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Fractional Weights and non-congruence subgroups

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Abstract

This note reviews the connection between the existence of fractional weight automorphic forms on real Lie groups, and the existence of non-congruence subgroups. It is intended to explain the simple results which are rarely even stated, and to avoid the complicated question of precisely where and why the congruence subgroup property fails. As a consequence, a new method is presented, for obtaining congruences between Eisenstein series and cusp forms in half-integral weight.

Let G be a (real) connected Lie group with a connected cyclic cover

$$1 \to \mu_n \to \tilde{G} \to G \to 1.$$

Here μ_n denotes the group of n-th roots of unity in $\mathbb{C}$. Suppose we have an arithmetic subgroup $\Gamma \subset G$. We shall discuss the following questions:

does Γ lift to a subgroup of $\tilde{G}$?

does Γ have a subgroup of finite index which lifts to $\tilde{G}$?

Example. Suppose the group G is $\mathrm{SL}_2(\mathbb{R})$. The fundamental group of G is $\mathbb{Z}$, and so for every $n \in \mathbb{N}$ there is a unique connected n-fold cover. For simplicity we shall assume that the arithmetic subgroup Γ is torsion-free.

A. If Γ has cusps then Γ is a free group. Therefore Γ lifts to every cover of G.

B. If Γ is cocompact then Peterson showed (see [7]) that Γ lifts to the n-fold cover if and only if n is a factor of the Euler characteristic $\chi(\Gamma)$. In particular for every n there is a Γ which lifts.

Very roughly speaking, Peterson's theorem is proved as follows. One finds a generator $\sigma \in H^2(G, \mathbb{Z})$ corresponding to the universal cover of G. A subgroup Γ lifts to the n-fold cover if and only if the image of σ in $H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ is a multiple of n. The image of σ in $H^2(G, \mathbb{R})$ is represented by an invariant 2-form on the upper half-plane. This 2-form turns out to be the Euler form. To find the image of σ in $H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ one integrates the 2-form over a fundamental domain for Γ. Hence by the Gauss-Bonnet theorem the image of σ in $H^2(\Gamma, \mathbb{Z})$ is $\chi(\Gamma)$. This implies the result.

1 Fractional weight multiplier systems

Let $\mathbb{C}^1$ denote the groups of complex numbers with absolute value 1. Suppose $w : G \times G \to \mu_n$ is a 2-cocycle representing the group extension $\tilde{G}$. By a weight w multiplier system on Γ, we shall mean a function $\chi : \Gamma \to \mathbb{C}^1$ such that

$$\chi(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)\chi(\gamma_1)\chi(\gamma_2).$$

In other words the image of w in $Z^2(\Gamma, \mathbb{C}^1)$ is the coboundary $\partial\chi$. If an arithmetic subgroup Γ lifts to $\tilde{G}$ then such a χ exists on Γ. We shall now prove a converse to this:

Proposition 1 *If there is a weight w multiplier system on an arithmetic subgroup $\Gamma \subset G$ then there is an arithmetic subgroup $\Gamma_0 \subset \Gamma$ which lifts to $\tilde{G}$.*

Proof. Suppose first that $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. In this case it is known (see [11]) that the commutator subgroup Γ' has finite index in Γ. From the exact sequence

$$1 \to \mu_n \to \mathbb{C}^1 \xrightarrow{n} \mathbb{C}^1 \to 1$$

we obtain a long exact sequence containing:

$$H^1(\Gamma, \mathbb{C}^1) \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

The image of w in $H^2(\Gamma, \mathbb{C}^1)$ is trivial, so w is the image of an element $\varphi \in H^1(\Gamma, \mathbb{C}^1)$. However $\varphi : \Gamma \to \mathbb{C}^1$ is just a character. Let $\Gamma_0 = \ker(\varphi)$. It follows that the restriction of w to Γ_0 is trivial, so Γ_0 lifts to $\tilde{G}$. Since $\Gamma_0 \supset \Gamma'$, it follows that Γ_0 is an arithmetic subgroup of G.

