

On uniqueness for Schrödinger maps with low regularity large data *

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Abstract

We prove that the solutions to the initial-valued problem for the 2-dimensional Schrödinger maps are unique in $C_t L_x^\infty \cap L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)$. For the proof, we follow McGahagan's argument with improving its technical part, combining Yudovich's argument.

1 Introduction

We consider the initial value problem for the Schrödinger map equation in the two dimensional case:

$$\begin{cases} \partial_t u = u \times \Delta u & \text{on } \mathbb{R}^2 \times \mathbb{R}, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^2, \end{cases} \quad (1.1)$$

where $u = u(x, t)$ is the unknown function from $\mathbb{R}^2 \times \mathbb{R}$ to a sphere

$$\mathbb{S}^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \subset \mathbb{R}^3,$$

and $\times$ denotes the vector product of vectors in $\mathbb{R}^3$.

In the physical context, (1.1) is considered as a mathematical model of the evolution of magnetization vectors in ferromagnetic materials. (In this context, (1.1) is usually called the Landau-Lifschitz equation. For more information, see [5] for example.)

The equation (1.1) has the energy conservation, where the energy is given by

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 \, dx.$$

Our main theorem is the uniqueness of solutions to (1.1) in the class of H^2 -valued solutions. The precise statement is the following:

Theorem 1.1. (*Uniqueness*) Let $u^{(0)}, u^{(1)} \in C(I : L^\infty(\mathbb{R}^2)) \cap L^\infty(I : \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2))$ be two solutions to (1.1), where $I \subset \mathbb{R}$ is a time interval including 0. Suppose that $u^{(0)}|_{t=0} = u^{(1)}|_{t=0}$ on $\mathbb{R}^2$. Then, $u^{(0)} = u^{(1)}$ on $\mathbb{R}^2 \times I$.

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Remark 1.1. (a) Theorem 1.1 is an improvement over that of McGahagan [8] in the range of uniqueness, whose class has been the largest ever. Typically, McGahagan's result implies the uniqueness in $C(I : L^\infty) \cap L^\infty(I : \dot{H}^1 \cap \dot{H}^{2+\epsilon})$ for $\epsilon > 0$.

(b) By the Sobolev embedding, our result implies the uniqueness of L^2 -localized strong solutions. More precisely, the solutions to (1.1) are unique in $C(I : Q + H^2)$, where Q is a fixed point in $\mathbb{S}^2$.

(c) Theorem 1.1 also implies the uniqueness proved previously by the author [11]. Under the equivariant conditions in two dimensions, he proved the uniqueness of solutions in the class $C(I : \dot{H}^1) \cap L^\infty(I : \dot{H}^2)$ near the harmonic family. Note that the auxiliary condition $C(I : L^\infty)$ follows from the embedding $\dot{H}^1 \subset L^\infty$ for equivariant maps.

(d) In [6], Kato proves the uniqueness of H^1 -valued solutions to the modified Schrödinger map equations under the Coulomb gauge condition. It is the equation which the differentiated field of the solutions to (1.1) satisfies, which is derived in [9]. The H^1 regularity for differentiated fields corresponds to the H^2 regularity for original maps. However, note that this result does not imply the uniqueness for the original equation, as mentioned in [6] and [9].

The framework of the proof of Theorem 1.1 is mainly based on the ideas of McGahagan [8]. In detail, we measure the difference of two solutions in a *geometrical* way, and show that it should be zero if the initial data coincides. More precisely, we consider the parallel transport of the derivatives of solutions along geodesics, and take the difference in the same tangent spaces. (See (2.2) below.) The advantage of this method is that we can avoid the emergence of extra derivatives on solutions, which leads to the uniqueness for H^3 -valued solutions. See Proposition 2.2 for the mechanism of the above phenomenon.

The differences from McGahagan's work are two points. First, we fixed the interval of the parameter of geodesics to $[0, 1]$, instead of using the arc length parameter which is used in McGahagan's argument. Indeed, in [8], the quantitative estimates for geodesics (see Lemma 2.1) is actually obtained by the former parameter. However, it is sensitive to change the parameter into the later one when the arc length is 0, namely, the two maps coincides with each other. Thus we rather perform the estimates without changing the parameter. Then, a difficult term emerges in the estimates which does not appear in [8], and it is necessary to exploit the geometrical structure of it. See the argument for the control of Q_2 in Section 3.

Second, we apply Yudovich's argument [13] to the energy method, Thanks to it, we can avoid the use of the Sobolev embedding for the control of L^∞ , which would require a little more regularity of the solutions. For the related result, see [3] and [4] for example.

The organization of the present paper is as follows. In Section 2, we introduce minimal geodesics and the related estimates from [8]. We also discuss the proof of the main theorem in the same section. In Section 3, we give the proof of the key estimate for our proof. Some technical lemmas is shown in Section 4.

