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On unramified pro-p Galois groups over cyclotomic $\mathbb{Z}_p$-extensions — A survey

By

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Abstract

For a fixed prime number p, we denote by k_∞ the cyclotomic $\mathbb{Z}_p$-extension of a given number field k. We expect that the Galois group $G(k_\infty)$ of the maximal unramified pro-p-extension over k_∞ would provide good information about the Galois groups of p-class field towers of number fields. In this paper, we will give an overview of some topics on $G(k_\infty)$ together with an announcement of some results in $p = 2$ case.

§ 1. Introduction

Let p be a fixed prime number and $\mathbb{Z}_p$ the ring of p-adic integers. For a given finite extension k of the field $\mathbb{Q}$ of rational numbers, we denote by k_∞ the cyclotomic $\mathbb{Z}_p$-extension of the number field k. The Galois group $\Gamma = \mathrm{Gal}(k_\infty/k)$ is isomorphic to the additive group of $\mathbb{Z}_p$ and has a topological generator γ. The main object of this paper is the Galois group

$$G(k_\infty) = \mathrm{Gal}(\widetilde{L}(k_\infty)/k_\infty)$$

of the maximal unramified pro-p-extension $\widetilde{L}(k_\infty)$ of k_∞. By choosing a suitable section $\Gamma \hookrightarrow \mathrm{Gal}(\widetilde{L}(k_\infty)/k) : \gamma \mapsto \widetilde{\gamma}$ of the natural exact sequence

$$1 \to G(k_\infty) \to \mathrm{Gal}(\widetilde{L}(k_\infty)/k) \to \Gamma \to 1$$

(which splits since Γ is a free pro-p group) such that $\widetilde{\gamma}$ is an element of the inertia subgroup of a prime lying above p, we define an action of Γ on $G(k_\infty)$ via the left conjugations by $\widetilde{\gamma}$, i.e., we define a continuous homomorphism

$$\phi : \Gamma \to \mathrm{Aut}\, G(k_\infty)$$

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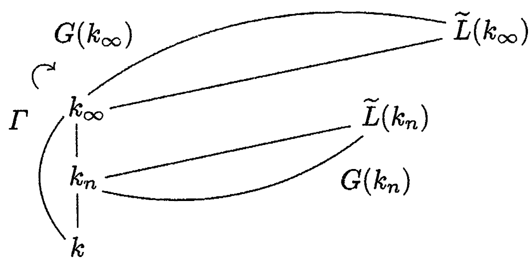
such that $\phi(\gamma)(g) = {}^{\gamma}g = \tilde{\gamma}g\tilde{\gamma}^{-1}$ for $g \in G(k_\infty)$. Then, the Galois group $G(k_\infty)$ is a pro-p-Γ operator group with ϕ (cf. [15] p.216, [23] I.1). To know the unramified pro-p Galois group $G(k_\infty)$ as a pro-p-Γ operator group is almost equivalent to knowing $\mathrm{Gal}(\widetilde{L}(k_\infty)/k) \simeq G(k_\infty) \rtimes \Gamma$ as a pro-p group.

For each integer $n \geq 0$, we denote by k_n the n-th layer of k_∞, i.e., the cyclic subextension of degree p^n over k. We are also interested in the Galois group $G(k_n) = \mathrm{Gal}(\widetilde{L}(k_n)/k_n)$ of the maximal unramified pro-p-extension $\widetilde{L}(k_n)$ of k_n. To borrow the words of Wingberg [23], the unramified pro-p Galois group is "one of the most mysterious objects in algebraic number theory". The sequence of unramified p-extensions associated to the commutator series of $G(k_n)$ is a classic object called p-class field tower of k_n. Especially, the abelianization of $G(k_n)$ is the Galois group of the Hilbert p-class field $L(k_n)$ over k_n, and the metabelian quotient of $G(k_n)$ is deeply related to the capitulation problem on the p-Sylow subgroup $A(k_n)$ ($\simeq \mathrm{Gal}(L(k_n)/k_n)$) of the ideal class group of k_n.

