

4. Remarks on Smoothings of Four-Spaces

Ken'ichi KUGA

One of the striking consequences of Michael Freedman's topological theory of 4-manifolds and Simon Donaldson's non-existence results of certain smooth 4-manifolds is the existence of an exotic smoothing of the euclidean 4-space  $\mathbb{R}^4$ . Then an example of a manifold with finitely generated homology groups admitting infinitely many smooth structures was found along the same lines [G] [K]. More recently Taubes showed the existence of uncountably many smooth structures on  $\mathbb{R}^4$  [T]. Actually Taubes' argument applies to a fairly large class of open 4-manifolds, and it may now be possible to expect uncountably many smooth structures on every non-compact 4-manifold.

In this informal note, we give some remarks and observations concerning smooth structures on non-compact 4-manifolds which seem to indicate the complicated nature of the problem: In §1 we provide a natural construction which possibly produces uncountably many smoothings and discuss some problems on the construction; In §2 we give some observations which shows some difficulties to reasonably topologize the set of smooth structures on a non-compact 4-manifold.

§1 Given a non-compact 4-manifold  $V^4$ , consider the following construction which is an immediate generalization of Taubes' construction to arbitrary non-compact manifold and actually produces uncountably

many smoothings in many cases (e.g. when an end of  $V$  is diffeomorphic to  $S^3 \times \mathbb{R}$ ).

1.1 Construction: Fix an exotic smoothing of  $\mathbb{R}^4$ , denoted  $R$ , which is standard on  $(-\infty, 0) \times \mathbb{R}^3$ , and a smooth properly embedded half-open ray  $A$  in  $V^4$ . Identifying an open tubular neighborhood of  $A$  with  $(-\infty, 0) \times \mathbb{R}^3$  in  $R$  so that the open end of  $A$  goes to  $0 \times \mathbb{R}^3$ , we can form an end connected sum of  $V^4$  with  $R$ , denoted  $V^4 \natural R$ , which is homeomorphic to  $V^4$ . Then we can define a continuously parametrized smooth submanifolds  $V(r)$  of  $V^4 \natural R$  for  $0 < r \leq \infty$  by setting  $V(r) = V^4 \cup (\text{open ball of radius } r \text{ centered at the origin in } R)$ , which are homeomorphic to  $V^4$ .

1.2 Remark: When the end of  $V$  is diffeomorphic to an end of a punctured definite 1-connected 4-manifold with non-standard intersection form, uncountably many distinct smoothings of  $V^4$  can be obtained by taking continuously parametrized parallel ends as in [T]. It is not clear, however, that the above construction (where  $R$  is connected to  $V$  along the standard structure) yields uncountably many smoothings in these cases.

1.3 Remark: If the smooth structure on the end of  $V^4$  is sufficiently complicated, the above construction fails. For example, set  $V^4 =$  a universal smoothing of  $\mathbb{R}^4$  in [FT]. Then, for any choice of  $R$ ,  $V(r)$ 's are all diffeomorphic to  $V^4$ , i.e., the above construction cannot produce any new smoothing.

1.4  $P = \#_{n=1}^{\infty} S^2 \times S^2$ : Also, if the end of  $V^4$  is topologically complicated, the above construction might fail. A candidate to this is an infinite connected sum of  $S^2 \times S^2$ . More specifically, consider a countable sequence of small disjoint 4-balls  $D_n^4$  in the standard  $\mathbb{R}^4$

centered at points  $(n,0,0,0)$ ,  $n = 1,2, \dots$ , and take connected sums with countably many copies of  $S^2 \times S^2$ , denoted  $(S^2 \times S^2)_n$ , at  $D_n^4$ 's.

The resulting manifold  $P^4 = \#_{n=1}^{\infty} (S^2 \times S^2)_n$  is an open smooth 4-manifold with one end whose homology groups are infinitely generated.

The following observation is an easy consequence of techniques in [FT] which shows the complicatedness of the smooth manifold  $P$ .