The above argument fails when $\mathrm{rk}_{\mathbb{R}}G = 1$ since Γ/Γ' is often infinite in this case. However since Γ is finitely generated, Γ/Γ' is a finitely generated abelian group, and so is of the form $F \oplus \mathbb{Z}^r$, where F is a finite abelian group. We extend our sequence one step to the left to give:

$$H^1(\Gamma, \mathbb{C}^1) \xrightarrow{\times n} H^1(\Gamma, \mathbb{C}^1) \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

This gives:

$$0 \to H^1(\Gamma, \mathbb{C}^1)/n \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

Note that we have

$$H^1(\Gamma, \mathbb{C}^1)/n = \mathrm{Hom}(F \oplus \mathbb{Z}^r, \mathbb{C}^1)/n = \mathrm{Hom}(F, \mathbb{C}^1)/n.$$

This implies

$$0 \to \mathrm{Hom}(F, \mathbb{C}^1)/n \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

We may therefore choose $\varphi : F \oplus \mathbb{Z}^r \to \mathbb{C}^1$ to be trivial on $\mathbb{Z}^r$. Hence $\ker(\varphi)$ again has finite index in Γ and the result follows as before. □

2 A trivial case

Suppose for a moment that the covering group $\tilde{G}$ is a linear group. In this case there is always some arithmetic subgroup Γ_0 of G which lifts to $\tilde{G}$. To see this, choose any arithmetic subgroup Γ of G and let $\tilde{\Gamma}$ be the preimage of Γ in $\tilde{G}$. Each element of the kernel μ_n is in $\tilde{\Gamma}$. For each of these elements apart from the identity, we can choose a congruence subgroup of $\tilde{\Gamma}$ not containing that element. Hence the intersection Γ_0 of all these congruence subgroups is a congruence subgroup with trivial intersection with μ_n. Thus Γ_0 is a lift to $\tilde{G}$ of a congruence subgroup of Γ.

3 A reformulation

In view of the above remark, it makes sense to assume that the group G is an (algebraically) simply connected linear group and that the covering group $\tilde{G}$ is non-linear. We shall make this restriction from now on.

In order to fix notation, we shall recall the definition of an arithmetic subgroup of the Lie group G. Suppose k is a totally real field with real places $v_1, \ldots, v_r$ and let $\mathcal{G}/k$ be an algebraic group such that

(i) $\mathcal{G}(k_{v_1})$ is isomorphic to G, and

(ii) $\mathcal{G}(k_{v_i})$ is compact for $i = 2, \ldots, r$.

We shall write $G(\mathcal{O})$ for the projection of $\mathcal{G}(\mathcal{O})$ onto G. By an arithmetic subgroup of G we mean a subgroup of G commensurable with some $G(\mathcal{O})$. As usual we let $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$.

Proposition 2 *Let $G/\mathbb{R}$ and $\mathcal{G}/k$ be as above.*

(i) Every topological cover $\tilde{\mathcal{G}}(k_\infty)$ of $\mathcal{G}(k_\infty)$ is of the form

$$\tilde{G} \oplus \mathcal{G}(k_{v_2}) \oplus \ldots \oplus \mathcal{G}(k_{v_r}),$$

for some unique cover $\tilde{G} \to G$.

(ii) An arithmetic subgroup Γ lifts from $\mathcal{G}(k_\infty)$ to $\tilde{\mathcal{G}}(k_\infty)$ if and only if its projection in G lifts to $\tilde{G}$.

