

Stable and unstable manifolds of diffeomorphisms with positive entropy

Naoya Sumi (鷲見 直哉)

Abstract

We show that C^2 -diffeomorphisms with positive entropy are chaotic in the sense of Li-Yorke. To do so we prove that these maps are $*$ -chaotic on the closure of stable manifolds for some points. The notion of " $*$ -chaos" was introduced by Kato and it is related to chaos in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of diffeomorphisms with positive entropy. Notions of chaos have been given by Li and Yorke [9], Devaney [2] and others. It is known that if a continuous map of interval has positive entropy, then it is chaotic according to the definition of Li and Yorke (cf. [1]).

In [6] Katok proved the following :let f be a $C^{1+\varepsilon}$ -diffeomorphism of a closed surface. If the topological entropy of f is positive, then there exists a hyperbolic set Γ such that the restriction of f into Γ is topologically conjugate to a subshift of finite type with positive entropy. This implies that f is chaotic in the sense of Li-Yorke.

However, Katok's theorem does not hold for the high dimensional case. Indeed, let f be a surface diffeomorphism with positive entropy and let $r : S^1 \rightarrow S^1$ be an irrational rotation. Then a product map $f \times r$ has the same positive entropy, but it does not have Γ as above because there are no periodic points of $f \times r$.

In this paper we show the following:

Theorem A *Let f be a C^2 -diffeomorphism of a closed C^∞ -manifold. If the topological entropy of f is positive, then f is chaotic in the sense of Li-Yorke.*

To my knowledge this theorem gives the most simplest sufficient condition for chaotic phenomena of high dimensional dynamical systems. It remains a question whether Theorem A is true for homeomorphisms. However this question is still unsolved.

Let M be a closed C^∞ -manifold and let d be the distance for M induced by a Riemannian metric $\|\cdot\|$ on M . A subset S of M is a *scrambled set* of f if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set S of f , then we say that f is *chaotic in the sense of Li-Yorke*. Li and Yorke showed in [9] that if $f: [0, 1] \rightarrow [0, 1]$ is a continuous map with a periodic point of period 3, then f is chaotic in this sense. In [9] there was the following one more condition: for any $x \in S$ and any periodic point $p \in M$, $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(p)) > 0$. But this condition is unnecessary because a scrambled set contains at most one point which does not satisfy this condition. For the examples and the properties of scrambled sets, the readers may refer to [1], [5], [11], [12], [13], [22], [23] and [24].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "**-chaos*" as follows: let F be a closed subset of M . A map $f: M \rightarrow M$ is **-chaotic* on F (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ such that if U and V are any nonempty open subsets of F with $U \cap V = \emptyset$ and N is any natural number, there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and
2. for any nonempty open subsets U, V of F and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U$, $y \in V$.

Such a set F is called a **-chaotic set*. If S is a scrambled set, then the closure of S , $\bar{S}$, is a **-chaotic set*. In [4] Kato showed that the converse is true.

Lemma 1.1 ([4], Theorem 2.4) *If $f: M \rightarrow M$ is continuous and is *-chaotic on F , then there F_σ -set $S \subset F$ such that S is a scrambled set of f and $\bar{S} = F$. If F is perfect (i.e. F has no isolated points), we can choose S such that it is a countable union of Cantor sets.*

To obtain Theorem A we need the following theorem.

Theorem B *Let f be a C^2 -diffeomorphism of a closed C^∞ -manifold M and let μ be an f -invariant ergodic Borel probability measure on M . If the metric entropy of μ is positive, then for μ -almost all $x \in M$ the following hold:*

- (a) $\overline{W^s(x)}$ is a perfect **-chaotic set*, and
- (b) $\overline{W^u(x)}$ contains a perfect **-chaotic set*.

Here $W^s(x)$ and $W^u(x)$ are defined by

$$W^s(x) = \{y \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\} \text{ and,}$$

$$W^u(x) = \{y \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0\}$$

respectively.

We notice that for μ -almost all $x \in M$, the above sets $W^s(x)$ and $W^u(x)$ are C^2 immersed manifolds under the assumptions of theorem B. Indeed, let f and μ be as above. For μ -almost all $x \in M$, there exist a splitting of the tangent space $T_x M = \oplus_{i=1}^{s(x)} E_i(x)$ and real numbers $\lambda_1(x) < \lambda_2(x) < \dots < \lambda_{s(x)}(x)$ such that

(a) the maps $x \mapsto E_i(x)$, $\lambda_i(x)$ and $s(x)$ are Borel measurable, moreover $E_i(f(x)) = D_x f(E_i(x))$ and $\lambda_i(x)$, $s(x)$ are f -invariant ($i = 1, \dots, s(x)$),

(b) $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda_i(x)$ ($0 \neq v \in E_i(x)$, $i = 1, \dots, s(x)$) and

(c) $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{s(x)} \lambda_i(x) \dim E_i(x)$

([14]). The numbers $\lambda_1(x), \dots, \lambda_{s(x)}(x)$ are called *Lyapunov exponents* of f at x . Since μ is ergodic, we can put $s = s(x)$, $\lambda_i = \lambda_i(x)$ and $m_i = \dim E_i(x)$ ($i = 1, \dots, s$) for μ -almost all $x \in M$.