Finally, we introduce some notations used in the paper. For a function space Y of space variables, we write $L^\infty(I : Y) := \{f : I \rightarrow Y : \|f\|_{L^\infty(I; Y)} \equiv \text{esssup}_{t \in I} \|f\|_Y < \infty\}$, $I \subset \mathbb{R}$. When $I = [0, T]$ for $T > 0$, we simply write it by $L_T^\infty Y_x$. For abbreviation, for a function space Y and a vector valued function

f , we write $f \in Y$ if and only if every component of f belongs to Y . We often interpret the time variable t as the variable with index 0 (see Lemma 2.1 for example). We define the commutator of two operators by $[A, B] = AB - BA$. For $f, g \in (L^2(\mathbb{R}^2))^3$, we define the inner product $\langle f, g \rangle = \int_{\mathbb{R}^2} f \cdot \bar{g} dx$. In our argument, we use the notation C for representing a constant, and the value varies in each situations. If we intend to indicate that such C depends on some specific quantity M , we write it by $C(M)$ for example.

2 Main argument

2.1 Geodesics

We only consider the case $I = [0, T]$ for $T > 0$. The negative direction in time follows from the similar argument. Let $u^{(0)}, u^{(1)} \in C(I : L^\infty) \cap L^\infty(I : \dot{H}^1 \cap \dot{H}^2)$ be two solutions with $u^{(0)}|_{t=0} = u^{(1)}|_{t=0}$. We may choose T so that $\|u^{(0)} - u^{(1)}\|_{L_{t,x}^\infty} < \pi$.

Then, for $(x, t) \in \mathbb{R}^2 \times [0, T]$, we consider the minimal geodesic from $u^{(0)}(x, t)$ to $u^{(1)}(x, t)$. In detail, we take a map $\gamma(s, x, t) : [0, 1] \times \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{S}^2$ such that

- $\gamma(0, x, t) = u^{(0)}(x, t)$, $\gamma(1, x, t) = u^{(1)}(x, t)$ in $(x, t) \in \mathbb{R}^2 \times [0, T]$.
- $\partial_{ss}\gamma + (\gamma_s \cdot \gamma_s)\gamma = 0$ in $[0, 1] \times \mathbb{R}^2 \times [0, T]$.

The following estimates for γ follows from the fact that the exponential map is C^∞ -diffeomorphism in the cut locus. See [Mc] in detail.

Lemma 2.1. (*Estimates for γ*) γ satisfies the followings:

- (1) $|\partial_k \gamma| \lesssim |\partial_k u^{\max}|$ ($k = 0, 1, 2$).
- (2) $|\partial_j \partial_k \gamma| \lesssim |\partial_j \partial_k u^{\max}| + |\partial_j u^{\max}| |\partial_k u^{\max}|$ ($j, k = 1, 2$).

Next, we consider the parallel transport along the geodesics. We define the operator $X(s, \sigma) : T_{\gamma(s, x, t)} \mathbb{S}^2 \rightarrow T_{\gamma(\sigma, x, t)} \mathbb{S}^2$ s.t. for $\xi \in T_{\gamma(s, x, t)} \mathbb{S}^2$, $X(s, \sigma)\xi$ is the parallel transport of ξ along $\gamma(s)$. $X(s, \sigma)$ is, by definition, the resolution operator for the following ODE:

$$D_s F \equiv \partial_s F + (F \cdot \gamma_s)\gamma = 0, \quad F : \mathbb{R} \rightarrow \mathbb{R}^3.$$

We frequently use the following commutator estimate for X and differential operators, which plays a central role in the argument.

Lemma 2.2. (*Commutator*) For $k = 0, 1, 2$, $s, \sigma \in [0, 1]$ and $F(x, t) \in T_{\gamma(s, x, t)} \mathbb{S}^2$, we have

$$[D_k, X(s, \sigma)] F = \int_\sigma^s X(s, s_1) R(\partial_s \gamma, \partial_k \gamma) X(s_1, \sigma) F ds_1, \quad (2.1)$$

where R is the Ricci curvature tensor, defined explicitly by

$$R(\xi_1, \xi_2)\xi_3 = (\xi_3 \cdot \xi_2)\xi_1 - (\xi_3 \cdot \xi_1)\xi_2$$

for ξ_1, ξ_2, ξ_3 in $T_p \mathbb{S}^2$ with $p \in \mathbb{S}^2$.

For the proof at classical level, see [8]. However, we need to be careful for the regularity. For example, in our argument in the next section, we will use Lemma 2.2 in the case when F is the third derivative of $u^{(0)}$. Since $u^{(0)}$ is only assumed to be in H^2 , such F cannot be defined in the classical sense. Therefore, we have to justify the lemma above even for some *distributional* vector fields. In Section 4, we provide a rigorous argument for this point by proving Lemma 2.2 in several distributional settings (see Lemma 4.3).

2.2 Yudovich's argument

In order to prove $u^{(0)} = u^{(1)}$, we apply Yudovich's argument to (1.1). The framework of the argument at formal level is the following: We first derive the following type of inequality for all $p > 2$:

$$\frac{d}{dt}(\text{difference}) \leq Cp(\text{difference})^{1-1/p},$$

where C is a constant independent of p . By solving the above inequality, we have

$$(\text{difference}) \leq (CT)^p,$$

where we recall that T is the length of time interval. If we choose T to be sufficiently small, by taking the limit $p \rightarrow \infty$, we can conclude that the difference is 0. We will proceed the above argument with caring about the singularity and regularity by combining penalization for instance.