If n is sufficiently large, there is a surjective homomorphism $G(k_\infty) \twoheadrightarrow G(k_n)$ induced from the restriction mapping. Then, we can regard $G(k_n)$ as a quotient of $G(k_\infty)$, and the structure of $G(k_n)$ is reflected by the relations of pro-p group $G(k_\infty)$ and the action of Γ. By the induced projective system, we have an isomorphism $G(k_\infty) \simeq \varprojlim G(k_n)$.

In this paper, we investigate the Galois group $G(k_\infty)$ by expecting that its structure as a pro-p-Γ operator group would give good information about the Galois groups $G(k_n)$ of p-class field towers of k_n. As the grounds of the expectations, we shall see some topics on the Galois groups $G(k_\infty)$ and $G(k_n)$ in the next section. In the third section, we will see some examples of explicitly presented (abelian or metabelian) $G(k_\infty)$.



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§ 2. Related topics

2.1. From abelian Iwasawa theory. By the action of Γ induced from ϕ, the abelianization $X(k_\infty)$ of $G(k_\infty)$ is considered as an Iwasawa module, i.e., a module over the complete group ring $\mathbb{Z}_p[[\Gamma]]$. The module $X(k_\infty)$ is identified with the Galois group of the maximal unramified abelian pro-p-extension $L(k_\infty)$ of k_∞, and it is proven by Iwasawa that $X(k_\infty)$ is finitely generated and torsion as a $\mathbb{Z}_p[[\Gamma]]$-module. Then, we can define the Iwasawa invariants $\lambda = \lambda(X(k_\infty)) = \dim_{\mathbb{Q}_p}(X(k_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, $\mu = \mu(X(k_\infty))$ and the characteristic polynomial

$$P(T) = \det\big((1+T)id - \gamma \,|\, X(k_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \big)$$

of the Iwasawa module $X(k_\infty)$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers (not p-th layer of $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$!). Based on the analogy with Alexander polynomial of a knot, it is pointed out in [14] that the Iwasawa polynomial $P(T)$ is also obtained in the words of pro-p Fox differential calculus if we have a presentation of $\mathrm{Gal}(\widetilde{L}(k_\infty)/k)$ explicitly.

For the cyclotomic $\mathbb{Z}_p$-extensions of any finite extensions of $\mathbb{Q}$, the vanishing of μ-invariants is conjectured by Iwasawa. Since "$\mu = 0$" is equivalent to the finiteness of the rank of $X(k_\infty)$ as a $\mathbb{Z}_p$-module, we can put this claim in the words about $G(k_\infty)$ as follows:

"$\mu = 0$" conjecture. The Galois group $G(k_\infty)$ is finitely generated as a pro-p group, i.e., the generator rank $d(G(k_\infty)) = \dim_{\mathbb{F}_p} H^1(G(k_\infty), \mathbb{Z}/p\mathbb{Z}) < \infty$.

Ferrero and Washington [3] proved that this conjecture is true if k is an abelian extension over $\mathbb{Q}$. This is an advantage of treating cyclotomic $\mathbb{Z}_p$-extensions.

Further, if k is a certain CM-field, the Iwasawa polynomial $P(T)$ is deeply related to the p-adic L-functions by the theorems of Mazur and Wiles [10] [22], namely "Iwasawa's main conjecture". Especially, if k is an imaginary quadratic field with the associated Dirichlet character χ ($\neq \omega$ the Teichmüller character), we have a power series $f(T) \in \mathbb{Z}_p[[T]]$ constructed from Stickelberger elements such that $(f(T)) = (2P(T))$ as a principal ideal of $\mathbb{Z}_p[[T]]$ and the Kubota-Leopoldt's p-adic L-function $L_p(s, \omega\chi) = f(\kappa(\gamma)^s - 1)$, where $\kappa : \Gamma \to \mathbb{Z}_p^\times$ is the restricted cyclotomic character.