Proof. Part (ii) is immediate from (i). To prove (i), we must show that for $i > 1$, the compact group $\mathcal{G}(k_{v_i})$ is (topologically) simply connected. Note that $\mathcal{G}(k_{v_i})$ is a compact real form of $\mathcal{G}(\mathbb{C}) = G(\mathbb{C})$, and is hence a maximal compact subgroup of $G(\mathbb{C})$. By the Iwasawa decomposition of $G(\mathbb{C})$, we know that $G(\mathbb{C})$ is homotopic to $\mathcal{G}(k_{v_i})$. However as $G/\mathbb{R}$ is (algebraically) simply connected, we know that $G(\mathbb{C})$ is simply connected. □

4 Metaplectic covers

Let $\mathcal{G}$ be a linear algebraic group over an algebraic number field k. We shall write $\mathbf{A}$ for the adèle ring of k. Let A be a finite Abelian group. By a *metaplectic extension* of $\mathcal{G}$ by A, we shall mean a topological central extension:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(\mathbf{A}) & \to & \mathcal{G}(\mathbf{A}) & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \mathcal{G}(k) & & \end{array},$$

which splits on the subgroup $\mathcal{G}(k)$ of k-rational points of $\mathcal{G}$. Suppose we have such an extension and let $\tilde{\mathcal{G}}(k_\infty)$ be the pre-image of $\mathcal{G}(k_\infty)$ in $\mathcal{G}(\tilde{\mathbf{A}})$. We therefore have an extension of Lie groups:

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \to \mathcal{G}(k_\infty) \to 1.$$

We shall show that this extension splits on a congruence subgroup of $\mathcal{G}(k_\infty)$.

To see this we let $\mathbb{A}_f$ denote the ring of finite adèles of k. As the map $\mathrm{pr} : \tilde{\mathcal{G}}(\mathbb{A}_f) \to \mathcal{G}(\mathbb{A}_f)$ is a topological covering, there is a neighbourhood U_1 of the identity in $\mathcal{G}(\mathbb{A}_f)$ such that $\mathrm{pr}^{-1}(U_1)$ is a disjoint union of homeomorphic copies of U_1. We may therefore choose a continuous section $\tau : U_1 \to \hat{U}_1$, where $\hat{U}_1$ is the copy of U_1 which contains the identity element of $\tilde{\mathcal{G}}(\mathbb{A}_f)$. Now define for $\alpha, \beta \in U_1$, $\sigma(\alpha, \beta) = \tau(\alpha)\tau(\beta)\tau(\alpha\beta)^{-1}$. Clearly σ is continuous on $U_1 \times U_1$ and has values in A. Furthermore $\sigma(1,1)$ is the identity element of A. Hence there is a neighbourhood U_2 of the identity in $\mathcal{G}(\mathbb{A}_f)$ such that σ is trivial on $U_2 \times U_2$. Now choose $U_3 \subset U_2$ to be a compact open subgroup of $\mathcal{G}(\mathbb{A}_f)$. On U_3 the section τ satisfies $\tau(\alpha\beta) = \tau(\alpha)\tau(\beta)$ and so the extension splits on U_3. Restricting the metaplectic extension we obtain:

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \circ \tau(U_3) \to \mathcal{G}(k_\infty) \oplus U_3 \to 1 .$$

(Remark: it is widely believed that the local factors of metaplectic groups always commute. This belief is false; some counterexamples are described in [8].) As U_3 commutes with $\mathcal{G}(k_\infty)$, it follows that the action of $\tau(U_3)$ by conjugation on $\tilde{\mathcal{G}}(k_\infty)$ is trivial in a neighbourhood of the identity of $\tilde{\mathcal{G}}(k_\infty)$. Therefore $\tau(U_3)$ acts by permuting the connected components of $\tilde{\mathcal{G}}(k_\infty)$. It follows that there is a subgroup U_4 of finite index in U_3, such that $\tau(U_4)$ commutes with $\tilde{\mathcal{G}}(k_\infty)$. We therefore have

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \oplus \tau(U_4) \to \mathcal{G}(k_\infty) \oplus U_4 \to 1 .$$

Now consider the congruence subgroup:

$$\Gamma = \mathcal{G}(k) \cap \Big(\mathcal{G}(k_\infty) \oplus U_4\Big).$$

As the metaplectic extension splits on $\mathcal{G}(k)$, we have by restriction:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) \oplus \tau(U_4) & \to & \mathcal{G}(k_\infty) \oplus U_4 & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \Gamma & & \end{array} .$$