In our argument, we choose the difference as the following, based on [8]:

$$\left\| u^{(0)} - u^{(1)} \right\|_{L_T^\infty L_x^2}^2 + \sum_{m=1,2} \left\| X(1,0) \partial_m u^{(0)} - \partial_m u^{(1)} \right\|_{L_T^\infty L_x^2}^2. \quad (2.2)$$

Each term is controlled in the following way:

Proposition 2.1. *Let $q = u^{(0)} - u^{(1)}$ and $M = \|\nabla u^{\max}\|_{L_T^\infty H_x^1}$. Then the following estimates hold true for all $p > 2$.*

- (i) $\left\| \partial_t \|q\|_{L_x^2}^2 \right\|_{L_T^\infty} \leq C(M) \|q\|_{L_T^\infty H_x^1}^2.$
- (ii) $\|\nabla q\|_{L_T^\infty L_x^2} \leq \sum_{m=1,2} \|X(1,0) \partial_m u^{(0)} - \partial_m u^{(1)}\|_{L_T^\infty L_x^2} + C(M) \sqrt{p} \|q\|_{L_T^\infty L_x^2}^{1-1/p}.$

Proposition 2.2. *Let $V_m = X(1,0) \partial_m u^{(0)} - \partial_m u^{(1)}$ for $m = 1, 2$, and $M = \|\nabla u^{\max}\|_{L_T^\infty H_x^1}$. Then the following estimate holds true for all $p > 2$.*

$$\|\langle \partial_t V_m, V_m \rangle\|_{L_t^\infty} \leq C(M)p \left(\|q\|_{L_T^\infty L_x^2}^2 + \sum_{m=1,2} \|V_m\|_{L_T^\infty L_x^2}^2 \right)^{1-1/p}. \quad (2.3)$$

The main part of our argument is the proof of Proposition 2.2, and we discuss it in the next section. Here we prove Proposition 2.1.

Proof of Proposition 2.1. We first note that our assumption of regularity implies $u^{(0)} - u^{(1)} \in W^{1,\infty}(I : L^2)$. See Lemma 4.4 in Section 4 for its proof. Hence for

a.a. $t \in I$, we have

$$\begin{aligned} \frac{d}{dt} \|q\|_{L_x^2}^2 &= 2\langle \partial_t q, q \rangle = 2\langle q \times \Delta u^{(0)}, q \rangle + 2\langle u^{(1)} \times \Delta q, q \rangle \\ &= -2 \sum_{j=1,2} \langle \partial_j u^{(1)} \times \partial_j q, q \rangle \\ &\lesssim_M \|q\|_{L_T^\infty H_x^1}^2, \end{aligned}$$

which gives (i) in the statement. For (ii), we first recall the following inequality: Here we recall the following inequality: For $f \in H^1$ and $r \in [2, \infty)$, we have

$$\|f\|_{L^r} \leq C\sqrt{r} \|f\|_{H^1}, \quad (2.4)$$

where C is independent of p and f . See for instance [10] for the detailed proof.

We decompose

$$\|\partial_m q\|_{L_x^2} \leq \|V_m\|_{L_x^2} + \left\| X(1,0) \partial_m u^{(0)} - \partial_m u^{(0)} \right\|_{L_x^2}$$

for $m = 1, 2$ and $t \in [0, 1]$. The second term is controlled by

$$\begin{aligned} \left\| \int_0^1 \partial_s X(s,0) \partial_m u^{(0)} ds \right\|_{L_x^2} &\leq \int_0^1 \left\| \left(X(s,0) \partial_m u^{(0)} \cdot \partial_s \gamma \right) \gamma \right\|_{L_x^2} ds \\ &\leq \left\| \partial_m u^{(0)} \right\|_{L_x^{4p}} \|l\|_{L_x^{4p/(2p-1)}} \end{aligned} \quad (2.5)$$

where $l = d_{\mathbb{S}^2}(u^{(0)}, u^{(1)})$ is the distance of $u^{(0)}$ and $u^{(1)}$ on the Riemannian manifold $\mathbb{S}^2$. Since we have

$$|l| \leq C|q|,$$

(2.5) is controlled by

$$C\sqrt{p} \left\| \partial_m u^{(0)} \right\|_{H^1} \|q\|_{L_x^{4p/(2p-1)}} \leq C(M)\sqrt{p} \|q\|_{L_x^2}^{1-1/p},$$

where we used (2.7) and the following inequality:

$$\|f\|_{L_x^{4p/(2p-1)}} \leq \|f\|_{L_x^2}^{1-1/p} \|f\|_{L_x^4}^{1/p}. \quad (2.6)$$

Thus we obtain (ii). $\square$

Proof of Theorem 1.1. We first note that $V_m \in L^\infty(I : H^1) \cap W^{1,\infty}(I : H^{-1})$ for $m = 1, 2$. We will check this regularity in Lemma 4.4 in Section 4.