2.2. Nonabelian Iwasawa type formulae. We define the lower central series of $G(k_\bullet)$ by putting $C^{(1)}(k_\bullet) = G(k_\bullet)$ and $C^{(i+1)}(k_\bullet) = [C^{(i)}(k_\bullet), G(k_\bullet)]$ for $i \geq 1$ inductively. The bracket means a topologically closed commutator subgroup. We also put the quotients $X^{(i)}(k_\bullet) = C^{(i)}(k_\bullet)/C^{(i+1)}(k_\bullet)$ for $i \geq 1$.

In [18], Ozaki defined the i-th Iwasawa module as the quotient $X^{(i)}(k_\infty)$ with the action of Γ induced from ϕ, and showed some basic properties. Especially, for each $i \geq 1$, $X^{(i)}(k_\infty) \simeq \varprojlim X^{(i)}(k_n)$ with respect to the restriction mappings. Note that $X^{(1)}(k_\infty) = X(k_\infty)$. If $\mu = 0$, the i-th Iwasawa module $X^{(i)}(k_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module with $\mu(X^{(i)}(k_\infty)) = 0$ for each $i \geq 1$. Then, the i-th Iwasawa λ-invariant is defined as $\lambda^{(i)} = \lambda(X^{(i)}(k_\infty)) = \dim_{\mathbb{Q}_p}(X^{(i)}(k_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

By considering the structure of $X^{(i)}(k_\infty)$ and putting $\widetilde{\lambda}^{(i)} = \sum_{j=1}^{i} \lambda^{(j)}$, Ozaki gave the following nonabelianization of Iwasawa's formula.

Theorem 2.1 (Ozaki [18]). *Assume that $\mu = 0$, and fix any $i \geq 1$. Then, there exists an integer $\widetilde{\nu}^{(i)}$ such that*

$$\#(G(k_n)/C^{(i+1)}(k_n)) = p^{\widetilde{\lambda}^{(i)} n + \widetilde{\nu}^{(i)}}$$

for all sufficiently large n.

Here, for each i, we denote by $n_0^{(i)}$ the minimal non-negative integer such that the above formula holds for all $n \geq n_0^{(i)}$.

The p-group $G(k_n)/C^{(i+1)}(k_n)$ is the maximal nilpotency-class-i quotient of $G(k_n)$. For $i = 1$, the formula above is well known as the Iwasawa's class number formula "$\#A(k_n) = p^{\lambda n + \mu p^n + \nu}$ $(n \gg 0)$" with $\mu = 0$ since $G(k_n)/C^{(2)}(k_n) \simeq A(k_n)$. In the case that $i = 2$, the asymptotic version "$\#(G(k_n)/C^{(3)}(k_n)) = p^{\widetilde{\lambda}^{(2)} n + o(1)}$ $(n \to \infty)$" has been proven by Fujii [5] under a certain condition.

The Ozaki's formula implies that the Galois groups $G(k_n)$ of p-class field towers also behave well Iwasawa-theoretically, i.e., the action of Γ on $G(k_\infty)$ controls the behavior of $G(k_n)$. Toward a nonabelianization of Iwasawa's main conjecture, Ozaki [17] asked that "What kind of p-adic functions relate to $X^{(i)}(k_\infty)$ and $G(k_\infty)$?". We are also interested in how p-adic L-functions relate to them.

2.3. Freeness and infinite p-class field towers. Based on the property that the Galois groups of p-class field towers are finitely presented, Golod and Shafarevich [7] gave a criterion for the infiniteness of p-class field towers. When the p-class field tower is infinite, we are interested in the cohomological dimension of the Galois group.

On the other hand, Ozaki [17] gave the following problem:

Problem 2.2. *Is the Galois group $G(k_\infty)$ always finitely presented as a pro-p group? Especially, the relation rank $r(G(k_\infty)) = \dim_{\mathbb{F}_p} H^2(G(k_\infty), \mathbb{Z}/p\mathbb{Z}) < \infty$?*

Though the general answer of this problem is not clear yet, Fujii and Okano [6] showed that $\#G(k_n) = \infty$ for sufficiently large n if $\infty > d(G(k_\infty))^2 \gg 4\,r(G(k_\infty))$, based on the idea of Wingberg [23]. Especially, they investigated the consequences under the assumption that $G(k_\infty)$ is a free pro-p group, i.e., $r(G(k_\infty)) = 0$.