Factoring out by U_4 and $\tau(U_4)$ in the above diagram, we obtain as required:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) & \to & \mathcal{G}(k_\infty) & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \Gamma & & \end{array} .$$

□

5 The congruence subgroup property

Let $\mathcal{G}/k$ be an absolutely simple and (algebraically) simply connected algebraic group over an algebraic number field k. We shall abbreviate $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. Assume also that $\mathcal{G}(k_\infty)$ is not topologically simply connected. The group $\mathcal{G}$ will be said to satisfy the *congruence subgroup property* if every arithmetic subgroup of $\mathcal{G}(k)$ is a congruence subgroup.

The question of whether congruence subgroups exist or not has been reformulated by Serre as follows. By the strong approximation theorem, we have

$$\mathcal{G}(\mathbb{A}_f) = \lim_{\leftarrow(\Gamma congruence)} G(k)/\Gamma.$$

Now define

$$\hat{\mathcal{G}}(\mathbb{A}_f) = \lim_{\leftarrow(\Gamma arithmetic)} G(k)/\Gamma.$$

There is a surjective map $\hat{\mathcal{G}}(\mathbb{A}_f) \to \mathcal{G}(\mathbb{A}_f)$. The kernel $C(\mathcal{G})$ of this map is called the *congruence kernel.* The congruence kernel is trivial if and only if all arithmetic subgroups are congruence subgroups. Serre has conjectured ([15]), that $C(\mathcal{G})$ is a finite subgroup of the centre of $\hat{\mathcal{G}}(\mathbb{A}_f)$ if and only if $\mathrm{rk}_{\mathbb{R}}(\mathcal{G}(k_\infty)) \geq 2$. Serre's conjecture in known for most groups of real rank ≥ 2. In particular the conjecture is known for all isotropic groups apart from groups of type ${}^2E_{6,1}$.

If Serre's conjecture holds for $\mathcal{G}$ of real rank ≥ 2, then our assumption that $\mathcal{G}(k_\infty)$ is not simply connected implies that

$$C(\mathcal{G}) \cong \mathrm{Hom}(\overline{\mathcal{G}(k)'}/\mathcal{G}(k)', \mathbf{C}^1),$$

where $\mathcal{G}(k)'$ is the commutator subgroup of $\mathcal{G}(k)$ and $\overline{\mathcal{G}(k)'}$ is its closure with respect to the subspace topology on $\mathcal{G}(k)$ induced from $\mathcal{G}(\mathbb{A}_f)$. In particular, if $\mathcal{G}(k)$ is perfect then $C(\mathcal{G})$ is trivial. Furthermore the triviality of $C(\mathcal{G})$ would follow from a conjecture of Platonov and Margulis (see [14]). This Conjecture is known in most cases. More precisely we have:

Theorem 1 (Congruence Subgroup Property) *Suppose $\mathcal{G}/k$ is absolutely simple and (algebraically) simply connected, but $\mathcal{G}(k_\infty)$ is not topologically simply connected. Suppose also that $\sum_{v|\infty} \mathrm{rk}_v \mathcal{G} \geq 2$. If either $\mathcal{G}/k$ is isotropic but not of type ${}^2E_{6,1}$, or $\mathcal{G}/k$ is anisotropic but not of type, E_6 or ${}^{3,6}D_4$, and not an outer form of type ${}^2A_n$ then $\mathcal{G}$ satisfies the congruence subgroup property*

The results and conjectures referred to above are more fully described in the useful survey [14].

6 A partial converse

We shall now prove a partial converse of the result of §4.

Theorem 2 *Let let $\mathcal{G}/k$ be absolutely simple and simply connected. Suppose there is a topological central extension*

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \to \mathcal{G}(k_\infty) \to 1,$$

which splits on some arithmetic subgroup Γ_0. If $\mathcal{G}$ satisfies the congruence subgroup property then this extension is the restriction to $\mathcal{G}(k_\infty)$ of a metaplectic extension of $\mathcal{G}$.