1.5 Proposition: If  $Q^4$  is a smooth 4-manifold topologically homeomorphic to  $P$  (i.e. a possibly different smoothing of  $P$ ). Then  $Q$  can be smoothly embedded into  $P$  in such a way that  $\text{int}(P - \text{Image}(Q))$  is topologically an open 4-ball.

Proof First observe that any smoothing of  $\mathbb{R}^4$ , say  $U$ , can be smoothly embedded into  $P$ . In fact, one can construct a proper h-cobordism consisting of (small) 2- and 3-handles between  $P$  and  $U \# (\#_{n=1}^{\infty} (S^2 \times S^2)_n)$  which is topologically a product and smoothly a product near  $\bigcup_{n=1}^{\infty} ((S^2 \times S^2)_n - D_n^4)$ . The smooth Whitney tricks may be performed after removing self-intersections of Whitney disks by Norman tricks in  $(S^2 \times S^2)_n - D_n^4$  in the middle level and we get a diffeomorphism  $U \# (\#_{n=1}^{\infty} (S^2 \times S^2)_n) \cong P$ . Then we get an embedding of  $U \cong U - (\text{tubular neighborhood of a smooth proper half-open arc which joins } D_n^4\text{'s in } U)$  into  $P$ .

Next consider the universal smoothing  $H$  of  $[0, \infty) \times \mathbb{R}^3$  constructed in [FT]. As above we can smoothly embed  $H$  into  $P$ , call the image  $H_+$ , so that the end of  $H_+$  goes to the end of  $P$  (i.e. a proper embedding). Consider a smooth proper h-cobordism  $W^5$  between  $P$  and  $Q$  consisting of (small) 2- and 3-handles which is topologically a product (Note that we can extend the smooth structure on  $P$  and  $Q$  to  $W$ , since the obstruction  $H^4(\text{relative}; \pi_3(\text{TOP/PL})) = 0$ ). Then as in [FT], resmooth-

ing a neighborhood  $N$  of properly embedded half-open ray  $A$  in  $H_+$  in  $P$  crossed with  $[0,1]$  in  $W^5$  by plugging  $Hx[0,1]$  into  $Nx[0,1]$  in  $W^5$ , smooth Whitney tricks may be performed after smoothing the cores of suitably chosen Casson Handles in the middle level using the resmoothing  $Hx(1/2)$ . Then we get a diffeomorphism  $(Q - (Nx(1))) \cup_2 (Hx(1)) \cong (P - (Nx(0))) \cup_2 (Hx(0))$ , and hence a smooth embedding of  $Q \cong (Q - Nx(1))$  into the latter manifold ( $P$  with  $Nx0$  replaced by  $Hx0$ ) so that the complement of the closure of the image is topologically homeomorphic to the open 4-ball. Finally this manifold,  $P$  with  $Nx0$  replaced by  $Hx0$ , is actually diffeomorphic to  $P$ , since  $Hx0$  is absorbed into the universal smoothing  $H_+$  originally embedded in  $P$ .

§ 2 Since there are uncountably many smoothings of a non-compact 4-manifold (at least for many cases), the following seems to be a natural question: Can we find a reasonable (Hausdorff, etc.) topology on the set of smooth structures on a non-compact 4-manifold ( $\mathbb{R}^4$ , for example)?

Let  $S_4$  be the set of smooth structures on  $\mathbb{R}^4$ . One can define a natural distance (admitting  $\infty$ ), call Lipschitz-Shikata distance, on  $S_4$  as follows: For  $U, V \in S_4$ , define  $d(U, V) \in [0, \infty]$  by  $d(U, V) = \inf_{d, \bar{d}} (\inf_h (\log \max(|h|, |h^{-1}|)))$ , where  $d$  and  $\bar{d}$  run through all complete riemannian metrics compatible with the smooth structures  $U$  and  $V$  respectively, and  $h$  runs through all onto homeomorphism  $U \rightarrow V$ , and  $|h| = \sup_{x \neq y, x \in U} (\bar{d}(h(x), h(y)) / d(x, y)) \in [0, \infty]$ . (It is non-trivial that  $d(U, V) = 0$  implies  $U = V$ , which is a consequence of the following proposition.) One can readily generalize a result of Shikata [S] to non-compact manifolds and get:

2.1 Proposition:  $S_4$  is a discrete space with respect to the Lipschitz-Shikata distance. In other words, if two exotic  $\mathbb{R}^4$ 's admit a sufficiently small lipeomorphism with respect to some riemannian metrics, then they are diffeomorphic.

