

Symplectic varieties of complete intersection and contact geometry

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A normal complex algebraic variety X is a *symplectic variety* if there is a holomorphic symplectic 2-form ω on the regular part X_{reg} of X and ω extends to a (possibly degenerate) holomorphic 2-form on a resolution $f : \tilde{X} \rightarrow X$.

Example: Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let G be its adjoint group. Let us consider the *adjoint quotient map* $\chi : \mathfrak{g} \rightarrow \mathfrak{g}/G$. If $\text{rank}(\mathfrak{g}) = r$, then $\mathfrak{g}/G$ is isomorphic to the r -dimensional affine space $\cong \mathbf{C}^r$. The nilpotent variety N is, by definition, the set of all nilpotent elements of $\mathfrak{g}$ and we have $N = \chi^{-1}(0)$. The nilpotent variety decomposes into the disjoint union of (finite number of) nilpotent orbits. There is a unique nilpotent orbit O_{reg} that is open dense in N , which we call the *regular nilpotent orbit*. Then $N = \overline{O_{reg}}$. The regular nilpotent orbit O_{reg} coincides with the regular part of N and it admits a holomorphic symplectic form ω_{KK} so called the *Kostant-Kirillov 2-form*. Then (N, ω_{KK}) is a symplectic variety. Moreover $N \subset \mathfrak{g}$ is defined as a complete intersection of r homogeneous polynomials (with respect to the standard $\mathbf{C}^*$ -action on $\mathfrak{g}$).

In this talk I characterize the nilpotent varieties of semisimple Lie algebras among affine symplectic varieties.

Let (X, ω) be a singular affine symplectic variety of dimension $2n$ embedded in an affine space $\mathbf{C}^{2n+r}$ as a complete intersection of r homogeneous polynomials. Assume that ω is also homogeneous, i.e. there is an integer l such that $t^*\omega = t^l \cdot \omega$ for $t \in \mathbf{C}^*$.

Main Theorem ([1]): *One has $(X, \omega) \cong (N, \omega_{KK})$, where N is the nilpotent variety of a semisimple Lie algebra $\mathfrak{g}$ together with the Kostant-Kirillov 2-form ω_{KK} .*

Before starting the proof, we give a few observations.

Observation 1. $l = 1$, where $l := wt(\omega)$.

Write X as $f_1 = \dots = f_r = 0$ with homogeneous polynomials $f_i(z_1, \dots, z_{2n+r})$. We put $a_i := \deg(f_i)$. As X is singular, we may assume that $a_i > 1$ for every i . The holomorphic volume form $\omega^n := \omega \wedge \dots \wedge \omega$ can be written, by the adjunction formula (or the residue formula), as

$$\omega^n = c \cdot \text{Res}(dz_1 \wedge \dots \wedge z_{2n+r} / (f_1, \dots, f_r))$$

with a nonzero constant c . Since X has only canonical singularities, we see that $wt(\omega) > 0$. Computing the weights of both sides, we get

$$0 < n \cdot wt(\omega) = 2n + r - \sum a_i < 2n.$$

The last inequality follows from the fact that $a_i > 1$ and so $\sum a_i > r$. Hence $wt(\omega) = 1$ and $\sum a_i = n + r$.

Observation 2. X has a $\mathbf{C}^*$ -equivariant crepant resolution $\pi : Y \rightarrow X$.