Remark 1 *In fact with some extra work one could replace the condition that all arithmetic subgroups are congruence subgroups by the weaker condition that the congruence kernel is finite. However, since $\mathcal{G}(k_\infty)$ is not topologically simply connected, it is conjectured that $C(\mathcal{G})$ is either infinite or trivial.*

Remark 2 *The theorem is essentially due to Deligne ([4]). Deligne makes the assumption that $\mathcal{G}(k)$ is perfect, which is slightly stronger than the congruence subgroup property here. However the assumptions are at least conjecturally equivalent.*

Proof. By the strong approximation theorem, $\mathcal{G}(k)$ is a dense subgroup of $\mathcal{G}(\mathbb{A}_f)$. We may therefore identify

$$\mathcal{G}(\mathbb{A}_f) = \varprojlim \mathcal{G}(k)/\Gamma,$$

where the limit is taken over the congruence subgroups, or equivalently over the arithmetic subgroups. We also define

$$\tilde{\mathcal{G}}(\mathbb{A}_f) = \varprojlim \tilde{\mathcal{G}}(k)/\tau(\Gamma),$$

where $\tilde{\mathcal{G}}(k)$ is the preimage of $\mathcal{G}(k)$ in $\tilde{\mathcal{G}}(k_\infty)$; Γ ranges over congruence subgroups of Γ_0 and $\tau : \Gamma_0 \to \tilde{\mathcal{G}}(k_\infty)$ is the splitting of the extension on Γ_0. For the moment we shall assume that $\tilde{\mathcal{G}}(\mathbb{A}(S))$ is a group.

The canonical projections $\tilde{\mathcal{G}}(k)/\tau(\Gamma) \to \mathcal{G}(k)/\Gamma$ induce a projection $\tilde{\mathcal{G}}(\mathbb{A}(S)) \to \mathcal{G}(\mathbb{A}(S))$. As $\tilde{\mathcal{G}}(\mathbb{A}(S))$ is a completion of $\tilde{\mathcal{G}}(k)$ it follows that we have a commutative diagramme:

$$\begin{array}{ccccccccc}
1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) & \to & \mathcal{G}(k_\infty) & \to & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \to & A & \to & \tilde{\mathcal{G}}(k) & \to & \mathcal{G}(k) & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & A & \to & \tilde{\mathcal{G}}(\mathbb{A}_f) & \to & \mathcal{G}(\mathbb{A}_f) & \to & 1.
\end{array}$$

Finally we define

$$\tilde{\mathcal{G}}(\mathbb{A}) = \left(\tilde{\mathcal{G}}(k_\infty) \oplus \tilde{\mathcal{G}}(\mathbb{A}_f)\right)/\Delta,$$

where $\Delta = \{(a, a) : a \in A\}$. As $(A \oplus A)/\Delta \cong A$, we have a central extension:

$$1 \to A \to \tilde{\mathcal{G}}(\mathbb{A}) \to \mathcal{G}(\mathbb{A}) \to 1.$$

The restriction of this extension to $\mathcal{G}(k_\infty)$ is our original extension. It remains show that this extension is metaplectic.

Choose any section $s : \mathcal{G}(k) \to \tilde{\mathcal{G}}(k)$ and define $t : \mathcal{G}(k) \to \left(\tilde{\mathcal{G}}(k) \oplus \tilde{\mathcal{G}}(k)\right)/\Delta$ by $t(\alpha) = (s(\alpha), s(\alpha))\Delta$. As the extensions are central we have $s(\alpha)s(\beta)s(\alpha\beta)^{-1} \in A$. Hence $t(\alpha)t(\beta)t(\alpha\beta)^{-1} \in \Delta$, so t is a homomorphism. This proves the theorem apart from the assertion that $\tilde{\mathcal{G}}(\mathbf{A}(S))$ is actually a group. □