Let $\epsilon > 0$ be arbitrary number. Let $\psi \in C^\infty(\mathbb{R})$ be a cut-off function, and define the operator $P_k = (\psi(\cdot/2^k))^*$. Then, for sufficiently large k uniformly in t , we have

$$\left| \|q\|_{L^2}^2 + \sum_{m=1,2} \langle V_m, P_k V_m \rangle + \epsilon \right| > 0.$$

Since $P_k V_m \in C(I : H^1)$, we have

$$\begin{aligned} &\partial_t \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \langle V_m, P_k V_m \rangle + \epsilon \right)^{1/p} \\ &= \frac{1}{p} \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \langle V_m, P_k V_m \rangle + \epsilon \right)^{1/p-1} \left(\partial_t \|q\|_{L^2}^2 + 2 \sum_{m=1,2} \operatorname{Re} \langle \partial_t V_m, P_k V_m \rangle \right), \end{aligned}$$

for all $t \in (0, T)$, and thus

$$\begin{aligned} & \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \langle V_m, P_k V_m \rangle + \epsilon \right)^{1/p} \Big|_{t=0}^{t=T} \\ &= \int_0^T \frac{1}{p} \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \langle V_m, P_k V_m \rangle + \epsilon \right)^{1/p-1} \left(\partial_t \|q\|_{L^2}^2 + 2 \sum_{m=1,2} \operatorname{Re} \langle \partial_t V_m, P_k V_m \rangle \right) dt. \end{aligned}$$

By taking the limit $k \rightarrow \infty$, it follows that

$$\begin{aligned} & \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \|V_m\|_{L^2}^2 + \epsilon \right)^{1/p} \Big|_{t=0}^{t=T} \\ &= \int_0^T \frac{1}{p} \left(\|q\|_{L^2}^2 + \sum_{m=1,2} \|V_m\|_{L^2}^2 + \epsilon \right)^{1/p-1} \operatorname{Re} \left(\partial_t \|q\|_{L^2}^2 + \sum_{m=1,2} \langle \partial_t V_m, V_m \rangle \right) dt, \end{aligned}$$

which can be justified by Lebesgue's dominant convergence theorem. Set

$$G(T) = \|q\|_{L_T^\infty L_x^2}^2 + \sum_{m=1,2} \|V_m\|_{L_T^\infty L^2}^2. \quad (2.7)$$

By Propositions 2.1 and 2.2, we have

$$(G(T) + \epsilon)^{1/p} \leq C(M)T.$$

If we choose T sufficiently small, $(C(M)T)^p \rightarrow 0$ as $p \rightarrow \infty$, and thus by letting $\epsilon \rightarrow 0$, we have

$$G(T) = 0,$$

which concludes the main theorem.

3 Proof of Proposition 2.2

In this section, we show Proposition 2.2. We first define J by the complex structure of $\mathbb{S}^2$, which can be explicitly written as $J\xi = p \times \xi$ for $\xi \in T_p \mathbb{S}^2$. From (1.1), V_m satisfies the following equation:

$$D_t V_m = J \sum_{k=1}^2 D_k^2 V_m + \sum_{\alpha=1}^4 R_\alpha,$$

where

$$\begin{aligned} R_1 &= [D_t, X] \partial_m u^{(0)} \\ R_2 &= - \sum_{k=1}^2 J [D_k, X] D_k \partial_m u^{(0)} \\ R_3 &= - \sum_{k=1}^2 J D_k [D_k, X] \partial_m u^{(0)} \end{aligned}$$

$$R_4 = \sum_{k=1}^2 \left\{ R(X \partial_m u^{(0)}, X \partial_k u^{(0)}) X \partial_k u^{(0)} - R(\partial_m u^{(1)}, \partial_k u^{(1)}) \partial_k u^{(1)} \right\}$$

It suffices to show the following estimate for $m = 1, 2$ and $t \in I$:

$$\sum_{\alpha=1}^4 |\langle R_\alpha, V_m \rangle| \leq C(M) p G(T)^{1-1/p},$$

where $G(T)$ is as in (2.7).

When $\alpha = 1$, we have

$$\begin{aligned} |\langle R_1, V_m \rangle| &= \left| \int_0^1 \langle X(1, s) R(\partial_s \gamma, \partial_t \gamma) X(s, 0) \partial_m u^{(0)}, V_m \rangle ds \right| \\ &\leq \int_0^1 \int_{\mathbb{R}^2} |\partial_s \gamma| |\partial_t \gamma| |\partial_m u^{(0)}| |V_m| dx ds \\ &\leq C \int_0^1 \int_{\mathbb{R}^2} l |\partial_t u^{\max}| |\partial_m u^{(0)}| |V_m| dx ds \\ &\leq C \|l\|_{L_x^{8p}} \|\partial_t u^{\max}\|_{L_x^2} \left\| \partial_m u^{(0)} \right\|_{L_x^{8p}} \|V_m\|_{L_x^{4p/(2p-1)}} \\ &\leq C(M) p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}, \end{aligned}$$