Proof Fix an isometric imbedding of  $V$  into the euclidean space  $\mathbb{R}^m$  for sufficiently large  $m$  (Nash imbedding theorem is valid for non-compact manifolds), and take a tubular neighborhood  $N$  of  $V$  in  $\mathbb{R}^m$  so that the fibers of  $\pi : N \rightarrow V$  are orthogonal to  $V$  in  $\mathbb{R}^m$ . Then, as usual, the continuous  $\mathbb{R}^m$ -valued function  $f = i \circ h : U \rightarrow V \hookrightarrow \mathbb{R}^m$  (where  $h : U \rightarrow V$  is a small lipeomorphism), is approximated by a  $C^\infty \mathbb{R}^m$ -valued function  $f_t(x) = \int_U f(y) g_t(x,y) dy$ ,  $x \in U$ , for  $t > 0$ , a coordinatewise integration over the riemannian manifold  $(U,d)$ , where  $g_t(x,y)$  is given by  $g_t(x,y) = g(d(x,y)/t) / (\int_U g(d(x,y)/t) dx)$  and  $g$  is a non-negative  $C^\infty$  function:  $\mathbb{R} \rightarrow [0, \infty)$  with compact support and  $g = \text{constant}$  near  $0 \in \mathbb{R}$ .

Consider the composition  $F(x,t) = \pi \circ f_t(x)$  for  $x \in U$ ,  $t > 0$ , and  $F(x,0) = h(x)$ , for  $x \in U$ , where  $\pi$  is the orthogonal projection of  $N$  onto  $V$ . Although this composition is not defined everywhere, there is a positive number  $t(L)$  for each compact subset  $L$  of  $U$  such that  $F(x,t)$  is well-defined on  $L \times [0, t(L)]$ , i.e.,  $f_t(x) \in N$  for  $x \in L$ ,  $0 \leq t \leq t(L)$ . Furthermore, Shikata's proof in [S] shows that there is a positive constant  $c$  (independent of the manifolds  $U$ ,  $V$  or the choice of isometric imbedding of  $V$ ) such that if  $|h|$  and  $|h^{-1}|$  are both  $< c$ , and if  $t(L) > 0$  is sufficiently small (depending on  $L$ ), then  $F$  defines a  $C^\infty$  embedding of  $(\text{neighborhood of } L) \times (0, t(L))$  for  $0 < t \leq t(L)$ .

Hence we can find codimension 0 compact submanifolds  $M_n, L_n$  of  $U$  for  $n = 1, 2, \dots$ , and a decreasing sequence  $t_1 > t_2 > \dots$  of positive numbers ( $0 < t_n < t(L_n)$ ) such that: (i)  $\bigcup_{n=1}^{\infty} M_n = \bigcup_{n=1}^{\infty} L_n = U$ ,

$M_n \subseteq L_n \subseteq M_{n+1}$ ; (ii)  $F|_{L_n x(t)}$  is a  $C^\infty$  embedding of  $L_n \cong L_n x(t)$  into  $V$  for  $0 < t \leq t_n$ ; (iii)  $F(M_n x[0, t_n]) \subset F(L_n x(t_n)) \subset F(M_{n+1} x(t_{n+1}))$ .