Remark 3 *As the above theorem fails for the group* $\mathrm{SL}_2/\mathbb{Q}$*, and we have not yet used the congruence subgroup property, we may deduce that in this case the completion* $\widetilde{\mathrm{SL}_2}(\mathbb{A}_f)$ *is not a group.*

6.1 A remark on profinite limits

Suppose G is an abstract group and we have a directed system $\mathcal{F}$ of subgroups $\Gamma \subset G$. We shall call $\mathcal{F}$ *normal* if for every $g \in G$ and every $\Gamma \in \mathcal{F}$ the subgroup $g^{-1}\Gamma g$ contains an element of $\mathcal{F}$. If $\mathcal{F}$ is a normal filtration then the profinite limit

$$\bar{G} = \varprojlim_{\Gamma \in \mathcal{F}} G/\Gamma.$$

is a group (with the group operation continuous and compatible with the canonical map

To complete the proof of the above theorem we must show that the system of subgroups

$$\mathcal{F} = \{\tau(\Gamma) : \Gamma \text{ is a congruence subgroup of } \Gamma_0\}$$

is normal in $\tilde{\mathcal{G}}(k)$. Choose any $\tilde{g} \in \tilde{\mathcal{G}}(k)$ and any congruence subgroup $\Gamma \subseteq \Gamma_0$. Let g be the projection of $\tilde{g}$ in $\mathcal{G}(k)$. We define a section $\tau^g : \Gamma^g \to \tilde{\mathcal{G}}(k)$ by $\tau^g(g^{-1}\gamma g) = \tilde{g}^{-1}\tau(\gamma)\tilde{g}$. Clearly the image of τ^g is $(\tau(\Gamma))^{\tilde{g}}$.

The intersection $\Gamma \cap \Gamma^g$ is a congruence subgroup. Furthermore on $\Gamma \cap \Gamma^g$ we have two splittings τ and τ^g. As our extension is central we easily verify that

$$\tau^g(\gamma) = \varphi(\gamma)\tau(\gamma), \quad \gamma \in \Gamma \cap \Gamma^g,$$

where $\varphi : \Gamma \cap \Gamma^g \to A$ is a homomorphism. Finally let $\Gamma_1 = \ker\varphi$. As A is finite, Γ_1 is an arithmetic subgroup of Γ_0. Hence, by the congruence subgroup property, Γ_1 is a congruence subgroup. The sections τ and τ^g coincide on Γ_1. Therefore $\tau(\Gamma_1) \subseteq \tau^g(\Gamma^{\tilde{g}}) = \tau(\Gamma)^{\tilde{g}}$. □

6.2 The classification of metaplectic extensions.

The above theorem is useful because the mataplectic extensions of absolutely simple, simply connected groups have been classified. For such a group G one defined the *metaplectic kernel* $M(\mathcal{G})$ to be the kernel of the restriction

$$H^2(\mathcal{G}(\mathbb{A}), \mathbb{C}^1) \to H^2(\mathcal{G}(k), \mathbb{C}^1).$$

This group is conjectured to be isomorphic to the Pontryagin dual of the group of roots on unity in the base field k. This conjecture is proved in almost all cases (see [13]). Thus if $\mathcal{G}(k)$ is not topologically simply connected then (in almost all cases) the metaplectic kernel has order 2. As a consequence we obtain the following.

Theorem 3 *Let $G/\mathbb{R}$ be absolutely simple and simply connected and let $\tilde{G} \to G$ be a connected n-fold cyclic cover. Let Γ be a congruence subgroup of G such that every subgroup of finite index in Γ is a congruence subgroup. Furthermore, in the case that G is a special unitary group, assume that the construction of Γ does not involve is a non-abelian division algebra. If Γ lifts to $\tilde{G}$ then $n \leq 2$.*

Proof. The special unitary case we have excluded is the only case in which the metaplectic kernel is not known. Let $\sigma \in H^2(G, \mu_n)$ correspond to the extension. As the extension is part of a metaplectic extension, we know that the image of σ in $H^2(G, \mathbb{C}^1)$ has order at most 2. However we have an exact sequence

$$H^1(G, \mathbb{C}^1) \to H^2(G, \mu_n) \to H^2(G, \mathbb{C}^1).$$

As G is perfect, it follows that σ has order at most 2 in $H^2(G, \mu_n)$. □

7 Examples

The descriptions of fundamental groups of Sp_{2n}, SU and SO given below are taken from [16]. The results for $\mathrm{Spin}(p, q)$ may be found in [6].