where we used Lemmas 2.1 and 2.2, (2.7) and (2.6). Similarly, when $\alpha = 2$, we have

$$\begin{aligned} |\langle R_2, V_m \rangle| &= \left| \sum_{k=1}^2 \int_0^1 \langle X(1, s) R(\partial_s \gamma, \partial_k \gamma) X(s, 0) D_k \partial_m u^{(0)}, J V_m \rangle ds \right| \\ &\leq C \int_0^1 \int_{\mathbb{R}^2} l |\nabla u^{\max}| |\nabla^2 u^{(0)}| |V_m| dx ds \\ &\leq C \|l\|_{L_x^{8p}} \|\nabla u^{\max}\|_{L_x^{8p}} \left\| \nabla^2 u^{(0)} \right\|_{L_x^2} \|V_m\|_{L_x^{4p/(2p-1)}} \\ &\leq C(M) p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}. \end{aligned}$$

R_4 is estimated in the following manner:

$$\begin{aligned} |\langle R_4, V_m \rangle| &\leq C \int_{\mathbb{R}^2} |\nabla u^{\max}|^2 (|V_1| + |V_2|) |V_m| dx ds \\ &\leq C \|\nabla u^{\max}\|_{L_T^\infty L_x^{2r}}^2 \| |V_1| + |V_2| \|_{L_T^\infty L_x^{4p/(2p-1)}}^2 \\ &\leq C(M) p \| |V_1| + |V_2| \|_{L_T^\infty L_x^2}^{2-2/p}. \end{aligned}$$

The most difficult part is the estimate for R_3 . We first consider the decom-

position as follows:

$$\begin{aligned}
\langle R_3, V_m \rangle &= \sum_{k=1}^2 \int_0^1 \langle [D_k, X(1, s)] R(\partial_s \gamma, \partial_k \gamma) X(s, 0) \partial_m u^{(0)}, JV_m \rangle ds \\
&\quad + \sum_{k=1}^2 \int_0^1 \langle R(D_k \partial_s \gamma, \partial_k \gamma) X(s, 0) \partial_m u^{(0)}, JX(s, 1) \rangle V_m \rangle ds \\
&\quad + \sum_{k=1}^2 \int_0^1 \langle R(\partial_s \gamma, D_k \partial_k \gamma) X(s, 0) \partial_m u^{(0)}, JX(s, 1) \rangle V_m \rangle ds \\
&\quad + \sum_{k=1}^2 \int_0^1 \langle R(\partial_s \gamma, \partial_k \gamma) [D_k, X(s, 0)] \partial_m u^{(0)}, JX(s, 1) \rangle V_m \rangle ds \\
&\quad + \sum_{k=1}^2 \int_0^1 \langle R(\partial_s \gamma, \partial_k \gamma) X(s, 0) D_k \partial_m u^{(0)}, JX(s, 1) \rangle V_m \rangle ds \\
&\equiv \sum_{\alpha=1}^5 Q_\alpha.
\end{aligned}$$

Each term except Q_2 is estimated as follows:

$$\begin{aligned}
|Q_1| &= \left| \sum_{k=1}^2 \int_0^1 \int_s^1 \langle X(1, \sigma) R(\partial_s \gamma, \partial_k \gamma) X(\sigma, s) R(\partial_s \gamma, \partial_k \gamma) X(s, 0) \partial_m u^{(0)}, JV_m \rangle d\sigma ds \right| \\
&\leq C \int_{\mathbb{R}^2} l^2 |\nabla u^{\max}|^3 |V_m| dx \\
&\leq \|l\|_{L_T^\infty L_x^{8p}} \|\nabla u^{\max}\|_{L_T^\infty L_x^{8p}} \|\nabla u^{\max}\|_{L_T^\infty L_x^4}^2 \|V_m\|_{L_T^\infty L_x^{4p/(2p-1)}} \\
&\leq C(M)p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}.
\end{aligned}$$

$$\begin{aligned}
|Q_3| &\leq C \int_{\mathbb{R}^2} l |\nabla^2 u^{\max}| |\nabla u^{(0)}| |V_m| \\
&\leq \|l\|_{L_T^\infty L_x^{8p}} \|\nabla^2 u^{\max}\|_{L_T^\infty L_x^2} \|\nabla u^{\max}\|_{L_T^\infty L_x^{8p}} \|V_m\|_{L_T^\infty L_x^{4p/(2p-1)}} \\
&\leq C(M)p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}.
\end{aligned}$$

$$\begin{aligned}
|Q_4| &= \left| \sum_{k=1}^2 \int_0^1 \int_s^1 \langle R(\partial_s \gamma, \partial_k \gamma) X(s, \sigma) R(\partial_s \gamma, \partial_k \gamma) X(\sigma, 0) \partial_m u^{(0)}, JV_m \rangle d\sigma ds \right| \\
&\leq C \int_{\mathbb{R}^2} l^2 |\nabla u^{\max}|^3 |V_m| dx \\
&\leq C(M)p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}.
\end{aligned}$$

$$\begin{aligned}
|Q_5| &\leq C \int_{\mathbb{R}^2} l |\nabla u^{\max}| |\nabla^2 u^{(0)}| |V_m| \\
&\leq C(M)p \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p}.
\end{aligned}$$

