

GENERALIZED VECTOR MEASURES AND FEYNMAN PATH INTEGRALS

お茶の水女大・理 竹尾富貴子(Fukiko TAKEO)

§1. Introduction.

Let us consider the following Cauchy problem

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} \Psi(t, x) = (-iH + V(t, x))\Psi(t, x) \\ \Psi(0, x) = g(x) \end{cases} \quad 0 < t < T, x \in \mathbb{R}^d$$

where $0 < T < \infty$, $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$, $V(t, \cdot) = \bar{V}(t)$ with

$\bar{V} \in C^1([0, T]; B(\mathbb{R}^d))$ and H is a self-adjoint operator on a Hilbert space $L^2(\mathbb{R}^d; \mathbb{C}^N)$.

Feynman[3] introduced the idea of path integral to make an intuitive representation of the Schroedinger equation. Various approaches to the "Feynman integral" have been taken by many mathematicians. In [1,2,6] they treated the Feynman integral by considering the analytic extension. K.Ito [5] gave the mathematical formulation of the Feynman integral by considering the Gauss measure in the Hilbert space. But those integrable functions are limited to a Fourier transform of a bounded complex measure or so on. In [7], I. Kluvanek defined the space of integrable functions which is complete with respect to the integrating seminorm depending on the norm of image of an operator μ_t . In a special case of a hyperbolic system which

includes the Dirac equation in two space-time dimensions, T. Ichinose [4] constructed a countably additive measure by using the L^∞ well-posedness of the Cauchy problem and gave the solution of the Cauchy problem by the Feynman integral with this measure.

In [8], we have constructed a $\mathfrak{L}(L^2(\mathbf{R}^d; \mathbf{C}^N))$ -valued generalized measure μ_t on the path space $\hat{X}_t$ and in case that $V(t, x)$ is independent of t , i.e. $V(t, x) = V(0, x)$, we gave the solution of $\Psi(t)$ of (1.1) by the Feynman integral.

In this paper, we shall examine the space $L(\mathfrak{L}, \beta)$ of integrable functions with respect to μ_t , which is defined as an extension of a tensor product space and is complete with respect to a seminorm β which does not depend on μ_t [Theorem 1]. $L(\mathfrak{L}, \beta)$ includes the function $F(X) = \exp \left\{ \int_0^t V(s, X(s)) ds \right\}$ with time-dependent potential $V(t, x)$. We shall also show that there is a kind of dominated convergence theorem with respect to μ_t [Theorems 2, 3] though it is not countably additive. By using this measure μ_t , we shall give the solution $\Psi(t)$ of (1.1) by the Feynman integral [Theorem 4]

$$\Psi(t) = \int_{\hat{X}_t} d\mu_t(X) \exp \left\{ \int_0^t V(s, X(s)) ds \right\} g(X(0)).$$

§2. Generalized vector measures μ_t on $\hat{X}_t$.

For $0 < t < \infty$, let $X_t = \prod_{[0,t]} \mathbb{R}^d$ be the product of the uncountably many copies of $\mathbb{R}^d$. Let Δ_n be a finite partition of the interval $[0,t]$ such that

$$\Delta_n: 0 = t_{0,n} < t_{1,n} < \dots < t_{2^n,n} = t, \quad \text{where } t_{j,n} = \frac{j}{2^n} t$$

and let σ_n be a mapping of X_t into itself such that

$$\sigma_n(X)(s) = \begin{cases} X(t_{j,n}) & \text{for } t_{j-1,n} < s \leq t_{j,n} (j=1,2,\dots,2^n) \\ X(0) & \text{for } s = 0 \end{cases}$$

for any $X \in X_t$. Let $\hat{X}_t$ be the subset of X_t such that

$$\hat{X}_t = \{ X \in X_t; X \in C([0,t]; \mathbb{R}^d) \text{ or } X \in \bigcup_{n=1}^{\infty} \sigma_n(X_t) \}.$$

