

Rankin-Selberg convolution for Cohen type Eisenstein series

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1 Introduction

In this report , we give a meromorphic continuation and a functional equation of a convolution product of two Cohen type Eisenstein series . Cohen type Eisenstein series is a real analytic automorphic form of half integral weight with respect to $\Gamma_0(4)$ with a parameter σ . When the parameter is fixed at a certain value , this convolution product appears as one of factors in an explicit formula of Koecher-Maass series associated with Siegel-Eisenstein series given by Ibukiyama and Katsurada[I-K].

On a proof , the convolution product is a convolution of the Mellin transformation of automorphic forms , we want to apply the Rankin-Selberg method . In this case , Cohen type Eisenstein series is not rapid decay as $y = \text{Im}z$ to infinity . So we use the Zagier's method which gives Rankin-Selberg method for the automorphic functions which are not of rapid decay [Z]. This method gives , for a $SL_2(\mathbf{Z})$ invariant function $G(z)$ on the upper half plane which satisfies an increasing condition such that $G(z) = G(z) - \sum_{i=1}^l \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i}(y)(z = x + iy)$ is rapid decay as $y \rightarrow \infty$, define $R(s)$ by $R(s) = \int_0^\infty \int_0^1 G(z) y^{s-2} dx dy$. Then this integral converges absolutely for $\text{Re}(s)$ sufficiently large , and by applying the unfolding method for a integral over the truncated domain , $R(s)$ can be expressed by sum of rational functions of s and integrals using Eisenstein series $E(z, s) = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$ over certain domain . Then analytic properties of $E(z, s)$ gives a meromorphic continuation and a functional equation of $R(s)$. S.Gupta[G] generalized it for invariant functions respect to congruence subgroup which satisfies the same type increasing condition at each cusps .

In section 2, we apply their method for an automorphic form $G(z)$ respect to congruence subgroup Γ ($G(\gamma z) = (cz+d)^k G(z)$ for $\gamma \in \Gamma$, k is not nesessaly zero) which satisfies the same type increasing condition at each cusps of Γ . And in section 3, as an application, we give a meromorphic continuation and a functional equation of a convolution product of Cohen type Eisenstein series. To obtain the functional equation, we use the plus condition of Fourier coefficients of Cohen type Eisenstein series and Kohnen's method on modular forms belonging to plus space.

2 Rankin-Selberg method for the automorphic forms which are not of rapid decay

Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the complex upper half plane and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ acts on $z \in H$ by $gz = \frac{az+b}{cz+d}$. Let Γ be a congruence subgroup which satisfies that

$$-I \in \Gamma \text{ and } \Gamma_{\infty}^+ = \Gamma \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\{x_1, \dots, x_h\}$ be the set of representative of non equivalent cusps of Γ . Take some $g \in SL_2(\mathbb{R})$ such that $g i\infty = x_i$ and put

$$\Gamma_i = \Gamma \cap g \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbb{R} \right\} g^{-1}, \quad \Gamma_i^+ = \Gamma \cap g \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbb{R} \right\} g^{-1}.$$

Γ_i is independent of a choice of such $g \in SL_2(\mathbb{R})$. By the assumption $-I \in \Gamma$, we can take and fix such $g_i \in SL_2(\mathbb{R})$; $g_i i\infty = x_i$ so that

$$\Gamma_i^+ = g_i \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle g_i^{-1}$$

for each i (especially we put $x_1 = i\infty, g_1 = I$).

Fix two integers k and p . For $z \in H$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, put $\nu(g, z) = \frac{(cz+d)^k}{|cz+d|^p}$. Let χ be a character of Γ . We assume that χ satisfies $\chi(-I) = (-1)^k$ for above k and $\chi(\Gamma_i^+) = \{1\}$ for all i .

Then we define Eisenstein series associated to the cusp x_i by

$$E_i(z, s, \chi) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} \overline{\chi(\gamma)} \nu(g_i^{-1} \gamma, z) \operatorname{Im}(g_i^{-1} \gamma z)^s.$$

This series converges absolutely for $\operatorname{Re}(s) + \frac{p-k}{2} > 1$.