(Sketch of Proof): Take a resolution $f : W \rightarrow X$ and apply the MMP (Minimal Model Program) to the morphism f . Then we get a partial crepant resolution $\pi : Y \rightarrow X$ where Y may possibly have $\mathbf{Q}$ -factorial terminal singularities. The $\mathbf{C}^*$ -action on X extends to a $\mathbf{C}^*$ -action in such a way that π is $\mathbf{C}^*$ -equivariant. The symplectic form ω induces a Poisson structure on the regular locus X_{reg} . By the normality of X , this Poisson structure extends to a Poisson structure on X . The pull-back of ω by π is a symplectic form on Y_{reg} because π is crepant. Then it induces a Poisson structure on Y_{reg} and it extends to a Poisson structure on Y . Now let us consider a *Poisson deformation* $\mathcal{Y} \rightarrow \Delta$ of Y . Then π extends to a birational morphism $\tilde{\pi} : \mathcal{Y} \rightarrow \mathcal{X}$ over Δ for a Poisson deformation $\mathcal{X} \rightarrow \Delta$ of X . If the Poisson deformation $\mathcal{Y}/\Delta$ is very general, then $\tilde{\pi}_t : Y_t \rightarrow X_t$ is an isomorphism for $t \neq 0$. Since X has only complete intersection singularities, X_t also does. On the other hand, we have $\text{Codim}_Y \text{Sing}(Y) \geq 4$ because Y has only terminal singularities. This implies that $\text{Codim}_{Y_t} \text{Sing}(Y_t) \geq 4$; and hence $\text{Codim}_{X_t} \text{Sing}(X_t) \geq 4$. Notice that X_t is a symplectic variety and such a symplectic variety X_t must be smooth by a proposition of Beauville. As $Y_t \cong X_t$, we have seen that Y_t is smooth. Finally, by the $\mathbf{Q}$ -factoriality of Y , we see that any Poisson deformation $\mathcal{Y} \rightarrow \Delta$ is a locally trivial flat deformation of Y . Therefore Y must be smooth and π is a crepant resolution.

We put $\mathbf{P}(X) := X - \{0\}/\mathbf{C}^*$ and $Z := Y - \pi^{-1}(0)/\mathbf{C}^*$. Then π induces a map $\bar{\pi} : Z \rightarrow \mathbf{P}(X)$. By using the fact that $wt(z_i) = 1$ for all i , we have:

Observation 3. $\bar{\pi} : Z \rightarrow \mathbf{P}(X)$ is a crepant resolution.

Notice that Z is a projective manifold of dimension $2n - 1$. An important fact is that Z has a *contact structure*.

Let W be a complex manifold of odd dimension $2n - 1$. A contact structure on W is an exact sequence of vector bundles

$$0 \rightarrow E \rightarrow \Theta_W \xrightarrow{\theta} M \rightarrow 0,$$

where M is a line bundle and the induced pairing $E \times E \rightarrow M$, $(x, y) \rightarrow \theta([x, y])$ is nondegenerate. Recall that a subbundle of Θ is called *integrable* if the bracket $[\cdot, \cdot]$ is closed in the subbundle. In this sense, E is a *highly non-integrable* subbundle of Θ . The line bundle M is called the *contact line bundle* and the twisted 1-form $\theta \in \Gamma(W, \Omega_W^1 \otimes M)$ is called the *contact form*. If W admits a contact structure, then $-K_W \cong M^{\otimes n}$ and $(d\theta)^{n-1} \wedge \theta$ is a nondegenerate $2n - 1$ -form on W .

Let us return to our situation. The $\mathbf{C}^*$ -bundle $X - \{0\} \rightarrow \mathbf{P}(X)$ restricts to the $\mathbf{C}^*$ -bundle $X_{reg} \rightarrow \mathbf{P}(X)_{reg}$. We put $L := \mathcal{O}_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}}$ and denote by $(L^{-1})^\times$ the $\mathbf{C}^*$ -bundle on $\mathbf{P}(X)_{reg}$ obtained from the dual line bundle L^{-1} by removing the 0-section. Then $X_{reg} \rightarrow \mathbf{P}(X)_{reg}$ can be identified with $p : (L^{-1})^\times \rightarrow \mathbf{P}(X)_{reg}$. The $\mathbf{C}^*$ -action on X_{reg} coincides with the natural $\mathbf{C}^*$ -action on $(L^{-1})^\times$ as a $\mathbf{C}^*$ -bundle. Let ζ be a vector field on $(L^{-1})^\times$ generating this $\mathbf{C}^*$ -action. We regard ω as a symplectic 2-form on $(L^{-1})^\times$ by the identification of X_{reg} with $(L^{-1})^\times$. Then one can write

$$\omega(\zeta, \cdot) = p^* \theta$$

with a twisted 1-form $\theta \in \Gamma(\mathbf{P}(X)_{reg}, \Omega_{\mathbf{P}(X)_{reg}}^1 \otimes L)$. We remark that p^*L has a natural trivialization on $(L^{-1})^\times$. This twisted 1-form θ determines a contact structure on $\mathbf{P}(X)_{reg}$.