Theorem 2.3 (Fujii-Okano [6]). *Let p be odd, and k a CM-field with the maximal totally real subfield k^+, and S the set of primes of k_∞ lying above p.*

(1) *Assume that $\#S = 1$, $\#A(k^+) = 1$ and $\dim_{\mathbb{F}_p}(3A(k)/3pA(k)) \geq 2$. If $G(k_\infty)$ is a free pro-p group, then $\#G(k_n) = \infty$ for all $n \geq 1$.*

(2) *Assume that $X(k_\infty^+) \simeq \mathbb{Z}/p\mathbb{Z}$, $\lambda \geq 1+2\sqrt{1+\delta+\#S}$ where $\delta = 1$ or 0 according to whether k contains a primitive p-th root ζ_p of unity or not. If $G(k'_\infty)$ is a free pro-p group for $k'_\infty = k_\infty L(k_\infty^+)$, then $\#G(k_n) = \infty$ for all sufficiently large n and $G(k_n)$ has an element of order p. (Especially, the cohomological dimension of $G(k_n)$ is infinite.)*

For each odd p, by using the result of [25], we can find infinitely many imaginary abelian extensions k of degree $2p$ satisfying the assumptions of (2) except for the freeness of $G(k'_\infty)$. In general, the freeness of $G(k_\infty)$ seems to be very delicate. Though the freeness for some CM-fields k (e.g., p-th cyclotomic field $k = \mathbb{Q}(\zeta_p)$) were treated in [23] (and [17] etc.), we have to pay attention to the pointing out (final Remark of [19]) and the results (announced in [20]) by Sharifi. Unfortunately, it seems that we have no concrete example of nonabelian free $G(k_\infty)$ yet.

If $G(k_\infty)$ is a nonabelian free pro-p group, we can see that $\lambda^{(i)}$ tends to infinity as $i \to \infty$. It is a considerable problem to find examples such that $\lambda^{(i)}$ (or $\widetilde{\lambda}^{(i)}$) are unbounded as $i \to \infty$.

2.4. p-adic analyticity and finite p-class field towers. For any finite dimensional vector space V_p over $\mathbb{Q}_p$ and any linear continuous representation $\rho : G(k_n) \to GL(V_p)$, it is conjectured (as a part of the conjecture by Fontaine and Mazur [4]) that the image of ρ is finite. In other words, this claim asserts that:

Fontaine-Mazur conjecture. The Galois group of p-class field tower has no infinite p-adic analytic quotient.

Since any finitely generated p-adic analytic pro-p group has an open powerful subgroup, we can replace the word "p-adic analytic" with "powerful" in the statement of this conjecture. As a weak version of this conjecture, we are also interested in the problem that whether the Galois group $G(k_n)$ itself can be infinite p-adic analytic (resp. powerful) or not. For this problem, Wingberg [24] proved the following by considering the Galois group $G(k_\infty)$.

Theorem 2.4 (Wingberg [24]). *Assume that p is odd and k is a CM-field containing ζ_p, and that $\mu = 0$ for k_∞. If n is sufficiently large and $G(k_n)$ is powerful, then $\#G(k_n) < \infty$.*

On the other hand, under the assumption that both "$\mu = 0$" conjecture and Fontaine-Mazur conjecture hold, we can easily show the following by the properties of p-adic analytic pro-p groups.