Let $h_\mu(f)$ denote the metric entropy of f (see [10] for definition). A well-known theorem of Margulis and Ruelle [21] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. $h_\mu(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i$. Since f has positive entropy, we have

$$0 < h_\mu(f) \leq \max\{\lambda_i\} = \lambda_s.$$

Therefore, by Pesin's stable manifold theorem ([3], [15], [17]), the set $W^u(x)$ is the image of a C^2 injective immersion of an euclidean space such that $T_x W^u(x) = \oplus_{\lambda_i > 0} E_i(x) (\neq \{0\})$ for μ -almost all $x \in M$. $W^u(x)$ is called an *unstable manifold*. Similarly, $W^s(x)$ is a C^2 immersed manifold because $W^s(x)$ is the unstable manifold of f^{-1} , which has positive entropy $h_\mu(f^{-1}) = h_\mu(f) > 0$. $W^s(x)$ is called a *stable manifold*.

Let us see how Theorem A follows from Theorem B. We denote as $h(f)$ the topological entropy of f (see [10] for definition). Then we know that $h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}_e(f)\}$ where $\mathcal{M}_e(f)$ is the set of all f -invariant ergodic Borel probability measures (cf [20]). Thus, if $h(f) > 0$, then we can choose $\mu \in \mathcal{M}_e(f)$ with $h_\mu(f) > 0$. Therefore, by Theorem B and Lemma 1.1, f is chaotic in the sense of Li-Yorke.

Remark that (a) of Theorem B does not hold for unstable manifolds in general. For example, the Smale horseshoe has unstable manifolds which intersect a stable manifold of a fixed point (cf. [18]). Since all points in the stable manifold converge to the fixed point, they do not satisfy the first condition of $*$ -chaos.

Now we shall give a sufficient condition that f is $*$ -chaotic on $\overline{W^u(x)}$ as follows.

Theorem C *If μ is an ergodic SRB measure, then both $\overline{W^s(x)}$ and $\overline{W^u(x)}$ are perfect $*$ -chaotic sets for μ -almost all $x \in M$.*

If ξ is a measurable decomposition of M , then a family $\{\mu_x^\xi | x \in M\}$ of Borel probability measures exists, and it satisfies the following conditions:

1. for $x, y \in M$ if $\xi(x) = \xi(y)$ then $\mu_x^\xi = \mu_y^\xi$, here $\xi(x)$ denotes a set containing x in ξ ,
2. $\mu_x^\xi(\xi(x)) = 1$ for μ -almost all $x \in M$,
3. for any Borel set A a function $x \mapsto \mu_x^\xi(A)$ is measurable and $\mu(A) = \int_M \mu_x^\xi(A) d\mu(x)$.

The family $\{\mu_x^\xi | x \in M\}$ is called a *canonical system of conditional measures* for μ and ξ (see [19] for more details).

An f -invariant Borel probability measure μ is called a *Sinai-Ruelle-Bowen measure* (SRB measure for abbreviation) provided

- (A) for μ -almost all $x \in M$, there exists a positive Lyapunov exponent of x ,
- (B) μ has a conditional measure that is absolutely continuous (with respect to the Lebesgue measure) on unstable manifolds, which is defined as follows:

From (A), the unstable manifold $W^u(x)$ is a C^2 submanifold for μ -almost all x in M . Let m_x^u denote the Lebesgue measure of $W^u(x)$. A measurable decomposition ξ of M is said to be *subordinate to unstable manifolds* if for μ -almost all x in M

- (C) $\xi(x) \subset W^u(x)$,
- (D) $\xi(x)$ contains an open neighborhood of x in $W^u(x)$.

We say that μ has an *absolutely continuous conditional measure* on unstable manifolds provided μ_x^ξ is absolutely continuous with respect to m_x^u for μ -almost all x in M if ξ is subordinate to unstable manifolds. It is known ([7], [8]) that μ satisfies (B) if and only if the following equation holds:

$$h_\mu(f) = \int \sum_{\lambda_i(x) > 0} \lambda_i(x) \dim E_i(x) d\mu(x).$$

This is sometimes known as *Pesin's formula*. For the examples and the stochastic properties of diffeomorphisms with SRB measures, the readers may refer to [25]. In a similar way we can define a measurable partition subordinate to stable manifolds.

2 Preliminaries

In this section we introduce f -invariant measurable partitions each of whose elements is contained in the closure of (un)stable manifolds. Let f be a C^2 -diffeomorphism of a closed C^∞ -manifold M and let μ be an f -invariant ergodic Borel probability measure on M with $h_\mu(f) > 0$. Denote as $\mathcal{B}$ the family of Borel sets.