For $F: \hat{X}_t \rightarrow \mathbb{C}$, define $F_{\sigma(n)}: \hat{X}_t \rightarrow \mathbb{C}$ by

$$F_{\sigma(n)}(X) = F(\sigma_n(X)) \quad \text{for any } X \in \hat{X}_t.$$

Let $\mathcal{B}$ be the set of Borel subsets of $\mathbb{R}^d$. For $n \in \mathbb{N}$ and $B_j \in \mathcal{B} (j=0,1,\dots,2^n)$, put $J(B_0, B_1, \dots, B_{2^n}) := \{X \in \hat{X}_t; X(t_{j,n}) \in B_j (j=0,1,\dots,2^n)\}$. Let $\mathcal{J} = \{J(B_0, B_1, \dots, B_{2^n}); n \in \mathbb{N}, B_j \in \mathcal{B}\}$ and $\mathcal{F}$ be the field generated by $\mathcal{J}$.

Let $\{U_t\}_{t \in \mathbb{R}}$ be a C_0 -group of unitary operators on $L^2(\mathbb{R}^d; \mathbb{C}^N)$. For $J = J(B_0, B_1, \dots, B_{2^n}) \in \mathcal{J}$, we shall define an operator $\mu_t(J) \in \mathfrak{L}(L^2(\mathbb{R}^d; \mathbb{C}^N))$ by

$$(\mu_t(J))g := U_{\delta_n} x_B U_{\delta_n} \cdots U_{\delta_n} x_{B_1} U_{\delta_n} x_{B_0} g \quad \text{for } g \in L^2(\mathbb{R}^d; \mathbb{C}^N),$$

where x_B is a multiplicative operator on $L^2(\mathbb{R}^d; \mathbb{C}^N)$ by the

characteristic function of the set B and $\delta_n = \frac{t}{2^n}$. Then μ_t can

be considered as a finitely additive $\mathfrak{L}(L^2(\mathbb{R}^d; \mathbb{C}^N))$ -valued measure defined on $\mathcal{J}$.

Now we shall consider the integral with respect to this measure μ_t .

$$\text{Put } x_m(x) = \begin{cases} 1 & \|x\| \leq m \\ 0 & \|x\| > m \end{cases} \quad \text{for } x \in \mathbb{R}^d.$$

For $\bar{a} \in \mathbb{C}^N$ and $J \in \mathcal{J}$, we shall write

$$(2.1) \quad \mu_t(J)\bar{a} := s\text{-}\lim_{m \rightarrow \infty} \mu_t(J)(\bar{a}x_m)$$

if the limit of the right-hand side exists

and we shall naturally use the integral as follows

$$(2.2) \quad \mu_t(J)\bar{a} = \int_{\hat{X}_t} d\mu_t(X) x_J(X)\bar{a}.$$

For $J = J(B_0, B_1, \dots, B_{2^n}) \in \mathcal{J}$ and relatively compact set

$C \in \mathcal{B}$, put $J \circ C := J(B_0 \cap C, B_1, \dots, B_{2^n})$. Then we have

$$(2.3) \quad \mu_t(J \circ C)\bar{a} = s\text{-}\lim_{m \rightarrow \infty} \mu_t(J \circ C)(\bar{a}x_m) = \mu_t(J)(\bar{a}x_C).$$

Let $\mathcal{S}_0$ be the space of $\mathcal{J}$ -measurable simple functions on $\hat{X}_t$.

For $g = \sum_{k=1}^r \bar{a}_k x_{C_k} \in L^2(\mathbb{R}^d; \mathbb{C}^N)$ ($\bar{a}_k \in \mathbb{C}^N$ and $C_k \in \mathcal{B}$ is

relatively compact) and $\Psi = \sum_{j=1}^q \alpha_j \chi_{J_j} \in \mathcal{J}_0$ ($\alpha_j \in \mathbb{C}$ and $J_j \in \mathcal{J}$),

we have

$$\int_{\hat{X}_t} d\mu_t(X) \Psi(X) g(X(0)) = \sum_{j=1}^q \alpha_j \sum_{k=1}^r \mu_t(J_j)(\bar{a}_k \chi_{C_k})$$

by using (2.2) and (2.3).