Proposition 2.5. *Assume that $\mu = 0$ for k_∞ and $G(k_\infty)$ is p-adic analytic. If Fontaine-Mazur conjecture (in the sense above) holds for $G(k_n)$, then $\#G(k_n) < \infty$.*

Proof. Put $H = \mathrm{Gal}(\widetilde{L}(k_n)/k_\infty \cap \widetilde{L}(k_n))$. Then H is an open subgroup of $G(k_n)$ and isomorphic to a quotient of $G(k_\infty)$. Since $G(k_\infty)$ has finite rank in the sense of [2] Definition 3.12 (cf. [2] Theorem 3.13, Corollary 8.33), H is also a pro-p group of finite rank (cf. [2] Exercise 3.1). Therefore, $G(k_n)$ is p-adic analytic. Since $G(k_n)$ has no infinite p-adic analytic quotient, $G(k_n)$ must be finite. □

The border between finite cases and infinite cases is one of the main theme in the study of p-class field towers. While the freeness of $G(k_\infty)$ provides criteria for infiniteness of $G(k_n)$ (Theorem 2.3, etc.), Proposition 2.5 implies that the p-adic analyticity of $G(k_\infty)$ provides criteria for finiteness of $G(k_n)$. Then, for $G(k_\infty)$, what is the border area between nearly free cases and p-adic analytic cases? It seems to be interesting problem to characterize number fields k with p-adic analytic $G(k_\infty)$ (including the cases that $G(k_\infty)$ becomes finite).

2.5. Greenberg's conjecture. For any totally real number field k, it is conjectured that $\#X(k_\infty) < \infty$ by Greenberg [8]. Since $X(k_\infty) = X^{(1)}(k_\infty) \simeq \varprojlim X^{(1)}(k_n)$, this claim is equivalent to that $\lambda = \mu = 0$, i.e., $X^{(1)}(k_\infty) \simeq X^{(1)}(k_n)$ for all $n \gg 0$. Since any finite unramified p-extension of k_∞ is also the cyclotomic $\mathbb{Z}_p$-extension of a certain totally real number field (which is actually a finite unramified p-extension of k_n for some n), we can extend this conjecture as follows:

Greenberg's conjecture (nonabelianized version). If k is a totally real number field, any open subgroup of $G(k_\infty)$ has finite abelianization (i.e., $G(k_\infty)$ satisfies "FIFA").

Let us call this property FIFA due to Boston [1]. The positive answers of this conjecture and Problem 2.2 imply that $G(k_\infty)$ is similar to the Galois groups of p-class field towers if k is totally real. From this point of view, Ozaki gave the following problem as a strong version of Greenberg's conjecture.

Problem 2.6. *If k is a totally real number field, $G(k_\infty) \simeq G(k_n)$ for $n \gg 0$?*

This claim is equivalent to that $\lambda = \mu = 0$ and $n_0^{(i)}$ is bounded as $i \to \infty$. If $G(k_\infty)$ is finite, this claim holds immediately. The finiteness of $G(k_\infty)$ is equivalent to the existence of a finite extension K over k such that $\#X(K_\infty) = 1$ (i.e., $\lambda = \mu = \nu = 0$ for K_∞). The abelian p-extensions K of $\mathbb{Q}$ with trivial $X(K_\infty)$ are completely characterized by Yamamoto ([25] etc.). It is also a considerable problem to characterize all finite (especially, p-)extensions K of $\mathbb{Q}$ with trivial $X(K_\infty)$.

If $G(k_\infty) \simeq G(k_n)$ for some $n \gg 0$, ϕ is not injective since $\phi(\gamma^{p^n}) = 1$. On the other hand, if ϕ is not injective, Γ^{p^n} acts on $G(k_\infty)$ trivially for all $n \gg 0$. Then, under the assumption that k is totally real and Leopoldt's conjecture holds for p and all subfields of $\widetilde{L}(k_\infty)$, we can show that $G(k_\infty)$ satisfies FIFA by using Proposition 1 of [8]. The injectivity of ϕ in the totally real case seems to be considerable as a problem between Greenberg's conjecture and Problem 2.6.