Denote by $i : \mathbf{P}(X)_{reg} \rightarrow \mathbf{P}(X)$ the inclusion map. Then $i_*\Omega_{\mathbf{P}(X)_{reg}}^1 = \bar{\pi}_*\Omega_Z^1$. Since $\Gamma(\mathbf{P}(X), i_*\Omega_{\mathbf{P}(X)_{reg}}^1 \otimes \mathcal{O}_{\mathbf{P}(X)}(1)) = \Gamma(Z, \Omega_Z^1 \otimes \bar{\pi}^*\mathcal{O}_{\mathbf{P}(X)}(1))$, the twisted 1-form θ can be regarded as an element of $\Gamma(Z, \Omega_Z^1 \otimes \bar{\pi}^*\mathcal{O}_{\mathbf{P}(X)}(1))$.

Proposition 1. *Z has a contact structure with the contact line bundle $\bar{\pi}^*\mathcal{O}_{\mathbf{P}(X)}(1)$.*

For a contact projective manifold Z , the following structure theorem was proved by Kebekus, Peternell, Sommese and Wisniewski.

Theorem 2: *Assume that Z is a contact projective manifold with $b_2(Z) > 1$ and K_Z not nef. Then Z is a projectivized cotangent bundle $\mathbf{P}(T^*M) := T^*M - (0 - \text{section})/\mathbf{C}^*$ of a projective manifold M . Moreover, the contact line bundle is isomorphic to $\mathcal{O}_{\mathbf{P}(T^*M)}(1)$.*

As $K_Z \cong \bar{\pi}^*\mathcal{O}_{\mathbf{P}(X)}(-n)$, it is not nef. Moreover, since X is a symplectic variety of complete intersection, $\text{Codim}_X \text{Sing}(X) = 2$. Thus, if $n \geq 2$, then $\mathbf{P}(X)$ has singularities and $\bar{\pi}$ has an exceptional locus. This implies that $b_2(Z) > 1$. When $n = 1$, it is easily checked that X is isomorphic to an A_1 -surface singularity $z_1^2 + z_2^2 + z_3^2 = 0$ in $\mathbf{C}^3$.

In the remainder we assume that $n \geq 2$. Then Z is a projectivized cotangent bundle $\mathbf{P}(T^*M)$ for some projective manifold by the theorem above. Which kind of manifold is M ?

First notice that $\mathcal{O}_{\mathbf{P}(T^*M)}(1) = \bar{\pi}^*\mathcal{O}_{\mathbf{P}(X)}(1)$. In particular, $\mathcal{O}_{\mathbf{P}(T^*M)}(1)$ is nef. The following theorem was proved by Demailly, Peternell and Schneider.

Theorem 3. *Let M be a projective manifold with $\mathcal{O}_{\mathbf{P}(T^*M)}(1)$ nef. Assume that $\chi(M, \mathcal{O}_M) \neq 0$. Then M is a Fano manifold.*

Actually it is conjectured that M is a rational homogeneous space under the same assumption; but it is still open except when $\dim M = 2$ or 3 . But, in our case, we can prove more:

Proposition 4. *Let M be a Fano manifold such that $|\mathcal{O}_{\mathbf{P}(T^*M)}(1)|$ is free from base points. Then M is a rational homogeneous space, that is, $M = G/P$ with a complex semisimple Lie group G and its parabolic subgroup P .*

(Sketch of Proof). By the assumption we see that the map

$$H^0(M, \Theta_M) \otimes \mathcal{O}_M \rightarrow \Theta_M$$

is surjective. Let G be the neutral component of $\text{Aut}(M)$. Then G is a linear algebraic group. By the surjectivity G acts transitively on M . Hence M can be written as G/P with a parabolic subgroup P . Let us prove that G is semisimple. Let $r(G)$ be the radical of G . Then $r(G)$ is contained in P . But this implies that $r(G)$ acts trivially on G/P . By the definition of G , G acts effectively on M . This implies that $r(G) = 1$.