Let $B(\mathbb{R}^d)$ be the space of complex-valued bounded Borel measurable functions on $\mathbb{R}^d$ and $B(\hat{X}_t: \otimes_{\pi}, \Delta_n)$ be the space of complex-valued functions F on $\hat{X}_t$ for which there exist $m \in \mathbb{N}$ and functions $f_{j,k} \in B(\mathbb{R}^d)$ ($j=0,1,\dots,2^n$ and $k=1,2,\dots,m$) such that $F(X) = \sum_{k=1}^m \prod_{j=0}^{2^n} f_{j,k}(X(t_{j,n}))$ for any $X \in \hat{X}_t$, equipped with π -norm:

$$\|F\|_{\pi} := \inf \sum_{k=1}^m \prod_{j=0}^{2^n} \|f_{j,k}\|_{\infty},$$

where the infimum is taken over all representations of F and $\|f\|_{\infty} = \sup \{|f(x)|; x \in \mathbb{R}^d\}$. Let $B(\hat{X}_t: \hat{\otimes}_{\pi}, \Delta_n)$ be the completion of $B(\hat{X}_t: \otimes_{\pi}, \Delta_n)$ with respect to π -norm.

For $F \in B(\hat{X}_t: \hat{\otimes}_{\pi}, \Delta_n)$ and $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$, there exist sequences $\{\Psi_n\} \subset \mathcal{J}_0$ and $\{g_n\}$ of $\mathbb{C}^N$ -valued simple functions on $\mathbb{R}^d$ such that $\lim_{n \rightarrow \infty} \|\Psi_n - F\|_{\pi} = 0$ and $\lim_{n \rightarrow \infty} \|g - g_n\|_2 = 0$.

So we shall define the integral of $F \in B(\hat{X}_t: \hat{\otimes}_{\pi}, \Delta_n)$ by

$$(2.4) \quad \int_{\hat{X}_t} d\mu_t(X) F(X) g(X(0)) = s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) \Psi_n(X) g_n(X(0)).$$

for $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$.

This is well-defined since the right hand side of (2.4) does not depend on sequences $\{\Psi_n\}$ and $\{g_n\}$ but only on g and F . We

shall define the space $B(\hat{X}_t; \hat{\otimes}_\pi)$ as the space of complex-valued

functions F on $\hat{X}_t$ such that $F_{\sigma(n)}$ belongs to $B(\hat{X}_t; \hat{\otimes}_\pi, \Delta_n)$

for each $n \in \mathbb{N}$ and $\sup_n \|F_{\sigma(n)}\|_\pi < \infty$. We shall define the

seminorm β on $B(\hat{X}_t; \hat{\otimes}_\pi)$ by

$$\beta(F) = \sup_n \|F_{\sigma(n)}\|_\pi$$

for $F \in B(\hat{X}_t; \hat{\otimes}_\pi)$.

A subset C of $\hat{X}_t$ is said to be β -null if $\chi_C \in B(\hat{X}_t; \hat{\otimes}_\pi)$ and $\beta(\chi_C) = 0$, where χ_C is the characteristic function of the set C . For functions f, g on $\hat{X}_t$, $f(X) = g(X)$ β -a.e. means that the set $\{X \in \hat{X}_t; f(X) \neq g(X)\}$ is β -null.

DEFINITION. We shall call a function $F \in B(\hat{X}_t; \hat{\otimes}_\pi)$ to be integrable with respect to μ_t if for any $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$, there exists a sequence $\{\Psi_n\}$ of $\mathcal{S}_0$ satisfying $F(X) = \lim_{n \rightarrow \infty} \Psi_n(X)$ β -a.e. and there exists $s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) \Psi_n(X) g(X(0))$, which does not depend on $\{\Psi_n\}$ but only on F .