The argument for Q_2 is far involved, since we cannot control $D_k \partial_s u$ directly by q or V_m . The only quantitative information of difference can be seen in the component in the $\partial_s \gamma$ direction, that is,

$$D_k \partial_s \gamma \cdot \partial_s \gamma = \frac{1}{2} \partial_k (l^2) = l \partial_k l. \quad (3.1)$$

It will be turned out, however, that the above information is sufficient to control Q_2 by exploiting the structure of Ricci tensor.

Now we fix $(x, t) \in \mathbb{R}^2 \times I$ satisfying $l(x, t) \neq 0$. Then we set

$$v = \frac{\partial_s \gamma + J \partial_s \gamma}{\sqrt{2}l}, \quad w = \frac{J \partial_s \gamma - \partial_s \gamma}{\sqrt{2}l}.$$

We note that v, w is an orthonormal frame on $T_\gamma \mathbb{S}^2$. By calculations, we have

$$\begin{aligned} & R(D_k \partial_s \gamma, \partial_k \gamma) X(s, 1) \partial_m u^{(0)} \cdot JX(s, 0) V_m \\ &= \{(\partial_k \gamma \cdot v)(D_k \partial_s \gamma \cdot w) - (\partial_k \gamma \cdot w)(D_k \partial_s \gamma \cdot v)\} \\ & \quad \cdot \left\{ (X(s, 0) \partial_m u^{(0)} \cdot v)(JX(s, 1) V_m \cdot w) - (X(s, 0) \partial_m u^{(0)} \cdot w)(JX(s, 1) V_m \cdot v) \right\} \end{aligned}$$

We set

$$A = (X(s, 0) \partial_m u^{(0)} \cdot v)(JX(s, 1) V_m \cdot w) - (X(s, 0) \partial_m u^{(0)} \cdot w)(JX(s, 1) V_m \cdot v).$$

Then A is independent of s . Thus, we have

$$\begin{aligned} & \int_0^1 R(D_k \partial_s \gamma, \partial_k \gamma) X(s, 1) \partial_m u^{(0)} \cdot JX(s, 0) V_m ds \\ &= A \int_0^1 \{(\partial_k \gamma \cdot v)(D_k \partial_s \gamma \cdot w) - (\partial_k \gamma \cdot w)(D_k \partial_s \gamma \cdot v)\} ds \\ &= \frac{A}{l^2} \int_0^1 \{(\partial_k \gamma \cdot \partial_s \gamma)(D_k \partial_s \gamma \cdot J \partial_s \gamma) - (\partial_k \gamma \cdot J \partial_s \gamma)(D_k \partial_s \gamma \cdot \partial_s \gamma)\} ds \end{aligned} \quad (3.2)$$

Since $D_k \partial_s \gamma \cdot J \partial_s \gamma = \partial_s (\partial_k \gamma \cdot J \partial_s \gamma)$, integration by parts gives that (3.2) is equal to

$$\begin{aligned} & -\frac{2A}{l^2} \int_0^1 (\partial_k \gamma \cdot J \partial_s \gamma)(D_k \partial_s \gamma \cdot \partial_s \gamma) ds + \frac{A}{l^2} (\partial_k \gamma \cdot \partial_s \gamma)(\partial_k \gamma \cdot J \partial_s \gamma) \Big|_{s=0}^{s=1} \\ & \equiv \frac{A}{l^2} \sum_{\alpha=1}^5 B_\alpha, \end{aligned}$$

where

$$\begin{aligned} B_1 &= -2 \int_0^1 (\partial_k \gamma \cdot J \partial_s \gamma)(\partial_k \partial_s \gamma \cdot \partial_s \gamma) ds, \\ B_2 &= (\partial_k q \cdot \partial_s \gamma|_{s=1})(\partial_k u^{(1)} \cdot J \partial_s \gamma|_{s=1}), \\ B_3 &= (\partial_k u^{(0)} \cdot \partial_s \gamma|_{s=0}^{s=1})(\partial_k u^{(1)} \cdot J \partial_s \gamma|_{s=1}), \\ B_4 &= (\partial_k u^{(0)} \cdot \partial_s \gamma|_{s=0})(\partial_k q \cdot J \partial_s \gamma|_{s=1}), \end{aligned}$$

$$B_5 = (\partial_k u^{(0)} \cdot \partial_s \gamma|_{s=0}) (\partial_k q \cdot J \partial_s \gamma|_{s=0}^{s=1}).$$