7.1 Symplectic groups

The symplectic group $\mathrm{Sp}_{2r}(\mathbb{R})$ of rank r is absolutely simple and algebraically simply connected. However it's topological fundamental group is $\mathbb{Z}$. Hence $\mathrm{Sp}_{2r}(\mathbb{R})$ has an n-fold cover for every $n \in \mathbb{N}$. If $r = 1$ then $\mathrm{Sp}_{2r}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and it follows from Peterson's result that all fractional weights occur. However if $r \geq 2$, then we only have forms of half-integral weight. This was pointed out in [4].

7.2 Spin groups

Let $p \geq q \geq 1$. The spin group $\mathrm{Spin}(p,q)$ has rank q. The group $\mathrm{Spin}(2,2)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R}) \oplus \mathrm{SL}_2(\mathbb{R})$, so is not absolutely simple.

If $p \geq q \geq 3$ then the topological fundamental group of $\mathrm{Spin}(p,q)$ is μ_2, so we have only a double cover of $\mathrm{Spin}(p,q)$.

For $p \geq 3$ the group $\mathrm{Spin}(p,2)$ is absolutely simple and simply connected. The fundamental group is $\mathbb{Z}$, so this group has an n-fold cover for every n. The congruence subgroup property holds in this case. Hence we have only half-integral weight forms on $\mathrm{Spin}(p,2)$.

7.3 Orthogonal groups

Let $p \geq q \geq 1$. The special orthogonal group $\mathrm{SO}(p,q)$ has rank q. The group has two connected components. Let $O^+(p,q)$ denote the connected component of the identity. For $p \geq 3$ the fundamental group of $O^+(p,2)^o$ is $\mathbb{Z}/2 \oplus \mathbb{Z}$.

The group $\mathrm{Spin}(p,2)$ is the double cover of $O^+(p,2)^o$ corresponding to the infinite cyclic subgroup of $\mathbb{Z} \oplus \mathbb{Z}/2$ generated by $(1,1)$. Thus the unique double cover $\widetilde{\mathrm{Spin}}(p,2)$ of $\mathrm{Spin}(p,2)$ is the cover of $O^+(p,2)$ corresponding to the subgroup generated by $(2,0)$. This shows that $\widetilde{\mathrm{Spin}}(p,2)$ is a $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-cover of $O^+(p,2)$ (rather than a $\mathbb{Z}/4$-cover).

If we had a form of fractional weight on $O^+(p,2)$, then we could pull the form back to a fractional weight on $\mathrm{Spin}(p,2)$. However this form would be a function on $\widetilde{\mathrm{Spin}}(p,2)$. Hence the original form would have to be of half-integral weight.

7.4 Congruences between modular forms

We shall end by pointing out a consequence of the above result using Borcherds products. Recall that a *nearly holomorphic* modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f(q)$ on the upper half-plane, which has the usual transformation behaviour, but which may have a pole at ∞. In other words the Fourier expansion is allowed a finite number of negative terms:

$$f(q) = \sum_{n \gg -\infty} b_n q^n.$$

Let f be a nearly holomorphic form of weight $1 - l/2$, normalized so that $b_n \in \mathbb{Z}$ for all $n < 0$. Corresponding to such an f there is an automorphic form Ψ on $\mathrm{SO}(2,l)^o$ given by a Borcherds product (see [2],[3]). The weight of Ψ is $b_0/2$. As we know that there are only half-integral weight forms on $\mathrm{SO}(2,l)^o$ ($l \geq 3$), we deduce the following:

Corollary 1 *Let $f(q) = \sum b_n q^n$ be a nearly holomorphic form on $\mathrm{SL}_2(\mathbb{Z})$ negative weight. If $b_n \in \mathbb{Z}$ for $n < 0$ then $b_0 \in \mathbb{Z}$.*

For a nearly holomorphic form f, we shall call the negative part of its Fourier expansion the *principal part.* The following result is proved in [3].