Lemma 2.1 (Proposition 3.1 [7]) *Let f and μ be as above. Then there exist measurable partitions ξ^s and ξ^u of M such that*

- (a) $\xi^s \leq f\xi^s$ and $\xi^u \leq f^{-1}\xi^u$,
- (b) ξ^s and ξ^u are subordinate to stable manifolds and unstable manifolds respectively,
- (c) both $\bigvee_{n=0}^{\infty} f^n \xi^s$ and $\bigvee_{n=0}^{\infty} f^{-n} \xi^u$ are the partitions into points,
- (d) for μ -almost all $x \in M$,

$$\bigcup_{n=0}^{\infty} f^{-n} \xi^s(f^n(x)) = W^s(x) \quad \text{and} \quad \bigcup_{n=0}^{\infty} f^n \xi^u(f^{-n}(x)) = W^u(x).$$

Lemma 2.2 (Corollary 5.3 [8]) *Let f and μ be as above and let ξ^σ ($\sigma = s, u$) be as in Lemma 2.1. Then,*

$$\begin{aligned} h_\mu(f) &= H_\mu(f\xi^s|\xi^s) = \int -\log \mu_x^{\xi^s}(f\xi^s(x)) d\mu(x) \\ &= H_\mu(f^{-1}\xi^u|\xi^u) = \int -\log \mu_x^{\xi^u}(f^{-1}\xi^u(x)) d\mu(x) \end{aligned}$$

where the family $\{\mu_x^{\xi^\sigma} | x \in M\}$ is a canonical system of conditional measures for μ and ξ^σ .

Let us introduce two measurable partitions defined by

$$\eta^s = \bigwedge_{i=0}^{\infty} f^{-i} \xi^s \quad \text{and} \quad \eta^u = \bigwedge_{i=0}^{\infty} f^i \xi^u.$$

By Lemma 2.1(a) and (d) we can easily check that $f\eta^\sigma = \eta^\sigma$ and $\eta^\sigma(x) \subset \overline{W^\sigma(x)}$ for μ -almost all x ($\sigma = s, u$). Let $\{\mu_x^\sigma | x \in M\}$ be a canonical system of conditional measures for μ and η^σ ($\sigma = s, u$). By Doob's theorem it follows that for $A \in \mathcal{B}$

$$\mu_x^s(A) = \lim_{n \rightarrow \infty} \mu_x^{f^{-n}\xi^s}(A) \quad \text{and} \quad \mu_x^u(A) = \lim_{n \rightarrow \infty} \mu_x^{f^n\xi^u}(A) \quad (\mu\text{-almost all } x).$$

Since $f\eta^\sigma = \eta^\sigma$ and f preserves μ , we have $\mu_x^\sigma(A) = \mu_{fx}^\sigma(fA)$ (μ -almost all x) for $A \in \mathcal{B}$ and $\sigma = s, u$.

Let $C(M)$ be the Banach space of continuous real-valued functions of M with the sup norm $\|\cdot\|$, and let $\mathcal{M}(M)$ be a set of all Borel probability measures on M with the weak topology. Since $C(M)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \dots\}$ which is dense in $C(M)$. For $\nu, \nu' \in \mathcal{M}(X)$ define

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n \|\varphi_n\|}.$$

Then ρ is a compatible metric for $\mathcal{M}(X)$ and $(\mathcal{M}(X), \rho)$ is compact (cf.[10]).

Lemma 2.3 *Let f , μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for $\varepsilon > 0$ and $\sigma = s, u$ there exists a closed set F_ε^σ with $\mu(F_\varepsilon^\sigma) \geq 1 - \varepsilon$ satisfying the map*

$$F_\varepsilon^\sigma \ni x \mapsto \mu_x^\sigma \in \mathcal{M}(X)$$

is continuous.

Proof. Let $\{\varphi_1, \varphi_2, \dots\}$ be as above. By the definition of conditional measures the map

$$M \ni x \mapsto \int \varphi_n d\mu_x^\sigma$$

is Borel measurable for $n \geq 1$. Thus, by Lusin's theorem, for $n \geq 1$ there exists a closed set F_n^σ with $\mu(F_n^\sigma) \geq 1 - \varepsilon/2^n$ satisfying

$$F_n^\sigma \ni x \mapsto \int \varphi_n d\mu_x^\sigma : \text{continuous.}$$

Then $F_\varepsilon^\sigma = \bigcap_{n=1}^\infty F_n^\sigma$ has the desired property. □

Lemma 2.4 *Let f , μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for μ -almost all $x \in M$ and $\sigma = s, u$, $\text{supp}(\mu_x^\sigma)$ has no isolated points.*

Proof. We will give the proof for $\sigma = u$ and so we here omit for $\sigma = s$ since the technique of the proof is similar.

If this lemma is false, $\text{supp}(\mu_x^u)$ has an isolated point for x belonging to some Borel set with positive measure. Let ξ^u be as in Lemma 2.1. Since $\text{diam}(f^{-k}\xi^u(x)) \rightarrow 0$ ($k \rightarrow \infty$) by Lemma 2.1 (c),

$$P_{-k} = \{x \in M : \mu_x^{f^{-k}\xi^u} \text{ is a point measure}\}$$

has positive μ -measure for k large enough. Since μ is f -invariant, we have $f_*^n \mu_x^{f^{-k}\xi^u} = \mu_{f^n x}^{f^{n-k}\xi^u}$ for μ -almost all x and $n \in \mathbb{Z}$. Then

$$\begin{aligned} f^n(P_{-k}) &= \{f^n(x) \in M : \mu_x^{f^{-k}\xi^u} \text{ is a point measure}\} \\ &= \{x \in M : f_*^n \mu_{f^{-n}x}^{f^{-k}\xi^u} \text{ is a point measure}\} \\ &= \{x \in M : \mu_x^{f^{n-k}\xi^u} \text{ is a point measure}\} \\ &= P_{n-k} \quad (n \in \mathbb{Z}). \end{aligned}$$

Put

$$P = \bigcap_{j \geq 1} \bigcup_{n \geq j} P_{n-k} = \bigcap_{j \geq 1} \bigcup_{n \geq j} f^n P_{-k}.$$

Since P is f -invariant and μ is ergodic, we have $\mu(P) = 1$.