Consider the function  $\bar{F}(x, t) = (F(x, t), t) \in Vx[0, \infty)$  for  $(x, t)$  in a neighborhood of  $X \equiv \bigcup_{n=1}^{\infty} (L_n x[0, t_n])$  in  $Ux[0, \infty)$ . Then  $\bar{F}$   $C^\infty$  embeds  $X - Ux(0)$  into  $Vx(0, \infty)$  (differentiability of  $\bar{F}$  is assured by the compactness of support of  $g$  in the definition of  $f_t(x)$ ). One can construct a  $C^\infty$  vector field  $\mathbb{X}$  on  $Vx(0, \infty)$  by partition of unity argument with the following properties: (i)  $dp_2(\mathbb{X}) = d/dt$  (where  $p_2$  is the projection  $Vx(0, \infty) \rightarrow (0, \infty)$ ); (ii)  $\mathbb{X} = d\bar{F}(\partial/\partial t)$  on  $\bar{F}(\bigcup_{n=1}^{\infty} M_n x(0, t_n])$ ; (iii)  $\mathbb{X} = \partial/\partial t'$  outside a neighborhood of  $\bigcup_{n=1}^{\infty} (F(L_n x(t_n)) \times (0, t_n])$ .

Fixing a number  $t_0 > t_1$ , let  $\Pi : V \times (0, \infty) \rightarrow V$  be the projection along the  $C^\infty$  flow  $\mathbb{X}$  onto  $V = Vx(t_0)$ , i.e.  $\Pi(y, t)$  is the unique intersection of  $Vx(t_0)$  with the trajectory of  $\mathbb{X}$  through  $(y, t)$ . Then the desired diffeomorphism  $\bar{h} : U \rightarrow V$  may be written as the union  $\bar{h}(x) = \bigcup_{n=1}^{\infty} \Pi \circ \bar{F}|_{M_n x(t_n)}(x, t_n)$ , which is well-defined from the construction. □

2.2 Remark: The above argument is valid for any non-compact manifolds of any dimension (the constant  $c$  depends only on the dimension).

2.3 Remark: The Lipschitz-Shikata distance is, hence, too strong.

There are some candidates for defining weaker topologies on  $S_4$ .

Consider the following spaces of embeddings: (i)  $E_1$  = topological embeddings of the unit open 4-ball into a universal smoothing  $U$  of  $\mathbb{R}^4$  in [FT]; (ii)  $E_2$  = topological embeddings of the unit open 4-ball into  $P = \#_{n=1}^{\infty} (S^2 \times S^2)_n$  described in 1.4; (iii)  $E_3$  =  $C^\infty$  proper embeddings of the universal smoothing  $H$  of the half-space  $[0, \infty) \times \mathbb{R}^3$  in [FT] into  $H$ ,  $f : H \rightarrow H$ . Then we have projections  $p_i : E_i \rightarrow S_4$ , defined by  $p_i(f) = \text{Image of } f \text{ with the induced smooth structure, for}$

$i = 1, 2$ , and  $p_3(f) = \text{Int}(H - \text{Image}(f))$  ( $p_1$  and  $p_3$  are surjective by [FT], and  $p_2$  is surjective from the argument in Proof of 1.5). Hence any topology on  $E_i$  induces a quotient topology on  $S_4$ . It seems, however, not easy to make this topology Hausdorff. For example, if we put compact-open topology on  $E_1$ , the only open sets of  $S_4$  will be the whole set and the empty set.

2.4 Remark: It would be nice if one could define a reasonable topology on  $S_4$  with possibly accessible homotopy groups. Related to this is the following naive question: Is there a reasonable topology on  $S_4$  such that the singular complex  $S(S_4)$  is identifiable with the Kan complex  $\text{DIFF}(\mathbb{R}^4)$  of sliced families of smooth structures on  $\mathbb{R}^4$  ( $S_4$  is the set of vertices  $\text{DIFF}(\mathbb{R}^4)^0$ )? Again, this topology cannot be Hausdorff, since a universal smoothing  $U$  is contained in any neighborhood of any element and the only neighborhood of the standard structure is the whole set.

## References

- [G] R. Gompf, Infinite families of Casson Handles and topological disks, *Topology* 23 (1984) 395-400.
- [K] K. Kuga, On immersed 2-spheres in  $S^2 \times S^2$ , to appear
- [T] C. Taubes, Gauge theory on asymptotically periodic 4-manifolds, to appear in *J. Diff. Geom.*
- [S] Y. Shikata, On a distance function on the set of differentiable structures, *Osaka J. Math.* 3 (1966), 65-79.