Theorem 4 *Let $b_{-1}, \ldots, b_{-n} \in \mathbb{C}$. There is a nearly holomorphic form of (integral) weight $2-k$ and principal part $b_{-1}q^{-1} + \ldots + b_{-n}q^{-n}$ if and only if for every weight k cusp form $f(q) = \sum a_i q^i$, we have*

$$\sum_{i=1}^{n} a_i b_{-i} = 0.$$

If such a nearly holomorphic form exists then its constant term is given by

$$b_0 = \sum_{i=1}^{n} c_i b_i,$$

where $E(q) = 1 + \sum_{i=1}^{\infty} c_i q^i$ is the weight k Eisenstein series, normalized so as to have constant term 1.

Using this characterization, we may reformulate our corollary as follows.

Corollary 2 *Let E be the (integral) weight k level 1 Eisenstein series normalized so that the coefficients are integers with no common factor. Then there is a cusp form f such that the coefficients of f are congruent to those of E modulo the constant term of E.*

The above result can be obtained by much more elementary methods; in fact it follows immediately from the fact that E_4 and E_6 have constant term 1. One can however obtain a similar result for the vector-valued, half-integral weight forms studied in [3] in the same way. Such congruences have been proved for scalar valued forms of weight $\frac{3}{2}$ and prime level in [10]. However as far a I know for general half-integral weight, this is a new result.

References

[1] A. Borel, N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, AMS Mathematical Surveys and Monographs 67 (2000).

[2] R. E. Borcherds, "Automorphic forms with singularities on Grassmannians", Invent. Math. 132 (1998) 491-562.

[3] J. H. Bruinier, *Borcherds Products on O(2,l) and Chern Classes of Heegner Divisors*, Springer LNM 1780 (2002).

[4] P. Deligne, "Extensions centrales non résiduellement finies de groupes arithmetiques", C. R. Acad. Sci. Paris 287 (1978) 203-208.

[5] P. Deligne, "Extensions Centrales de Groupes Algébriques Simplement Connexes et Cohomologie Galoisienne", Publ. Math. I.H.E.S. 84 (1996) 35-89.

[6] A. J. Hahn, O. T. O'Meara, *The Classical Groups and K-Theory*, Springer Verlag 1989

[7] D. A. Hejhal, "The Selberg trace formula for $PSL(2,\mathbb{R})$" Springer Lecture Notes in Mathematics, 1001 (1983).

[8] R. Hill, "Space forms and Higher Metaplectic Groups" Math. Ann. 310 (1998) 735-775.

[9] D. Kazhdan, "Some applications of the Weil representation", J. Analyse Math. 32 (1977) 235-248.

[10] W. Kohnen, Antoniadis "Congruences between cusp forms and Eisenstein series of half-integral weight" Abh. Math. Sem. Univ. Hamburg 57 (1987) 157-164.

[11] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer Verlag 1991.

[12] G. Prasad, M. S. Raghunathan, "On the congruence subgroup problem: determination of the 'metaplectic kernel' ", Invent. Math. 71 (1983) 21-42.

[13] G. Prasad, A. S. Rapinchuk, "Computation of the Metaplectic Kernel", Publ. Math. I.H.E.S. 84 (1996) 91-187.

[14] A. S. Rapinchuk, "The congruence subgroup problem", Contemp. Math. 243 (1999) 175-188.

[15] J. P. Serre, "Le problème des groupes de congruence pour SL_2", Ann. of Math., 92 (1970) 489-527.

[16] D. Witte, *Arithmetic Groups and Locally Symmetric Spaces*, (preprint book `arXiv:math.DG/0106063`).