For $x \in P$ there exists an increasing sequence $\{n_i\}_{i \geq 0}$ such that $x \in P_{n_i}$ for $i \geq 0$. Since $\mu_x^u = \lim_{n \rightarrow \infty} \mu_x^{f^n \xi^u} = \lim_{i \rightarrow \infty} \mu_x^{f^{n_i} \xi^u}$ and $\mu_x^{f^{n_i} \xi^u}$ is a point measure for i , so is μ_x^u . Therefore μ_x^u is a point measure for μ -almost all $x \in M$.

Since ξ^u is finer than η^u and μ_x^u is a point measure for μ -almost all $x \in M$, so is $\mu_x^{\xi^u}$. Thus $\mu_x^{\xi^u}(f^{-1}\xi^u(x)) = 1$ for μ -almost all x . Therefore

$$h_\mu(f) = \int -\log \mu_x^{\xi^u}(f^{-1}\xi^u(x)) d\mu(x) = 0$$

by Lemma 2.2. This is a contradiction. $\square$

Let $B(x, r)$ and $U(x, r)$ denote the closed and open balls in M with center $x \in M$ and radius $r > 0$ respectively.

Lemma 2.5 *Let f , μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for μ -almost all $x \in M$*

$$\overline{W^s(y)} = \overline{W^s(x)} \quad (\mu_x^s\text{-almost all } y \in M).$$

Proof. Let ξ^s be as in Lemma 2.1. Then we have that for μ -almost all $x \in M$

$$\xi^s(y) \subset \overline{W^s(x)} \quad (\mu_x^s\text{-almost all } y).$$

Indeed, let d_x^s denote the distance induced by the Riemannian metric on $W^s(x)$. Then there exist an increasing family $\{\Lambda_\ell\}_{\ell \geq 1}$ of closed sets of M with $\mu(\cup_{\ell \geq 1} \Lambda_\ell) = 1$ and a sequence $\{A_\ell\}_{\ell \geq 1}$ of positive numbers satisfying that

(e) for each $x \in \Lambda_\ell$ there exists $\varepsilon = \varepsilon(x) > 0$, such that

$$B(x, \varepsilon) \cap \Lambda_\ell \ni y \mapsto W_{A_\ell}^s(y) = \{z \in W^s(y) : d_x^s(z, y) \leq A_\ell\}$$

is continuous with respect to the Hausdorff metric d_H : i.e.

$$\lim_{\Lambda_\ell \ni y \rightarrow x} d_H(W_{A_\ell}^s(y), W_{A_\ell}^s(x)) = 0,$$

(f) for each $x \in \Lambda_\ell$, $\xi^s(x) \subset W_{A_\ell}^s(x)$

(cf. [7], [15], [16]). Take arbitrary $y \in \text{supp}(\mu_x^s | \Lambda_\ell)$ ($\ell \geq 1$). Let $\varepsilon = \varepsilon(y) > 0$ be as in (e) and let $0 < r < \varepsilon$. Recall that for μ -almost all $x \in M$

$$\mu_x^s(\cup_{\ell \geq 1} \Lambda_\ell) = 1 \quad \text{and} \quad \mu_x^s | \Lambda_\ell = \lim_{n \rightarrow \infty} \mu_x^{f^{-n}\xi^s} | \Lambda_\ell \quad (\ell \geq 1).$$

Since $U(y, r)$ is open, we have $\mu_x^{f^{-n}\xi^s}(U(y, r) \cap \Lambda_\ell) > 0$ for n large enough. So we can take $y' \in U(y, r) \cap \Lambda_\ell \cap f^{-n}\xi^s(x)$. Since $y' \in f^{-n}\xi^s(x) \subset W^s(x)$, we have $W_{A_\ell}^s(y') \subset W^s(x)$. Since $y' \in U(y, r) \cap \Lambda_\ell$ and r is arbitrary, it follows that

$$\lim_{r \rightarrow 0} d_H(W_{A_\ell}^s(y'), W_{A_\ell}^s(y)) = 0.$$

Therefore $\xi^s(y) \subset W_{A_\ell}^s(y) \subset \overline{W^s(x)}$.

From this fact it follows that for $n \geq 0$ and μ -almost all $x \in M$

$$\xi^s(f^n(y)) \subset \overline{W^s(f^n(x))} \quad (\mu_x^s\text{-almost all } y).$$

Thus

$$W^s(y) = \cup_{n \geq 0} f^{-n} \xi^s(f^n y) \subset \cup_{n \geq 0} f^{-n} (\overline{W^s(f^n(x))}) \subset \overline{W^s(x)}$$

for μ_x^s -almost all y . On the other hand, by the definition of conditional measures, $\mu_y^s = \mu_x^s$ for $y \in \eta^s(x)$. This implies that $\overline{W^s(y)} = \overline{W^s(x)}$ for μ_x^s -almost all y . $\square$

3 Proof of Theorem B(a)

The purpose of this section is to show Theorem B(a). Let f be a C^2 -diffeomorphism of a closed C^∞ -manifold M and let μ be an f -invariant ergodic Borel probability measure on M with positive entropy. As described in §1 the stable manifold $W^s(x)$ is a C^2 immersed manifold for μ -almost all $x \in M$ and so the closure of $W^s(x)$, $\overline{W^s(x)}$, is perfect.