By using (3.1), we have

$$|B_1| \leq Cl^2 |\partial_k l| |\partial_k u^{\max}|.$$

The other terms are controlled as follows:

$$|B_2| + |B_4| \leq Cl^2 |\partial_k q| |\nabla u^{\max}| \leq Cl^2 \left(|V_k| + |\partial_k u^{(0)} l| \right) |\partial_k u^{(1)}|,$$

$$|B_3| \leq C |\partial_k u^{(0)}| |\nabla u^{(1)}| l \int_0^1 |\partial_{ss} \gamma| ds \leq C |\partial_k u^{\max}|^2 l^3,$$

$$|B_5| \leq C |\partial_k u^{(0)}| |\nabla u^{(1)}| l \left| (\gamma|_{s=0}^{s=1} \times \partial_s \gamma|_{s=1} - \gamma|_{s=0} \times \partial_s \gamma|_{s=0}^{s=1}) \right| \leq C |\partial_k u^{\max}|^2 l^3.$$

On the other hand, A is bounded by

$$C |\nabla u^{\max}| |V_m|.$$

Hence, (3.2) is bounded by

$$C \{ |u^{\max}| |\partial_k l| + |\nabla u^{\max}| |V_m| + |\nabla u^{\max}|^2 l \} |\nabla u^{\max}| |V_m|.$$

Here we recall the following estimate obtained in [8]:

$$|\partial_k l| \leq |V_k|. \quad (3.3)$$

Writing

$$\mathcal{A}(t) = \{x \in \mathbb{R}^2 : l(x, t) = 0\},$$

we have

$$\begin{aligned} |Q_2| &= \left| \sum_{k=1}^2 \int_{\mathbb{R}^2 - \mathcal{A}(t)} \int_0^1 R(D_k \partial_s \gamma, \partial_k \gamma) X(s, 1) \partial_m u^{(0)} \cdot JX(s, 0) V_m ds dx \right| \\ &\leq C \int_{\mathbb{R}^2 - \mathcal{A}(t)} |u^{\max}|^2 (|V_1| + |V_2|)^2 dx + C \int_{\mathbb{R}^2 - \mathcal{A}(t)} |u^{\max}|^3 |V_m|^2 dx \\ &\leq C(M)p \left(\|V_1\| + \|V_2\| \|V_1\|_{L_T^\infty L_x^2}^{2-2/p} + \|q\|_{L_T^\infty H_x^1}^{1-1/p} \|V_m\|_{L_T^\infty L_x^2}^{1-1/p} \right), \end{aligned}$$

which completes the proof. $\square$

4 Appendix: Some technical lemmas

We first check the following inequality from harmonic analysis:

Lemma 4.1. *For $f, g \in \mathcal{S}(\mathbb{R}^2)$ and $\epsilon > 0$, we have*

$$\|fg\|_{H^{-1}} \leq C \|f\|_{H^{-1}} \|g\|_{L^\infty \cap \dot{H}^{1+\epsilon}}, \quad (4.1)$$

where $C > 0$ is independent of f, g .

Proof. By Leibniz rule (see [7] for example), we have for $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} fg\varphi dx \right| &\leq \|f\|_{H^{-1}} \|g\varphi\|_{H^1} \\ &\leq \|f\|_{H^{-1}} (\|g\|_{L^\infty} (\|\varphi\|_{L^2} + \|\nabla\varphi\|_{L^2}) + \|\nabla g\|_{L^{2/(1-\epsilon)}} \|\varphi\|_{L^{2/\epsilon}}) \\ &\leq \|f\|_{H^{-1}} \|g\|_{L^\infty \cap \dot{H}^{1+\epsilon}} \|\varphi\|_{H^1}, \end{aligned}$$

which leads to (4.1). $\square$

In particular, for $f \in H^{-1}$ and $g \in L^\infty \cap \dot{H}^{1+\epsilon}$ with $\epsilon > 0$, we can define the multiplication $fg \in H^{-1}$ by the usual distributional sense, namely,

$$\langle fg, \varphi \rangle_{H^{-1}, H^1} = \langle f, g\varphi \rangle_{H^{-1}, H^1}$$

for $\varphi \in H^1$. This is consistent with the classical definition, thus the Leibniz rule is also justified for such distributions. See [11] for the related discussions.

Let γ be as in the previous sections. We show that the parallel transport can be defined for distributional tangent vectors:

Lemma 4.2. *Let Y be either (i) $L_T^\infty H_x^{-1}$ for $r \in (-\infty, 2)$, or (ii) $W_T^{1,\infty} H_x^{-1} \cap L_T^\infty H_x^1$. Moreover, let $\sigma \in [0, 1]$ and $\xi \in Y^3$ satisfying $\xi \cdot \gamma|_{s=\sigma} = 0$. Then a unique solution $F(s) \equiv X(s, \sigma)\xi \in C_s^1 Y$ to the following ODE exists:*

$$D_s F(s) = 0, \quad F(\sigma) = \xi. \quad (4.2)$$

Proof. We will show in the case $\sigma = 0$, where the other case can be proved similarly. We rewrite (4.2) in the integral form:

$$F(s) = \xi - \int_0^s (F(s') \cdot \partial_s \gamma(s')) \gamma(s') ds' \equiv \Phi(F)(s)$$