Further for another cusp x_j , we put

$$E_{i,j}(z, s, \chi) = \nu(g_j, z) E_i(g_j z, s, \chi).$$

Then this has a period 1 and has a Fourier expansion. Especially its constant term can be written as

$$a_{0,i,j}(y, s, \chi) = \delta_{i,j} y^s + \varphi_{i,j}(s - \frac{k-p}{2}, \chi) y^{1-s+k-p},$$

where $\delta_{i,j}$ is the Kronecker delta, and $\varphi_{i,j}(s, \chi)$ is a certain function. Then we put

$$\Phi(s, \chi) = (\varphi_{i,j}(s, \chi))_{1 \leq i, j \leq l}.$$

Theorem 1 We use assumptions and notations as above.

Let $\xi(z)$ be a continuous function on H which satisfies the following two conditions (a) and (b).

(a)

$$\xi(\gamma z) = \overline{\chi(\gamma)} \nu(\gamma, z) \xi(z) = \overline{\chi(\gamma)} \frac{(cz + d)^k}{|cz + d|^p} \xi(z),$$

for all $z \in H, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(b) For each i ($i = 1, \dots, h$), put $\xi_i(z) = \nu(g_i, z)^{-1} \xi(g_i z)$ and assume that $\xi_i(z) = \psi_i(y) + O(y^{-N})$ for any N as $y \rightarrow \infty$, where $\psi_i(y)$ is a function of the form

$$\psi_i(y) = \sum_{m_i: \text{finite sum}} \frac{c_{i,m_i}}{n_{i,m_i}!} y^{\alpha_{i,m_i}} \log^{n_{i,m_i}}(y) \quad , \quad c_{i,m_i}, \alpha_{i,m_i} \in \mathbb{C}, n_{i,m_i} \in \mathbb{N} \geq 0.$$

For such function $\xi(z)$, define the Rankin-Serberg transformation of $\xi(z)$ at the cusp x_i by

$$R_i(s, \xi) = \int_0^\infty \int_0^1 (\xi_i(z) - \psi_i(y)) y^{s-2} dx dy.$$

Then we claim ,

(i) $R_i(s, \xi)$ converges absolutely for $\text{Re}(s)$ sufficiently large .

When we take (it is possible) the fundamental domain D for the action of Γ in the form

$$D = D_0 \cup (\cup_{j=1}^h g_j D^Y),$$

where D_0 is a relative compact open subset in H and

$$D^Y = \{z = x + iy; 0 < x < 1, y > Y\}$$

for sufficiently large Y , and put

$$D_Y = \{x + iy; 0 < x < 1, y \leq Y\}, d\mu = \frac{dx dy}{y^2},$$

then we have a following expression .

$$\begin{aligned} R_i(s, \xi) = & \sum_{j=1}^h \left\{ \int \int_{D^Y} \xi_j(z) (E_{i,j}(z, s, \chi) - a_{0,i,j}(y, s, \chi)) d\mu \right. \\ & + \int \int_{D^Y} (\xi_j(z) - \psi_j(y)) a_{0,i,j}(y, s, \chi) d\mu \\ & + \int \int_{D^Y} \psi_j(y) \varphi_{ij}(s - \frac{k-p}{2}, \chi) y^{1-s+k-p} d\mu \quad \left. \right\} \\ & + \int \int_{D_0} \xi(z) E_i(z, s, \chi) d\mu \\ & - \int \int_{D_Y} \psi_i(y) y^s d\mu. \end{aligned}$$

(ii) If we assume further that each Eisenstein series $E_i(z, s, \chi)$ ($i = 1, \dots, h$) can be meromorphically continued to all s -plane, then so is $R_i(s, \xi)$ and we have a following functional equation .

$$\begin{aligned} \mathcal{R}(s, \xi) &= \begin{pmatrix} R_1(s, \xi) \\ \vdots \\ R_h(s, \xi) \end{pmatrix} \\ &= \Phi(s - a, \chi) \mathcal{R}(1 - s + 2a, \xi). \end{aligned}$$

Proof is same as in Gupta[G] .

3 An application for Cohen type Eisenstein series

We apply above result in case

$$\Gamma = \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) : c \equiv 0 \pmod{4} \right\},$$

and $\xi(z) = F_1(z)\overline{F_2(z)}$ where $F_j(z)$ ($j = 1, 2$) are Cohen type Eisenstein series defined as follows .