$$\begin{tikzcd}
\boxed{\exists K/k \text{ finite}\,;\, \#X(K_\infty) = 1} \arrow[r, Leftrightarrow] & \boxed{\#G(k_\infty) < \infty} \arrow[d, Rightarrow] & \boxed{0 \neq r(G(k_\infty)) < \infty} \\
\boxed{\lambda = \mu = 0,\ \sup\{n_0^{(i)}\} < \infty} \arrow[r, Leftrightarrow] & \boxed{\forall n \gg 0\,;\, G(k_\infty) \simeq G(k_n)} \arrow[d, Rightarrow] \arrow[r, Rightarrow] \arrow[ur, Rightarrow] & \boxed{G(k_\infty) \text{ satisfies FIFA}} \\
\boxed{\text{Leopoldt's conjecture}} \arrow[r, phantom, "+"] & \boxed{\#\mathrm{Ker}\phi \neq 1} \arrow[ur, Rightarrow, "*"] &
\end{tikzcd}$$

For an imaginary quadratic field k in which p splits, the unique $\mathbb{Z}_p^{\oplus 2}$-extension $\widetilde{k}$ of k is unramified over k_∞. It is conjectured (as Greenberg's generalized conjecture) that the abelianization of $\mathrm{Gal}(\widetilde{L}(k_\infty)/\widetilde{k})$ is pseudo-null as a $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/k)]]$-module. If this is true, $G(k_\infty)$ is not a nonabelian free pro-p group (cf. [17] etc.). On the other hand, we can find many examples for which $\widetilde{L}(k_\infty) = \widetilde{k}$, i.e., $G(k_\infty) \simeq \mathbb{Z}_p$ (not FIFA!) but $\#\mathrm{Im}\phi = 1$. (The arrow $\overset{*}{\Rightarrow}$ above depends on the totally reality of k.)

§ 3. Explicitly presented examples

3.1. Abelian examples. If $X(k_\infty)$ is a $\mathbb{Z}_p$-module of rank 1, then $G(k_\infty) \simeq X(k_\infty)$, i.e., $G(k_\infty)$ is also a cyclic pro-p group. In the case that $X(k_\infty)$ is not cyclic, it is not a trivial problem whether $G(k_\infty)$ is abelian or not. The abelianity of $G(k_\infty)$ is equivalent to the vanishing of second Iwasawa module $X^{(2)}(k_\infty)$. As an easiest case, we can show the following with nontrivial examples.

Proposition 3.1. *Let p be odd and k a CM-field containing ζ_p, and assume that $\mu = 0$. If $\lambda = 1$, then $G(k_\infty) \simeq \mathbb{Z}_p \oplus \mathbb{Z}/p^m\mathbb{Z}$ with some $m \geq 0$.*

Proof. Put $X^+ = X(k_\infty^+)$ for the maximal real subfield k^+ of k, and let X^- be the minus part of $X(k_\infty)$. Since p is odd, $X(k_\infty) \simeq X^+ \oplus X^-$. Since $\mu = 0$, X^- is a free $\mathbb{Z}_p$-module ([21] Corollary 13.29). By Leopoldt's Spiegelungssatz ([21] Theorem 10.11), we know that $\lambda(X^+) \leq \mathrm{rank} X^+ \leq \mathrm{rank} X^- = \lambda(X^-)$. Since $\lambda(X^+) + \lambda(X^-) = \lambda = 1$ by our assumption, we know that $G(k_\infty^+) \simeq X^+ \simeq \mathbb{Z}/p^m\mathbb{Z}$ with some $m \geq 0$ and $X^- \simeq \mathbb{Z}_p$. Then, $K_\infty^+ = \widetilde{L}(k_\infty^+)$ is an unramified finite cyclic p-extension of k_∞^+. Put

$K_\infty = k_\infty K_\infty^+$. Note that K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of a certain CM-field K, and that $\#X(K_\infty^+) = 1$. By Kida's formula [9], we know that $\mu(X(K_\infty)) = 0$ and $\lambda(X(K_\infty)) = 1$. Since $G(K_\infty) \simeq X(K_\infty) \simeq \mathbb{Z}_p$, we can see that $\widetilde{L}(k_\infty) = \widetilde{L}(K_\infty) = L(k_\infty)$. Therefore, $G(k_\infty) \simeq X(k_\infty) \simeq \mathbb{Z}_p \oplus \mathbb{Z}/p^m\mathbb{Z}$. □

By using the result of Yamamoto [25], Kida's formula [9] and Proposition 3.1, we can easily find infinitely many abelian sextic fields k containing $\mathbb{Q}(\sqrt{-3})$ such that $G(k_\infty) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}/3\mathbb{Z}$ in $p = 3$ case.