By the proposition $Z = \mathbf{P}(T^*(G/P))$. Let us consider the moment map $\mu : T^*(G/P) \rightarrow \mathfrak{g}^*$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g}^* \cong \mathfrak{g}$. Then $\text{Im}(\mu)$ coincides with a nilpotent orbit closure $\bar{O}$ of $\mathfrak{g}$. One can take the projectivization $\bar{\mu} : \mathbf{P}(T^*(G/P)) \rightarrow \mathbf{P}(\bar{O})$. We compare this map with $\bar{\pi} : Z \rightarrow \mathbf{P}(X)$. The embeddings $\mathbf{P}(\bar{O}) \rightarrow \mathbf{P}(\mathfrak{g})$ and $\mathbf{P}(X) \rightarrow \mathbf{P}^{2n+r-1}$ determine the tautological line bundles $O_{\mathbf{P}(\bar{O})}(1)$ and $O_{\mathbf{P}(X)}(1)$. One can check that $\bar{\mu}^*O_{\mathbf{P}(\bar{O})}(1) = O_{\mathbf{P}(T^*(G/P))}(1)$ and $\bar{\pi}^*O_{\mathbf{P}(X)}(1) = O_{\mathbf{P}(T^*(G/P))}(1)$. Moreover both $\bar{\mu}$ and $\bar{\pi}$ coincide with the morphisms determined by the complete linear system $|O_{\mathbf{P}(T^*(G/P))}(1)|$. This implies that $(\mathbf{P}(X), O_{\mathbf{P}(X)}(1)) \cong (\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1))$ as polarized varieties. Therefore $X \cong \bar{O}$ as a $\mathbf{C}^*$ -varieties.

The final task is to show that $\bar{O}$ is the nilpotent variety N of $\mathfrak{g}$. For simplicity we only discuss the case when $\mathfrak{g}$ is an exceptional simple Lie algebra. See [1] for other cases. We are now assuming that $\bar{O}$ is complete intersection in $\mathfrak{g}$. Then one can construct a G -equivariant morphism $f : \mathfrak{g} \rightarrow V$ from $\mathfrak{g}$ to a G -representation V with $\dim V = \text{Codim}_{\mathfrak{g}} \bar{O}$ in such a way that $f^{-1}(0) = \bar{O}$. By the argument above the embedding $X \rightarrow \mathbf{C}^{2n+r}$ is identified with the embedding $\bar{O} \rightarrow \mathfrak{g}$. Recall that we have $\sum a_i = n + r$. As $a_i > 1$ for each i , we have $2r \leq n + r$; hence $r \leq n$. Note that $\dim \mathfrak{g} = 2n + r$ and $\text{Codim}_{\mathfrak{g}} \bar{O} = r$. Thus we have

$$\dim V \leq 1/3 \cdot \dim \mathfrak{g}.$$

When $\mathfrak{g}$ is exceptional, there is no non-trivial irreducible G -representation that satisfies this inequality. Thus V is a direct sum of trivial G -representations. In particular, $\bar{O}$ is the common zeros of some adjoint invariant polynomials. On the other hand, N is the common zeros of *all* adjoint invariant polynomials. Hence $\bar{O}$ contains N . As all nilpotent orbits are contained in N , $\bar{O}$ is contained in N . Therefore $\bar{O} = N$.

REFERENCES

- [1] Y. Namikawa, *On the structure of homogeneous symplectic varieties of complete intersection*, arXiv: 1201.5444, to appear in Invent. Math.