For each $\sigma \in \mathbf{C}$, an odd integer k , define Eisenstein series $E(k, \sigma, z)$ as

$$E(k, \sigma, z) = y^{\sigma/2} \sum_{d=1, \text{odd}}^{\infty} \sum_{c=-\infty}^{\infty} \left(\frac{4c}{d}\right) \epsilon_d^{-k} (4cz + d)^{k/2} |4cz + d|^{-\sigma}$$

where for $\gamma = \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \in \Gamma_0(4)$, we put

$$j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)} = \epsilon_d^{-1} \left(\frac{4c}{d}\right) (4cz + d)^{1/2} \quad , \quad \theta(z) = \sum_{n \in \mathbf{Z}} e^{2\pi i n^2 z}$$

is well known automorphic factor of $\Gamma_0(4)$. This $E(k, \sigma, z)$ converges absolutely and uniformly for $-k+2\sigma-4 > 0$. And put $E^*(k, \sigma, z) = E(-\frac{1}{4z})(-2iz)^{k/2}$. Then Cohen type Eisenstein series is defined by

$$F(k, \sigma, z) = E(k, \sigma, z) + 2^{k/2-\sigma} (e(k/8) + e(-k/8)) E^*(k, \sigma, z).$$

This satisfies

$$F(k, \sigma, \gamma z) = j(\gamma, z)^{-k} F(k, \sigma, z) \quad \gamma \in \Gamma_0(4)$$

and has a Fourier expansion as

$$F(k, \sigma, z) = y^{\sigma/2} + \sum_{d=-\infty}^{\infty} c(d, \sigma, k) y^{\sigma/2} e(dx) \tau_d(y, \frac{\sigma-k}{2}, \frac{\sigma}{2}),$$

where $e(x) = e^{2\pi i x}$ and $\tau_d(y, \alpha, \beta)$ is same as in [I-S] . $\tau_d(y, \alpha, \beta)$ relate to the Whittaker function $W_{\alpha, \beta}(y) = y^{\alpha} e^{-\frac{y}{2}} \omega(y, \frac{1}{2} + \alpha + \beta, \frac{1}{2} - \alpha + \beta)$, where $\omega(y, \alpha, \beta) = \frac{y^{\beta}}{\Gamma(\beta)} \int_0^{\infty} (1+u)^{\alpha-1} u^{\beta-1} e^{-yu} du$ as

$$\tau_d(y, \alpha, \beta) = i^{\beta-\alpha}$$

$$\times \begin{cases} (2\pi)^\alpha \Gamma(\alpha)^{-1} d^{\alpha-1} e^{-2\pi d y} (2y)^{-\beta} (4\pi d y)^{\frac{\beta-\alpha}{2}} e^{2\pi d y} W_{\frac{\alpha-\beta}{2}, \frac{\alpha+\beta-1}{2}}(4\pi d y), \\ (2\pi)^\beta \Gamma(\beta)^{-1} |d|^{\beta-1} e^{-2\pi |d| y} (2y)^{-\alpha} (4\pi |d| y)^{\frac{\alpha-\beta}{2}} e^{2\pi |d| y} W_{\frac{\beta-\alpha}{2}, \frac{\beta+\alpha-1}{2}}(4\pi |d| y), \\ (2\pi)^{\alpha+\beta} \Gamma(\alpha)^{-1} \Gamma(\beta)^{-1} \Gamma(\alpha + \beta - 1) (4\pi y)^{1-\alpha-\beta}, \end{cases}$$

It is known that , $W_{\alpha,\beta}(y)$ is continued holomorphically to whole $(\alpha, \beta) \in \mathbb{C}^2$, and

$$\lim_{y \rightarrow \infty} \sup_{(\alpha,\beta) \in K} |y^{-\alpha} e^{\frac{y}{2}} W_{\alpha,\beta}(y) - 1| = 0$$

for any compact subset K in $\mathbb{C}^2$.

Fourier coefficient $c(d, \sigma, k)$ is explicitly calculated in [I-S] and it is proven in [I-S]p10 that Fourier coefficients satisfy the plus condition ; if $(-1)^{\frac{k+1}{2}} d \equiv 2, 3 \pmod{4}$ then $c(d, \sigma, k) = 0$.