For odd p and $k = \mathbb{Q}(\zeta_p)$, it is announced by Sharifi [20] that $G(k_\infty)$ is abelian if $p < 1000$ (and there exists $p > 1000$ such that $G(k_\infty)$ is nonabelian!). Especially, $G(k_\infty) \simeq \mathbb{Z}_p^{\oplus 2}$ for $p = 157$, and $G(k_\infty) \simeq \mathbb{Z}_p^{\oplus 3}$ for $p = 461$. Further, for odd p, Okano [16] characterized an imaginary quadratic field k with noncyclic abelian $G(k_\infty)$ as follows:

Theorem 3.2 (Okano [16]). *For odd p and an imaginary quadratic field k, $G(k_\infty)$ is noncyclic abelian if and only if $\lambda = 2$ and $A(k)$ is generated by the ideal classes containing some power of a prime ideal above p. Then, $G(k_\infty) \simeq \mathbb{Z}_p^{\oplus 2}$.*

For odd p and imaginary quadratic fields k, the abelianity of $G(k_\infty)$ and the powerfulness of $G(k_\infty)$ are equivalent (cf. [24] Proposition 2.1). Also in $p = 2$ case, all imaginary quadratic fields k with abelian $G(k_\infty)$ are characterized by Ozaki and author [13]. Especially, the following case is related with the Iwasawa polynomial $P(T)$.

Theorem 3.3 ([13]). *For $p = 2$ and an imaginary quadratic field $k = \mathbb{Q}(\sqrt{-q})$ with a prime number $q \equiv 15 \pmod{32}$, $G(k_\infty)$ is abelian if and only if $P(-1) \equiv 1 \pmod 4$. Then, $G(k_\infty) \simeq \mathbb{Z}_2^{\oplus 3}$.*

On the other hand, as a corollary of the results of Gen Yamamoto ($p = 2$ version of [25]), we can find infinitely many real quadratic fields k with $G(k_\infty) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ (cf. e.g., [11]).

3.2. Metabelian examples in $p = 2$ case. Throughout this subsection, we put $p = 2$ and denote commutators by $[x, y] = x^{-1}y^{-1}xy$. For an imaginary quadratic field k with $\lambda = 1$, we can obtain an explicit presentation of $G(k_\infty)$ which is not necessarily abelian.

Theorem 3.4 ([12]). *Let $p = 2$ and $k = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field with positive squarefree integer $m \equiv 1 \pmod 4$, and put a real quadratic field $K^+ = \mathbb{Q}(\sqrt{m})$. If $\lambda = 1$ for k_∞, then*

$$G(k_\infty) = \left\langle a, b \mid [a, b] = a^{-2}, \ a^{2^{N+1}} = 1 \right\rangle^{\text{pro-}2}$$

where 2^N is the order of $G(K_\infty^+)$ which is finite cyclic.

Corollary 3.5. *$X^{(i)}(k_\infty) \simeq \mathbb{Z}/2\mathbb{Z}$ for $2 \le i \le N+1$, and $\#X^{(i)}(k_\infty) = 1$ for $N+2 \le i$. Especially, $\widetilde{\lambda}^{(i)} = 1$, $\lambda^{(i)} = 0$ for all $i \ge 2$ and $\sup\{n_0^{(i)}\} < \infty$.*

In Theorem 3.4, the metacyclic $G(k_\infty)$ is nonabelian if and only if $N \ge 1$, and such cases exist. For example, $N = 1$ if $m = 13 \cdot 29$.