Let η^s and $\{\mu_x^s | x \in M\}$ be as in §2. By Lemma 2.4, $\text{supp}(\mu_x^s)$ has no isolated points for μ -almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 *If μ_x^s is not a point measure for μ -almost all $x \in M$, then $\overline{W^s(x)}$ is a $*$ -chaotic set for μ -almost all $x \in M$.*

Proof. Fix $0 < \varepsilon < 1$ and let F_ε^s be as in Lemma 2.3. By assumption we can take and fix $x_0 \in \text{supp}(\mu|_{F_\varepsilon^s})$ such that $\mu_{x_0}^s$ is not a point measure. Since $\text{supp}(\mu_{x_0}^s)$ is not one point, there are disjoint open sets O_1 and O_2 of M satisfying that

$$\begin{aligned} d(O_1, O_2) &= \inf\{d(x, y) : x \in O_1, y \in O_2\} > \delta \text{ and} \\ \mu_{x_0}^s(O_i) &> \delta \quad (i = 1, 2) \end{aligned} \tag{1}$$

for some $\delta > 0$. By Lemma 2.3 we can choose $\varepsilon' > 0$ such that

$$\mu_x^s(O_i) > \delta \quad (i = 1, 2) \tag{2}$$

for $x \in U(x_0, \varepsilon') \cap F_\varepsilon^s$. Put $K = \cap_{n=0}^\infty \cup_{k \geq n} f^{-k}(U(x_0, \varepsilon') \cap F_\varepsilon^s)$. Since $\mu(U(x_0, \varepsilon') \cap F_\varepsilon^s) > 0$, by ergodicity of μ we have $\mu(K) = 1$.

Take arbitrary δ' with $0 < \sqrt{\delta'} < \min\{\mu(U(x_0, \varepsilon') \cap F_\varepsilon^s), \delta\}$. Let ξ^s be as in Lemma 2.1 and put

$$A_m^s(n) = \left\{ x \in M \left| \begin{array}{l} d_H(f^{-[k/2]} \xi^s(f^{[k/2]} x), \overline{W^s(x)}) \leq 1/m, \\ \text{diam}(f^{k-[k/2]} \xi^s(f^{[k/2]} x)) \leq 1/m \quad (k \geq n) \end{array} \right. \right\} \tag{3}$$

for $n, m \geq 1$. Then $A_m^s(n) \subset A_m^s(n+1)$ and $\mu(\cup_{n=0}^\infty A_m^s(n)) = 1$ by Lemma 2.1 (c) and (d). Thus there exists an increasing sequence $\{n_m\}$ such that $\mu(A_m^s(n_m)) \geq 1 - \delta'/2^m$ ($m \geq 1$). Since

$$\int \mu_x^s(\cap_{m=1}^\infty A_m^s(n_m)) d\mu = \mu(\cap_{m=1}^\infty A_m^s(n_m)) \geq 1 - \sum_{m=1}^\infty \delta'/2^m = 1 - \delta',$$

we can find a Borel set $C_{\delta'}^s \subset M$ with $\mu(C_{\delta'}^s) \geq 1 - \sqrt{\delta'}$ satisfying

$$\mu_x^s(\cap_{m=1}^{\infty} A_m^s(n_m)) \geq 1 - \sqrt{\delta'} \quad (4)$$

for $x \in C_{\delta'}^s$. To obtain the conclusion it suffices to show that $\overline{W^s(x)}$ is a $*$ -chaotic set for $x \in K \cap C_{\delta'}^s$, because δ' is arbitrary.

For $x \in K \cap C_{\delta'}^s$, by the definition of K , there exists a sequence of positive integers $\{k_m\}_m$ with $k_m > n_m$ such that $f^{k_m}(x) \in U(x_0, \varepsilon') \cap F_{\varepsilon}^s$. Then by (4) and (2)

$$\begin{aligned} \mu_x^s(A_m^s(k_m)) &\geq \mu_x^s(\cap_{m=1}^{\infty} A_m^s(k_m)) \geq \mu_x^s(\cap_{m=1}^{\infty} A_m^s(n_m)) \geq 1 - \sqrt{\delta'}, \\ \mu_x^s(f^{-k_m}(O_i)) &= \mu_{f^{k_m}(x)}^s(O_i) > \delta > \sqrt{\delta'} \quad (i = 1, 2, m \geq 1). \end{aligned}$$

Thus we have $\mu_x^s(A_m^s(k_m) \cap f^{-k_m}(O_i)) > 0$ for $i = 1, 2$ and $m \geq 1$. From Lemma 2.5 we may assume that for $m \geq 1$ and $i = 1, 2$ there exists a point $y_i = y_i(m) \in A_m^s(k_m) \cap f^{-k_m}(O_i)$ such that

$$\overline{W^s(y_i)} = \overline{W^s(x)}.$$

By (3) we have

$$\begin{aligned} d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(x)}) &= d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(y_i)}) \leq 1/m, \\ \text{diam}(f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i)) &\leq 1/m \quad (i = 1, 2, m \geq 1). \end{aligned} \quad (5)$$