For odd integer k_j and $\sigma_j \in \mathbb{C}$ ($j = 1, 2$) , we put $F_j(z) = F(k_j, \sigma_j, z)$.

Theorem 2 If $k_1 \equiv k_2 \pmod{4}$, $-k_j + 2\sigma_j - 4 > 0$ ($j = 1, 2$) ,

$$\Omega(s) = 2^{2s} \pi^{-s} \Gamma(s + \frac{k_1}{2}) \zeta(2s + \frac{k_1}{2} + \frac{k_2}{2}) R_\infty(s, F_1 \overline{F_2})$$

($\Gamma(s)$ is gamma function and $\zeta(s)$ is Riemann zeta function) can be meromorphically continued to all s -plane and satisfies

$$\Omega(s) = \Omega(1 - \frac{k_1}{2} - \frac{k_2}{2} - s).$$

(proof) We will check the conditions in Theorem 1 for $\Gamma = \Gamma_0(4)$, and $\xi(z) = F_1(z) \overline{F_2(z)}$.

$$F_1(\gamma z) \overline{F_2(\gamma z)} = \frac{(cz + d)^{\frac{k_2-k_1}{2}}}{|cz + d|^{k_2}} F_1(z) \overline{F_2(z)} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

is the condition (a) if we put $p = k_2, k = \frac{k_2-k_1}{2}$, and $\chi = id \pmod{4}$. As the set of representative of non equivalent cusps , and the element $g \in SL_2(\mathbb{R})$ which transforms $i\infty$ to each cusps , we take $\{i\infty, 0, \frac{1}{2}\}$ and

$$g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_0 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}, g_1 = \begin{pmatrix} 1 & -1/2 \\ 2 & 0 \end{pmatrix}.$$

Each Eisenstein series $E_i(z, s) = E_i(z, s, id)$ associated to cusp have a meromorphic continuation to whole s -plane by Fourier expansion .

Some components of the matrix $\Phi(s, id)$ which we need for a functional equation are given by

$$\phi_{\infty, \infty}(s) = \frac{2^{1-4s}}{1-2^{-2s}} \frac{\zeta(2(1-s))\Gamma((1-s) - \frac{k}{2})\pi^{-(1-s)}}{\zeta(2s)\Gamma(s - \frac{k}{2})\pi^{-s}} , \quad k = \frac{k_2 - k_1}{2}$$

$$\phi_{\infty, 0}(s) = \phi_{\infty, 1}(s) = 2^{-2s} \frac{1 - 2^{1-2s}}{1 - 2^{-2s}} \frac{\zeta(2(1-s))\Gamma((1-s) - \frac{k}{2})\pi^{-(1-s)}}{\zeta(2s)\Gamma(s - \frac{k}{2})\pi^{-s}}.$$

To check the condition (b) , and to obtain a Dirichlet series expansion of Rankin-Selberg transformations , we use a Fourier expansion of $(2z)^{\frac{k}{2}} F(k, \sigma, g_i z)$ ($i = 0, 1$) which can be obtained by a method of Kohnen [K]. Next result is proven in [I-S] .

Proposition 1 (I-S, Lemma 3.1) *For any function $f(z)$ on H , define*

$$(f|_k W_4)(z) = f(-\frac{1}{4z})(-2iz)^{k/2} , \quad (f|U_4)(z) = \frac{1}{4} \sum_{\nu=0}^3 f(\frac{z+\nu}{4})$$

Then for $F(z) = F(k, \sigma, z)$,

$$F|_k W_4 = 2^{\frac{k+1}{2}} (-1)^{\frac{k^2-1}{8}} F|U_4 \quad (1)$$

$$(F|U_4)(z) = \frac{1}{2} (F(\frac{z}{4}) + F(\frac{z+2}{4})) \quad (2)$$

Using these relations , we obtain Fourier expansion of $(2z)^{\frac{k}{2}} F(k, \sigma, g_i z)$.

