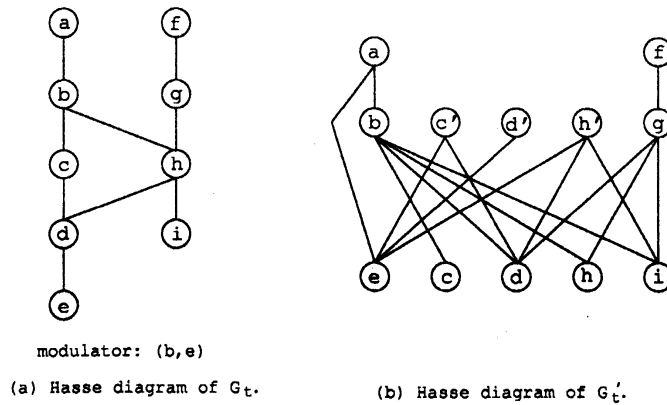


Figure 2: An example of reduction.

$B\} \cup \{(w', s) \mid (w, s) \in E_t, w \in B\} \cup \{(r, s) \mid (r, s) \in E_t - M, r \notin B, s \notin B\}$. Let the pairs of vertices be $P = \{\langle w, w' \rangle \mid w \in B\}$ and $k' = k$.

Now check that G'_t is really a transitive graph. Let $B' = \{w' \mid w \in B\}$. Note that there is no outgoing edges from vertices in B and no incoming edges from vertices in B' . Consider two edges $(a, b), (b, c) \in E'_t$. When $a, b, c \in V' - (B \cup B')$, both (a, b) and (b, c) are edges of E_t . As G_t is a transitive graph, there exists an edge $(a, c) \in E_t$, and thus $(a, c) \in E'_t$. When $c \in B$, there also exists an edge $(a, c) \in E_t$, and thus $(a, c) \in E'_t$. It is similar for the case when $a \in B'$. Fig.2 is an example of the G_t and G'_t represented as Hasse diagrams.

If we identify w and w' for all $w \in B$ on G' , the obtained graph is the same as G . Therefore, it is obvious that G' has a pair coloring with k colors iff G is k -colorable. $\square$

On the pair coloring problem, the number of pairs can be regarded as a parameter. However, as the number of pairs does not depend on the size of the modulator, this reduction is not a parameterized reduction.

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