So we shall write

$$\int_{\hat{X}_t} d\mu_t(X) F(X) g(X(0)) = s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) \Psi_n(X) g(X(0)).$$

Let $B([0, t] \times \mathbb{R}^d)$ be the space of bounded Borel measurable functions θ on $[0, t] \times \mathbb{R}^d$ such that $\tilde{\theta}(s) = \theta(s, \cdot) \in B(\mathbb{R}^d)$ is piecewise continuous on $[0, t]$.

Let S be the set of those functions Ψ on $\hat{X}_t$ for which there exist $m \in \mathbb{N}$, $C_k \in \mathcal{B}([0, t] \times \mathbb{R}^d)$ (= set of Borel subsets of $[0, t] \times \mathbb{R}^d$) ($k=1, 2, \dots, m$) such that $\Psi(X) = \prod_{k=1}^m \int_0^t \chi_{C_k}(s, X(s)) ds$.

Let $\mathcal{J}$ be the linear span of $\mathcal{J}_0 \cup S$.

Let $L(\mathcal{J}, \beta)$ be the space of functions F of $B(\hat{X}_t; \hat{\mathcal{O}}_\pi)$ for which there exists a sequence $\{F_j\} \subset \mathcal{J}$ such that $\lim_{j \rightarrow \infty} \beta(F - F_j) = 0$. Then we have

Proposition. For $F \in L(\mathcal{J}, \beta)$ and $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$, there exists a sequence $\{h_n\}$ of $\mathcal{J}_0$ such that

- i) $F(X) = \lim_{n \rightarrow \infty} h_n(X)$ β -a.e. and
- ii) $s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) h_n(X) g(X(0))$ exists.

Proof. For $F \in L(\mathcal{J}, \beta)$, there exists a sequence $\{F_j\} \subset \mathcal{J}$, such that $\beta(F - F_j) < \frac{1}{2^j}$. For $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$ and $F_j \in \mathcal{J}$,

$s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) (F_j)_{\sigma(n)}(X) g(X(0))$ exists. For any $\varepsilon > 0$,

there exists $h_{j,n} \in \mathfrak{L}_0$ such that $\|(F_j)_{\sigma(n)} - h_{j,n}\|_{\pi} < \varepsilon$. So we can find $\{h_n\} \subset \mathfrak{L}_0$ satisfying the desired conditions. //

The above proposition shows that the space $L(\mathfrak{L}, \beta)$ consists of integrable functions with respect to μ_t and we have

Theorem 1. A C_0 -group $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on $L^2(\mathbb{R}^d; \mathbb{C}^N)$ induces a $\mathfrak{L}(L^2(\mathbb{R}^d; \mathbb{C}^N))$ -valued generalized measure μ_t on $\hat{X}_t$ such that the space $L(\mathfrak{L}, \beta)$ consisting of an integrable function with respect to μ_t is complete with respect to the seminorm β .

§3. The property of the generalized measure μ_t .

The generalized measure μ_t defined at §2 is not countably additive but it has a kind of convergence theorem as shown below.

DEFINITION. We shall call a sequence $\{f_n\} \subset B(\mathbb{R}^d)[\text{resp. } B([0, t] \times \mathbb{R}^d)]$ to be (*)-sequentially compact if for any subsequence $\{f_{n(j)}\}$ of $\{f_n\}$, there exists a subsequence

$\{f_{n(j(k))}\}$ of $\{f_{n(j)}\}$ such that $f_{n(j(k))}(x)$ [resp. $f_{n(j(k))}(s, x)$] converges to some function $g(x) \in B(R^d)$ for any $x \in R^d \setminus N$ [resp. $g(s, x) \in B([0, t] \times R^d)$ for any $(s, x) \in [0, t] \times (R^d \setminus N)$] with $\nu(N) = 0$ as $k \rightarrow \infty$, where ν is the Lebesgue measure on R^d .