Proposition 2 *If $F(z) = F(k, \sigma, z) = \sum_{d=-\infty}^{\infty} c(d, y) e(dx)$ then*

$$(2z)^{k/2} F(g_i z) = e(k/8) (-1)^{(k^2-1)/8} 2^{(k+1)/2} \sum_{d \equiv i \pmod{2}} c(d, y/4) e(dx/4)$$

(proof) For $i = 0, 1$, put $F^i(z) = \sum_{d \equiv i \pmod{2}} c(d, y/4) e(dx/4)$. Then (2) implies $F|U_4(z) = F^0(z) \dots (2')$. By definition of W_4 and (1), (2') , we obtain

$$(-2iz)^{\frac{k}{2}} F(g_0 z) = (-1)^{(k^2-1)/8} 2^{(k+1)/2} F^0(z) \quad (3)$$

This proves the case $i = 0$. Next by definition ,

$$F^0(z) + F^1(z) = F\left(\frac{z}{4}\right) \quad (4)$$

$$F\left(z + \frac{1}{2}\right) = F^0(4z) - F^1(4z) \quad (5)$$

So

$$\begin{aligned} & (-2iz)^{\frac{k}{2}} F(g_1 z) \\ = & (-2iz)^{k/2} F\left(-\frac{1}{4z} + \frac{1}{2}\right) \\ = & (-2iz)^{\frac{k}{2}} \left(F^0\left(-\frac{1}{z}\right) - F^1\left(-\frac{1}{z}\right)\right) \quad (\text{by (5)}) \\ = & (-2iz)^{\frac{k}{2}} \left(2F^0\left(-\frac{1}{z}\right) - F\left(-\frac{1}{4z}\right)\right) \quad (\text{use (4) for } F^1\left(-\frac{1}{z}\right)) \\ = & (-1)^{(k^2-1)/8} 2^{(k+1)/2} \left(F\left(\frac{z}{4}\right) - F^0(z)\right) \quad (\text{use (3) for each terms}) \\ = & (-1)^{(k^2-1)/8} 2^{(k+1)/2} F^1(z) \quad \text{by (4)} \end{aligned}$$

this gives the case $i = 1$.

If we take

$$\psi_\infty(y) = c_1(0, y) \overline{c_2(0, y)}, \quad \psi_1(y) = 0$$

$$\psi_0(y) = e\left(\frac{k_1 - k_2}{8}\right) \prod_{i=1}^2 (-1)^{(k_i^2-1)/8} 2^{(k_i+1)/2} c_1(0, y/4) \overline{c_2(0, y/4)},$$

where $c_j(0, y)$ is 0-th Fourier coefficient of $F_j(z)$, then by the facts on $\tau_d(y, \alpha, \beta)$ stated above and I-S Corollary 2.4;

$$c(0, y) = y^{\sigma/2} + 2^{3k/2-3\sigma+7/2} (-1)^{(k^2-1)/8} \pi \frac{\Gamma(\sigma - k/2 - 1) \zeta(2\sigma - k - 2)}{\Gamma((\sigma - k)/2) \Gamma(\sigma/2) \zeta(2\sigma - k - 1)} y^{1-\sigma/2+k/2}$$

, the condition(b) in theorem is satisfied . Thus each Rankin-Selberg transformation at each cusps $R_j(s, F_1 \overline{F_2}) (j = \infty, 0, 1)$ is continued meromorphically to all s-plane and these become

$$R_\infty(s, F_1 \overline{F_2}) = \sum_{d \neq 0} c(d, \sigma_1, k_1) \overline{c(d, \sigma_2, k_2)} I_d(s, \sigma_1, \sigma_2, k_1, k_2)$$

$$\begin{aligned}
R_i(s, F_1 \overline{F_2}) &= e\left(\frac{k_1 - k_2}{8}\right) \prod_{i=1}^2 (-1)^{(k_i^2 - 1)/8} 2^{(k_i + 1)/2} 4^{s-1} \\
&\quad \times \sum_{d \neq 0, d \equiv i \pmod{2}} c(d, \sigma_1, k_1) \overline{c(d, \sigma_2, k_2)} I_d(s, \sigma_1, \sigma_2, k_1, k_2)
\end{aligned}$$