To show that $\overline{W^s(x)}$ is a $*$ -chaotic set, suppose that nonempty open sets U_1 and U_2 satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \overline{W^s(x)} \neq \emptyset \quad (j = 1, 2).$$

By (5) we may assume that

$$\begin{aligned} y_i &\in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j \neq \emptyset, \\ f^{k_m}(y_i) &\in f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \subset O_i \quad (1 \leq i, j \leq 2) \end{aligned} \quad (6)$$

if m is sufficiently large. Take

$$a_{i,j} = a_{i,j}(m) \in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j$$

for $1 \leq i, j \leq 2$. Then we have that for $1 \leq i, j \leq 2$

$$a_{i,j} \in U_j, \quad d(f^{k_m}(a_{1,1}), f^{k_m}(a_{2,2})) > \tau \quad \text{and} \quad d(f^{k_m}(a_{1,1}), f^{k_m}(a_{1,2})) < 1/m$$

by (1), (5) and (6). Since m is arbitrary, $\overline{W^s(x)}$ is a $*$ -chaotic set for $x \in K \cap C_{\delta'}^s$. □

4 Proof of Theorem B(b)

In this section we will prove Theorem B (b). Let f, μ, η^u and $\{\mu_x^u | x \in M\}$ be as in §2. By Lemma 2.4, $\text{supp}(\mu_x^u)$ has no isolated points for μ -almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 2 *If μ_x^u is not a point measure for μ -almost all $x \in M$, then $\text{supp}(\mu_x^u) \subset \overline{W^u(x)}$ is a $*$ -chaotic set for μ -almost all $x \in M$.*

Proof. Fix $0 < \varepsilon < 1$ and let F_ε^u be as in Lemma 2.3. By assumption we can take and fix $x_0 \in \text{supp}(\mu|F_\varepsilon^u)$ such that $\mu_{x_0}^u$ is not a point measure. Choose two distinct points $y_1, y_2 \in \text{supp}(\mu_{x_0}^u)$ and put $\tau = d(y_1, y_2)/2 (> 0)$. Take arbitrarily $0 < r < \tau/2$ and choose $\delta = \delta(r) > 0$ such that

$$\mu_{x_0}^u(U(y_i, r)) > \delta \quad (i = 1, 2). \quad (7)$$

Remark that

$$d(U(y_1, r), U(y_2, r)) = \inf\{d(x, y) : d(x, y_1) < r, d(y, y_2) < r\} > \tau. \quad (8)$$

Since $U(y_i, r)$ ($i = 1, 2$) are open, by (7) there exists a large integer $M = M(r) > 0$ such that

$$\nu(U(y_i, r)) > \delta = \delta(r) \quad (i = 1, 2) \quad (9)$$

for $\nu \in \mathcal{M}(M)$ with $\rho(\nu, \mu_{x_0}^u) < 1/M$. We can find $\varepsilon' = \varepsilon'(r) > 0$ such that

$$\rho(\mu_x^u, \mu_{x_0}^u) < 1/2M = 1/2M(r) \quad (x \in U(x_0, \varepsilon') \cap F_\varepsilon^u). \quad (10)$$

Note that ε' depends on r .

Let ξ^u be as in Lemma 2.1 and put

$$B_m^u(n) = \left\{ x \in M \left| \begin{array}{l} \rho(\mu_x^{f^k \xi^u}, \mu_x^u) \leq 1/m, \\ \text{diam}(f^{-k+[k/2]} \xi^u(f^{-[k/2]} x)) \leq 1/m \quad (k \geq n) \end{array} \right. \right\} \quad (11)$$

for $n, m \geq 1$. Then $B_m^u(n) \subset B_m^u(n+1)$ and $\mu(\bigcup_{n=0}^\infty B_m^u(n)) = 1$, by Lemma 2.1 (c) and Doob's theorem. Thus there exists an increasing sequence $\{n_m\}$ such that $\mu(B_{n_m}^u(n_m)) \geq 1 - 1/2^{m+1}$ ($m \geq 1$). Since $\mu(\bigcap_{k=m}^\infty B_k^u(n_k)) \geq 1 - 1/2^m$ for $m \geq 1$, we can find a Borel set D_m^u with $\mu(D_m^u) \geq 1 - 2^{-m/2}$ satisfying

$$\mu_x^u(\bigcap_{k=m}^\infty B_k^u(n_k)) \geq 1 - 2^{-m/2} \quad (x \in D_m^u). \quad (12)$$

Put

$$K_r = \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty \left(\bigcap_{n=0}^\infty \bigcup_{\ell \geq n} f^{-\ell}(U(x_0, \varepsilon'(r)) \cap F_\varepsilon^u \cap D_m^u) \right) \quad (0 < r < \tau/2).$$

Then $\mu(K_r) = 1$ ($0 < r < \tau/2$) by ergodicity of μ . To obtain the conclusion it suffices to show that $\text{supp}(\mu_x^u)$ is a $*$ -chaotic set for $x \in K = \bigcap_{n \geq 1} K_{1/n}$.