It suffices to show that Φ is a contraction in $C([0, s] : Y)$ for $s \ll 1$ depending only on M as in Proposition 2.2. In both cases of (i) and (ii), we have

$$\|\Phi(F)\|_{L_s^\infty Y} \leq \|\xi\|_Y + C(M)s \|F\|_{L_s^\infty Y}$$

$$\|\Phi(F_1) - \Phi(F_2)\|_{L_s^\infty Y} \leq C(M)s \|F_1 - F_2\|_{L_s^\infty Y}$$

by using Lemmas 2.1 and 4.1, which proves the lemma. $\square$

Lemma 4.3. *The followings are true.*

(i) *Let $\sigma \in [0, 1]$ and $F \in L_T^\infty L_x^2$ with $F \cdot \gamma|_{s=\sigma} = 0$. Then, (2.1) holds for $k = 1, 2$.*

(ii) *Let $\sigma \in [0, 1]$ and $F \in W_T^{1,\infty} H_x^{-1} \cap L_T^\infty H_x^1$ with $F \cdot \gamma|_{s=\sigma} = 0$. Then, (2.1) holds for $k = 0$.*

Proof. By Lemma 4.2, $[D_k, X]F \in C_s^1 L_T^\infty H^{-1}$, and the derivatives of the both sides of (2.1) with respect to s coincides with each other. Since both sides are 0 when $s = \sigma$, the conclusion follows from the uniqueness of the solution of ODEs. $\square$

Next we check the regularity of q and V_m .

Lemma 4.4. *The following properties hold true.*

(i) $q \in W^{1,\infty}(I : L^2)$.

(ii) $V_m \in L^\infty(I : H^1) \cap W^{1,\infty}(I : H^{-1})$ for $m = 1, 2$.

Proof. (i) From (1.1), we have $\partial_t q = q \times \nabla u^{(0)} - u^{(1)} \times q \in L_T^\infty L_x^2$. In particular, it follows that $q \in W_T^{1,\infty}(L_x^2 \cup L_x^\infty) \cap C_T L^\infty$. Therefore, by using Theorem 1.4.35 in [1], we have $q(t, \cdot) = \int_0^t \partial_t q dt$ for all $t \in I$, which leads to the desired conclusion.

(ii) By (i), it suffices to show that $X(1, 0)\partial_m u^{(0)} - \partial_m u^{(0)} \in L_T^\infty H^1 \cap W_T^{1,\infty} H^{-1}$. However, it immediately follows from

$$X(1, 0)\partial_m u^{(0)} - \partial_m u^{(0)} = - \int_0^1 (X(s, 0)\partial_m u^{(0)} \cdot \partial_s \gamma) \gamma ds \in L_T^\infty H^1 \cap W_T^{1,\infty} H^{-1},$$

where the last regularity property follow from Lemma 4.2. $\square$

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References

- [1] T. Cazenave and A. Haraux, "An introduction to semilinear evolution equations," Oxford lecture series in mathematics and its applications, 13, The Clarendon Press, New York, 1998.
- [2] N. H. Chang, J. Shatah, and K. Uhlenbeck, *Schrödinger maps*, Comm. Pure Appl. Math., 53 (2000), 590-602.
- [3] M. Hayashi, *A note on the nonlinear Schrödinger equation in a general domain*, Nonlinear Anal., 173 (2018), 99-122.
- [4] M. Hayashi and T. Ozawa *Well-posedness for a generalized derivative nonlinear Schrödinger equation*, J. Differ. Equations, 261 (2016), 5424-5445.
- [5] A. Hubert and R. Schäfer, "Magnetic Domains: The Analysis of Magnetic Microstructures," Springer-Verlag, Berlin, 1998.
- [6] J. Kato, *Existence and uniqueness of the solution to the modified Schrödinger map*, Math. Res. Lett., 12 (2005), 171-186.
- [7] F. Linares and G. Ponce, "Introduction to nonlinear dispersive equations. Second edition," Springer-Verlag, New York, 2015.
- [8] H. McGahagan, *An approximation scheme for Schrödinger maps*, Commun. Part. Diff. Eq., 32 (2007), 375-400.
- [9] A. Nahmod, A. Stefanov, and K. Uhlenbeck, *On Schrödinger maps*, Comm. Pure Appl. Math., 56 (2003), 114-151.

- [10] T. Ogawa, *A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations*, Nonlinear Anal.-Theor., 14 (1990), 765-769.
- [11] I. Shimizu, *Remarks on local theory for Schrödinger maps near harmonic maps*, to be appeared in Kodai Math. J., arXiv: 1806.02365.
- [12] P.-L. Sulem, C. Sulem, and C. Bardos, *On the continuous limit for a system of classical spins*, Comm. Math. Phys., 107 (1986), 431-454.
- [13] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid*, Ž. Vyčisl. Mat. I Mat. Fiz., 3 (1963), 1032-1066.