Then we have the following convergence theorems.

Theorem 2. For $k, m \in N$ and $\{F_n\}_{n=0}^\infty \subset B(\hat{X}_t : \otimes_\pi, \Delta_m)$

with $F_n(X) = \sum_{\ell=1}^K \prod_{j=0}^{2^m} f_{j, \ell, n}(X(t_{j, m}))$, suppose

$\sup_n \|F_n\|_\pi < \infty$, $\lim_{n \rightarrow \infty} F_n(X) = F_0(X)$ a.e. on $R^{(2^m+1)d}$ and

$\{f_{j, \ell, n}; 1 \leq j \leq m, 1 \leq \ell \leq K, n \in N\}$ is $(*)$ -sequentially compact.

Then we have

$$\int_{\hat{X}_t} d\mu_t(X) F_0(X) g(X(0)) = s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) F_n(X) g(X(0))$$

for any $g \in L^2(R^d; C^N)$

Proof. By the assumption, there exist a subsequence

$\{n(k)\}$ and sequences $\{\tilde{f}_{j, \ell, n(k)}\} \subset B(R^d)$ and $\{h_{j, \ell}\} \subset B(R^d)$ satisfying

$$\sup \{\|\tilde{f}_{j, \ell, n(k)}\|_\infty; 1 \leq j \leq m, 1 \leq \ell \leq K, k \in N\} < \infty,$$

$$F_{n(k)}(X) - F_0(X) = \sum_{\ell=1}^K \prod_{j=0}^{2^m} \tilde{f}_{j, \ell, n(k)}(X(t_{j, m}))$$

$$\lim_{k \rightarrow \infty} \tilde{f}_{j, \ell, n(k)}(x) = h_{j, \ell}(x) \text{ a.e. on } R^d \text{ and}$$

for any $q=1, \dots, K$, there exists $j_q \in \{1, \dots, m\}$ such that

$\lim_{k \rightarrow \infty} \tilde{f}_{j_q, q, n(k)}(\dot{x}) = 0$ a.e. on $\mathbb{R}^d$. By using the property of

μ_t , we have $s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) (F_{n(k)}(X) - F_0(X))g(X(0)) = 0$. By

the property of $(*)$ -sequential compactness, we have

$$s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) (F_n(X) - F_0(X))g(X(0)) = 0. \quad //$$

For a function $F \in B(\hat{X}_t; \hat{\mathcal{O}}_\pi)$ we shall call $F(X) = 0$ α -a.e.

if $F_{\mathfrak{S}(n)}(X) = 0$ a.e. on $\mathbb{R}^{(2^n+1)d}$ for any $n \in \mathbb{N}$.

Then we have

Theorem 3. For $\{\theta_n\}_{n=0}^\infty \subset B([0, t] \times \mathbb{R}^d)$, put $F_n(X) =$

$\exp \int_0^t \theta_n(s, X(s)) ds$ for any $X \in \hat{X}_t$ and $n = 0, 1, \dots$. Suppose

$\lim_{n \rightarrow \infty} F_n(X) = F_0(X)$ α -a.e. and $\{\theta_n\}_{n=0}^\infty$ is $(*)$ -sequentially compact.

Then we have $F_n \in L(\mathfrak{J}, \beta)$ and

$$\int_{\hat{X}_t} d\mu_t(X) F_0(X) g(X(0)) = s\text{-}\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) F_n(X) g(X(0))$$

for any $g \in L^2(\mathbb{R}^d; \mathbb{C}^N)$.