To do this fix $x \in K_r$ ($r = 1/n, n \geq 1$) and suppose that nonempty open sets U_1 and U_2 satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \text{supp}(\mu_x^u) \neq \emptyset \quad (j = 1, 2).$$

Choose $m_0 > 0$ with

$$0 < 2^{-m_0/2} < \min\{\mu_x^u(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2M.$$

Since $x \in K_r$, by the definition of K_r , there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}_k$ with $\ell_k > n_k$ such that

$$f^{\ell_k}(x) \in U(x_0, \varepsilon'(r)) \cap F_\varepsilon^u \cap D_{m_1}^u \quad (k \geq 1). \quad (13)$$

Since

$$\begin{aligned} \mu_x^u(f^{-\ell_k}(B_k^u(n_k))) &\geq \mu_x^u(f^{-\ell_k}(\cap_{k=m_1}^\infty B_k^u(n_k))) \\ &= \mu_{f^{\ell_k}(x)}^u(\cap_{k=m_1}^\infty B_k^u(n_k)) \\ &\geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1) \end{aligned}$$

by (12), we have

$$\mu_x^u(U_j \cap f^{-\ell_k}(B_k^u(n_k))) \geq \mu_x^u(U_j) - 2^{-m_0/2} > 0 \quad (k \geq m_1).$$

Then, by the definition of $\{\mu_x^u | x \in M\}$, we can choose

$$z_j = z_j(k) \in U_j \cap f^{-\ell_k} B_k^u(n_k)$$

with $\eta^u(x) = \eta^u(z_j)$ for $j = 1, 2$ and $k \geq m_1$. Thus we have $\eta^u(f^{\ell_k}(x)) = \eta^u(f^{\ell_k}(z_j))$ and so $\mu_{f^{\ell_k}(x)}^u = \mu_{f^{\ell_k}(z_j)}^u$. Since $f^{\ell_k}(z_j) \in B_k^u(n_k) \subset B_k^u(\ell_k)$, by (11) we have

$$\rho(\mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}, \mu_{f^{\ell_k}(z_j)}^u) = \rho(\mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}, \mu_{f^{\ell_k}(z_j)}^u) \leq 1/k \leq 1/m_0 \leq 1/2M, \quad (14)$$

$$\text{diam}(f^{-\ell_k + [\ell_k/2]}\xi^u(f^{\ell_k - [\ell_k/2]}(z_j))) \leq 1/k \quad (15)$$

for $j = 1, 2$ and $k \geq m_1$. Thus, by (9), (10), (13) and (14),

$$\mu_{z_j}^{\xi^u}(f^{-\ell_k}U(y_i, r)) = \mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}(U(y_i, r)) > \delta.$$

Since $\text{supp}(\mu_{z_j}^{\xi^u}) \subset \xi^u(z_j)$, we have

$$\xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset$$

for $1 \leq i, j \leq 2$ and $k \geq m_1$. For k large enough, by (15) we may assume

$$z_j \in f^{-\ell_k + [\ell_k/2]}\xi^u(f^{\ell_k - [\ell_k/2]}(z_j)) \subset U_j.$$

Therefore

$$\begin{aligned} U_j \cap f^{-\ell_k}U(y_i, r) &\supset f^{-\ell_k + [\ell_k/2]}\xi^u(f^{\ell_k - [\ell_k/2]}(z_j)) \cap f^{-\ell_k}U(y_i, r) \\ &\supset \xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset \end{aligned}$$

for $1 \leq i, j \leq 2$ and k large enough. Take

$$b_{i,j} = b_{i,j}(k) \in f^{-\ell_k}(U(y_i, r)) \cap U_j$$

for $1 \leq i, j \leq 2$. Then we have that for $1 \leq i, j \leq 2$

$$b_{i,j} \in U_j, \quad d(f^{\ell_k}(b_{1,1}), f^{\ell_k}(b_{2,2})) > \tau \quad \text{and} \quad d(f^{\ell_k}(b_{1,1}), f^{\ell_k}(b_{1,2})) < r = 1/n$$

by (8). This implies that $\text{supp}(\mu_x^u)$ is a $*$ -chaotic set for $x \in K = \bigcap_{n \geq 1} K_{1/n}$. $\square$

5 Proof of Theorem C

The purpose of this section is to show Theorem C. Let f be a C^2 -diffeomorphism of a closed C^∞ -manifold M and let μ be an ergodic SRB measure. As described in §1 the Pesin's formula holds: i.e. $h_\mu(f) = \sum_{\lambda_i > 0} \lambda_i m_i$. Thus we have $h_\mu(f) \geq \max\{\lambda_i\} > 0$ because μ satisfies the condition (A) mentioned in §1. Therefore, by Theorem B, $\overline{W^s(x)}$ is a $*$ -chaotic set for μ -almost all $x \in M$.

To show that $\overline{W^u(x)}$ is a $*$ -chaotic set we need the following lemma.