Further, we have the following as an example of nonmetacyclic metabelian $G(k_\infty)$.

Theorem 3.6 ([12]). *Let $p = 2$ and $k = \mathbb{Q}(\sqrt{-q_1q_2})$ an imaginary quadratic field with prime numbers $q_1 \equiv 3 \pmod 8$, $q_2 \equiv 7 \pmod{16}$. Then, we have a presentation*

$$G(k_\infty) = \left\langle a,\, b,\, c \;\middle|\; [a,b] = a^{-2},\ [b,c] = a^2,\ [a,c] = 1 \right\rangle^{\text{pro-}2}$$

such that ${}^\gamma a = a$, ${}^\gamma b = bc$, ${}^\gamma c = a^{C_1} b^{-C_0} c^{1-C_1}$, where C_1, $C_0 \in \mathbb{Z}_2$ are the coefficients of the Iwasawa polynomial $P(T) = T^2 + C_1T + C_0$.

Corollary 3.7. *$X^{(i)}(k_\infty) \simeq \mathbb{Z}/2\mathbb{Z}$ for all $i \ge 2$. Especially, $\widetilde{\lambda}^{(i)} = 2$, $\lambda^{(i)} = 0$ for all $i \ge 2$ and $\sup\{n_0^{(i)}\} = \infty$.*

The Galois group $G(k_\infty)$ in Theorem 3.6 is 2-adic analytic, especially a Poincaré pro-2 group of dimension 3. According to Proposition 2.5 and Fontaine-Mazur conjecture, the Galois groups $G(k_n)$ of 2-class field towers should be finite. In fact, $G(k_n)$ are finite since $G(k_\infty)$ is metabelian. Further, by using the explicit action of γ on $G(k_\infty)$, we can calculate the presentations of $G(k_n)$ for $n \ge 1$ under some assumptions as follows. (It is well known that $G(k)$ is abelian.)

Corollary 3.8 ([12]). *If $(q_1/q_2) = -1$, i.e., q_1 is not quadratic residue modulo q_2, then*

$$G(k_1) = \left\langle a,\, b,\, c \;\middle|\; [a,b] = a^{-2},\ [b,c] = a^2 = b^2 = c^2,\ [a,c] = a^4 = 1 \right\rangle.$$

Further, if $(q_1/q_2) = -1$ and $C_1 \equiv 0 \pmod 4$,

$$G(k_n) = \left\langle a,\, b,\, c \;\middle|\; [a,b] = a^{-2},\ [b,c] = a^2,\ [a,c] = a^{2^{n+1}} = b^{2^{n+1}} = c^{2^n} = 1 \right\rangle$$

for all $n \ge 2$.

For all pairs (q_1, q_2) with $q_1q_2 < 5000$, one can see that $P(T) \equiv T^2 + (1 + (q_1/q_2))T + (1 - (q_1/q_2)) \pmod 4$ by the numerical computation of Stickelberger elements. Then, one can expect that always $C_1 \equiv 0 \pmod 4$ if $(q_1/q_2) = -1$, but it is not clear yet.

Under the stronger assumptions that $(q_1/q_2) = -1$ and $C_1 \equiv 0 \pmod 4$, there is another proof of the metabelianity of $G(k_\infty)$ of Theorem 3.6. It is parallel to the proof (of if-part) of Theorem 3.3, which is based on the calculation of "$\mathrm{Gal}(L(k_n)/\mathbb{Q})$" and the decomposition subgroups of some primes. By putting $K = k(\sqrt{-1}, \sqrt{-q_1})$ and

$F = \mathbb{Q}(\sqrt{-1}, \sqrt{-q_1})$, one can see that $\mathrm{Gal}(L(K_n)/F)$ has a presentation which is very similar to the presentation of "$\mathrm{Gal}(L(k_n)/\mathbb{Q})$" in the proof of Theorem 3.3.

Finally, concerning Greenberg's conjecture, we remark that there are infinitely many real quadratic fields k with finite dihedral $G(k_\infty)$ in $p = 2$ case (cf. [11]).

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