Proof. By the property of (*)-sequential compactness, there exist a subsequence $\{n_j\}$ and $\tilde{\theta} \in B([0, t] \times \mathbb{R}^d)$ such that

$$(3.1) \quad \lim_{j \rightarrow \infty} \theta_{n_j}(s, x) = \tilde{\theta}(s, x) \quad \text{for any } (s, x) \in [0, t] \times (\mathbb{R}^d \setminus N)$$

with $\nu(N) = 0$. Then we have $\lim_{j \rightarrow \infty} F_{n_j}(X) = \exp \int_0^t \tilde{\theta}(s, X(s)) ds$

α -a.e., which implies $F_0(X) = \exp \int_0^t \tilde{\theta}(s, X(s)) ds$ α -a.e. Put

$$G_{n,k}(X) = \sum_{\ell=0}^k \frac{1}{\ell!} \left(\int_0^t \theta_n(s, X(s)) ds \right)^\ell \quad \text{and} \quad \tilde{G}_k(X) = \sum_{\ell=0}^k \frac{1}{\ell!}$$

$$\left(\int_0^t \tilde{\theta}(s, X(s)) ds \right)^\ell. \quad \text{By the definitions of the integral and } \mu_t,$$

$$\text{we have } \int_{\hat{X}_t} d\mu_t(X) \tilde{G}_k(X) g(X(0)) = \sum_{\ell=1}^k \int_0^t \int_0^{s_\ell} \dots \int_0^{s_2} U_{t-s_\ell} \tilde{\theta}(s_\ell)$$

$$U_{s_\ell-s_{\ell-1}} \tilde{\theta}(s_{\ell-1}) \dots U_{s_2-s_1} \tilde{\theta}(s_1) U_{s_1} g ds_1 ds_2 \dots ds_m + g \quad \text{for } g \in$$

$L^2(\mathbb{R}^d; \mathbb{C}^N)$. So by (3.1), we have

$$\lim_{n \rightarrow \infty} \int_{\hat{X}_t} d\mu_t(X) G_{n,k}(X) g(X(0)) = \int_{\hat{X}_t} d\mu_t(X) \tilde{G}_k(X) g(X(0)). \quad \text{By}$$

using the relation $\lim_{k \rightarrow \infty} \beta(F_n - G_{n,k}) = 0$, we have the desired result. //

§4. The Feynman path integral.

Now we shall consider the Cauchy problem described at §1.

By using the above theorem, we have

Theorem 4. Let H be a self-adjoint operator on a Hilbert space $L^2(\mathbb{R}^d; \mathbb{C}^N)$ and $0 < T < \infty$. Suppose $\bar{V} \in C^1([0, T]; B(\mathbb{R}^d))$, $V(t, \cdot) = \bar{V}(t)$ and g belongs to the domain of iH . Then the solution $\Psi(t, \cdot)$ of the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} \Psi(t, x) = (-iH + \theta(t, x))\Psi(t, x) \\ \Psi(0, x) = g(x) \end{cases} \quad 0 < t < T, x \in \mathbb{R}^d$$

is expressed as follows;

$$\Psi(t, \cdot) = \int_{\hat{X}_t} d\mu_t(X) \exp\left\{\int_0^t V(s, X(s)) ds\right\} g(X(0)).$$

Proof. iH generates a C_0 -group $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on $L^2(\mathbb{R}^d; \mathbb{C}^N)$ such that $U_t = e^{iHt}$. By Theorem 1, the C_0 -group induces a $\mathfrak{L}(L^2(\mathbb{R}^d; \mathbb{C}^N))$ -valued generalized measure μ_t on $\hat{X}_t$. Since $F(X) = \exp\left\{\int_0^t V(s, X(s)) ds\right\}$ belongs to $L(\mathfrak{L}, \beta)$, there exists its integral with respect to μ_t and put

$$\bar{\Psi}(t) = \int_{\hat{X}_t} d\mu_t(X) F(X) g(X(0)).$$

Then it holds that $\bar{\Psi}(t) = U_t \bar{\Psi}(0) + \int_0^t U_{t-s} \bar{V}(s) \bar{\Psi}(s) ds$ and $\bar{\Psi}(t)$ is the solution of the above Cauchy problem. //

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