Lemma 5.1 ([8], Corollary 6.1.4) *Let μ be an ergodic measure satisfying Pesin's formula, let ξ^u be as in Lemma 2.1 and let ψ be the density of $\mu_x^{\xi^u}$ with respect to m_x^u . Then at μ -almost all x , ψ is a strictly positive function on $\xi^u(x)$ and $\log \psi$ is Lipschitz along W^u -leaves.*

Let η^u and $\{\mu_x^u | x \in M\}$ be as in §2. Then, by Proposition 2, $\text{supp}(\mu_x^u) \subset \overline{W^u(x)}$ is a $*$ -chaotic set for μ -almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 3 *If μ is an SRB measure, then $\text{supp}(\mu_x^u) = \overline{W^u(x)}$ for μ -almost all $x \in M$.*

Proof. We first show that $\xi^u(x) \subset \text{supp}(\mu_x^u)$ for μ -almost all $x \in M$. Since ξ^u is finer than η^u , for μ -almost all $z \in M$

$$\int \mu_x^{\xi^u}(\text{supp}(\mu_z^u)) d\mu_z^u(x) = \mu_z^u(\text{supp}(\mu_z^u)) = 1.$$

Then $\mu_x^{\xi^u}(\text{supp}(\mu_z^u)) = 1$ for μ_z^u -almost all x . Since $\text{supp}(\mu_z^u)$ is closed, by Lemma 5.1 we have that

$$\xi^u(x) \subset \text{supp}(\mu_z^u) = \text{supp}(\mu_x^u)$$

for μ_z^u -almost all x . Therefore $\xi^u(x) \subset \text{supp}(\mu_x^u)$ for μ -almost all x .

Since $f_*\mu_x^u = \mu_{f(x)}^u$ for μ -almost all x , we have $f(\text{supp}(\mu_x^u)) = \text{supp}(\mu_{f(x)}^u)$ for μ -almost all x . By Lemma 2.1 (d)

$$\begin{aligned} W^u(x) &= \bigcup_{n=0}^{\infty} f^n \xi^u(f^{-n}(x)) \\ &\subset \bigcup_{n=0}^{\infty} f^n(\text{supp}(\mu_{f^{-n}(x)}^u)) \\ &\subset \text{supp}(\mu_x^u) \end{aligned}$$

for μ -almost all $x \in M$. Therefore $\text{supp}(\mu_x^u) = \overline{W^u(x)}$ for μ -almost all $x \in M$. $\square$

References

- [1] L.S. Block and W.A. Coppel, *Dynamics in one dimension*, Lecture Notes in Math., Springer-Verlag, 1513, 2151-2157.
- [2] R. Devaney, *An introduction to chaotic dynamical systems*, Benjamin/Cummings, 1986.
- [3] A. Fathi, M.R. Herman and J-C. Yoccoz, *A proof of Pesin's stable manifold theorem*, in Geometric Dynamics, Springer Lecture Notes in Math. 1007, 117-215 (1983), 689-698.
- [4] H. Kato, *On scrambled sets and a theorem of Kuratowski on independent sets*, Proc. Amer. Math. Soc., 126 (1998), 2151-2157.
- [5] H. Kato, *Chaotic continua of (continuum-wise) expansive homeomorphisms and chaos in the sense of Li and Yorke*, Fund. Math., 145 (1994), 261-279.
- [6] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.E.S., 51 (1980), 137-174.
- [7] F. Ledrappier and J.M. Strelcyn, *A proof of the estimation from below in Pesin entropy formula*, Ergod. Theory & Dynam. Syst., 2 (1982), 203-219.
- [8] F. Ledrappier and L.S. Young, *The metric entropy of diffeomorphisms*, Ann. Math., 122 (1985), 509-574.
- [9] T.Y. Li and J.A. Yorke, *Period three implies chaos*, Amer. Math. Monthly, 82 (1975), 985-992.
- [10] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer, Berlin, 1987.
- [11] F.R. Marotto, *Snap-back repellers imply chaos in $\mathbb{R}^n$* , J. Math. Analysis and Appl., 63 (1978), 199-223.
- [12] M. Misiurewicz, *Chaos almost everywhere*, Lecture Notes in Math. 1163 (1985), 125-130.
- [13] M. Misiurewicz and J. Smítal, *Smooth chaotic maps with zero entropy*, Ergod. Th. & Dynam. Sys., 8 (1988), 421-424.
- [14] V.I. Oseledec, *A multiplicative ergodic theorem: Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), 197-231.
- [15] Y.B. Pesin, *Families of invariant manifolds corresponding to non-zero characteristic exponents*, Math. of the USSR, Izvestija, 10 (1978), 1261-1305.

- [16] Y.B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russ. Math. Surveys, 32:4 (1977), 55-114.
- [17] C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc., 312 (1989), 1-54.
- [18] C. Robinson, *Dynamical systems: stability, symbolic dynamics, and chaos*, CRC Press, 1994.
- [19] V.A. Rohlin, *On the fundamental ideas of measure theory*, A.M.S. Transl., (1)10 (1962), 1-52.
- [20] V.A. Rohlin, *Lectures on the theory of entropy of transformations with invariant measures*, Russ. Math. Surveys, 22:5 (1967), 1-54.
- [21] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Bras. Math., 9 (1978), 83-87.
- [22] J. Smítal, *A chaotic function with some extremal properties*, Proc. Amer. Math. Soc., 87:1 (1983), 54-56.
- [23] J. Smítal, *A chaotic function with a scrambled set of positive Lebesgue measure*, Proc. Amer. Math. Soc., 97 (1984), 50-54.
- [24] J. Xiong and Z. Yang, *Chaos caused by a topologically mixing map*, World Scientific, Advanced Series in Dynamical Systems Vol 9 (1991), 550-572.
- [25] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Annals of Mathematics 147 (1998), 558-650.