($i=0,1$) for $Re(s)$ sufficiently large, where

$$\begin{aligned}
I_d(s, \sigma_1, \sigma_2, k_1, k_2) &= \int_0^\infty \tau_d(y, \frac{\sigma_1 - k_1}{2}, \frac{\sigma_1}{2}) \overline{\tau_d(y, \frac{\sigma_2 - k_2}{2}, \frac{\sigma_2}{2})} y^{\frac{\sigma_1 + \overline{\sigma_2}}{2} + s - 2} dy \\
&= |d|^{\frac{\sigma_1 + \overline{\sigma_2}}{2} - \frac{k_1 + k_2}{2} - 1 - s} e\left(\frac{k_1 - k_2}{8}\right) (2\pi)^{\sigma_1 + \overline{\sigma_2} - \frac{k_1 + k_2}{2}} (4\pi)^{-\frac{\sigma_1 + \overline{\sigma_2}}{2} - s + 1} \\
&\quad \times \int_0^\infty y^{\frac{k_1 + k_2}{4} - 2 + s} W_{\frac{-sgn(d)k_1}{4}, \frac{\sigma_1}{2} - \frac{k_1}{4} - \frac{1}{2}}(y) W_{\frac{-sgn(d)k_2}{4}, \frac{\overline{\sigma_2}}{2} - \frac{k_2}{4} - \frac{1}{2}}(y) dy \\
&\quad \times \begin{cases} \Gamma(\frac{\sigma_1 - k_1}{2})^{-1} \overline{\Gamma(\frac{\sigma_2 - k_2}{2})}^{-1}, & \text{for } d > 0 \\ \Gamma(\frac{\sigma_1}{2})^{-1} \overline{\Gamma(\frac{\sigma_2}{2})}^{-1}, & \text{for } d < 0 \end{cases}
\end{aligned}$$

So by Theorem 1, we obtain

$$\begin{aligned}
&R_\infty(s, F_1 \overline{F_2}) \\
&= \left\{ e\left(\frac{k_1 - k_2}{8}\right) \prod_{i=1}^2 (-1)^{(k_i^2 - 1)/8} 2^{(k_i + 1)/2} 4^{-\frac{k_1 + k_2}{2} - s} \phi_{\infty,0}\left(s + \frac{k_1 + k_2}{4}\right) \right. \\
&\quad \left. + \phi_{\infty,\infty}\left(s + \frac{k_1 + k_2}{4}\right) \right\} R_\infty\left(1 - s - \frac{k_1 + k_2}{2}, F_1 \overline{F_2}\right) \\
&= \frac{2^{2(1-s-\frac{k_1+k_2}{2})}}{2^{2s}} \frac{\pi^s}{\pi^{1-s-\frac{k_1+k_2}{2}}} \frac{\Gamma(1-s-\frac{k_2}{2})}{\Gamma(s+\frac{k_1}{2})} \frac{\zeta(2-2s-\frac{k_1}{2}-\frac{k_2}{2})}{\zeta(2s+\frac{k_1}{2}+\frac{k_2}{2})} \\
&\quad \times R_\infty\left(1 - s - \frac{k_1}{2} - \frac{k_2}{2}, F_1 \overline{F_2}\right),
\end{aligned}$$

in the second equation, we use $k_1 \equiv k_2 \pmod{4}$.

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We set $k_j = \sigma_j = 2\kappa_j$ where $\kappa_j \in \mathbf{Z} + \frac{1}{2}$, > 2 is a half integer for $j = 0, 1$, and observe by the formula of Fourier coefficients in [I-S] that if $\sigma \in \mathbf{R}$ then

$e(\frac{k}{8})c(d, \sigma, k) \in \mathbf{R}$ for $d \neq 0$, then Dirichlet series $D_{2\kappa_1-1}^*(s) \otimes D_{2\kappa_2-1}^*(s)$ which appears in an explicit formula of Koecher-Maass series for Siegel-Eisenstein series ([I-K]) is equal to

$$(-1)^{2\kappa_1-1} \pi^{1-2\kappa_1-2\kappa_2} \prod_{i=1}^2 \Gamma(\kappa_i) \Gamma\left(\frac{2\kappa_i-1}{2}\right) \zeta(2\kappa_i-1) \\ \times \frac{(4\pi)^s}{\Gamma(s)} \zeta(2s - \kappa_1 - \kappa_2 + 2) R_\infty(s - \kappa_1 - \kappa_2 + 1, F_1 \overline{F_2}).$$

By the explicit form in [I-K] and Theorem 2, we can check the functional equation of Koecher-Maass series of Siegel-Eisenstein series.

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