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On sufficient conditions for Carathéodory functions

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Abstract

By using the method of differential subordinations, we derive certain sufficient conditions for Carathéodory functions. Our results extend and improve some results due to Nunokawa et al. [*Indian J. Pure Appl. Math.*, 33(2002), 1385 - 1390], Owa and Obradović [*Bull. Austral. Math. Soc.*, 41(1990), 487 - 494], Li and Owa [*Indian J. Pure Appl. Math.*, 33(2002), 313 - 318], and Tuneski [*Internat. J. Math. Math. Sci.*, 23(2000), 521 - 527].

1 Introduction

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z)=1+\sum_{n=1}^{\infty}p_n z^n$$

which are analytic in the unit disk $\mathbb{E}=\{z\,|\,|z|<1\}$. If $p(z)$ in $\mathcal{P}$ satisfies Re $(p(z))>0$ for $z\in\mathbb{E}$, then we say that $p(z)$ is the Carathéodory function.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{E}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$, written $f(z)\prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{E}$ such that $|w(z)|\leqq|z|$ and $f(z)=g(w(z))$ $(z\in\mathbb{E})$. If $g(z)$ is univalent in $\mathbb{E}$, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{E})\subset g(\mathbb{E})$.

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in $\mathbb{E}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order α in $\mathbb{E}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad(z\in\mathbb{E})$$

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for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order α in $\mathbb{E}$. Also we denote by $\mathcal{S}^*(0) = \mathcal{S}^*$. For $-1 \leqq a \leqq 1$, $-1 \leqq b \leqq 1$ and $a \neq b$, a function $f(z)$ in $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(a,b)$ if satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz} \qquad (z \in \mathbb{E}).$$

The class $\mathcal{S}^*(a,b)$ can be reduced to several well known classes of starlike functions by selecting special values for a and b . In particular ,

$$\mathcal{S}^*(1-2\alpha,-1) = \mathcal{S}^*(2\alpha-1,1) = \mathcal{S}^*(\alpha) \qquad (0 \leqq \alpha < 1).$$

For Carathéodory functions , Nunokawa et al. [3] have given the following two theorems .

Theorem A. *If $p(z) \in \mathcal{P}$ satisfies*

$$\alpha(p(z))^2 + \beta zp'(z) \prec \frac{2\alpha+\beta}{2}\left(\frac{1+z}{1-z}\right)^2 - \frac{\beta}{2},$$

where $\beta > 0$ and $\alpha > -\frac{\beta}{2}$, then $\operatorname{Re}(p(z)) > 0 \quad (z \in \mathbb{E})$

Theorem B. *Let $p(z) \in \mathcal{P}$ and $w(z)$ be analytic in $\mathbb{E}$ with $w(0) = \alpha$ and $w(z) \neq ik$ $(k \in \mathbb{R}, z \in \mathbb{E})$. If*

$$\alpha(p(z)) + \beta\frac{zp'(z)}{p(z)} \prec w(z),$$

where $\alpha > 0$, $\beta > 0$ and $k^2 \geq \beta(2\alpha+\beta)$, then $\operatorname{Re}(p(z)) > 0 \quad (z \in \mathbb{E})$.

For the starlikeness of functions in $\mathcal{A}$, the following results have been proved .

Theorem C ([4]). *If $f(z) \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > -\frac{1}{2} \quad (z \in \mathbb{E}),$$

then $f(z) \in \mathcal{S}^*$.

Theorem D ([1]). *If $f(z) \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > 0 \quad (z \in \mathbb{E}),$$

then $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Theorem E ([5]). *If $f(z) \in \mathcal{A}$ satisfies $f'(z) \neq 0$ and*

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec 2 - \frac{2}{(1-z)^2} \qquad (z \in \mathbb{E}),$$

then $f(z) \in \mathcal{S}^*$.

Theorem F ([5]). *If $f(z) \in \mathcal{A}$ satisfies $f'(z) \neq 0$ and*

$$\left|\frac{f(z)f''(z)}{(f'(z))^2}\right| < 2 \quad (z \in \mathbb{E}),$$

then $f(z) \in \mathcal{S}^*$.

In this paper we shall generalize or refine the above results .
To derive our results , we need the following lemma due to Miller and Mocanu [2] .

Lemma. *Let* $g(z)$ *be analytic and univalent in* $\mathbb{E}$, *and let* $\theta(w)$ *and* $\phi(w)$ *be analytic in a domain* $\mathbb{D}$ *containing* $g(\mathbb{E})$, *with* $\phi(w) \neq 0$ *when* $w \in g(\mathbb{E})$. *Set*

$$Q(z) = zg'(z)\phi(g(z)), h(z) = \theta(g(z)) + Q(z)$$

and suppose that
(i) $Q(z)$ *is univalent and starlike in* E , *and*
(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\theta'(g(z))}{\phi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{E})$.
If $p(z)$ *is analytic in* $\mathbb{E}$, *with* $p(0) = g(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(g(z)) + zg'(z)\phi(g(z)) = h(z),$$

then $p(z) \prec g(z)$ *is the best dominant of the subordination* .

2 Main results

Our main result is contained in

Theorem 1. *Let* a , b , λ *and* μ *satisfy either*
(i) $0 < a = -b \leqq 1$, $\lambda > -\frac{1}{2}$, $\mu \in \mathbb{C}$ *and* $\operatorname{Re}(\mu) \geqq 0$, *or*
(ii) $-1 \leqq b < a \leqq 1$, $\lambda \geqq 0$, $\mu \in \mathbb{C}$ *and* $\operatorname{Re}(\mu) \geqq -2\lambda\frac{(1-a)}{(1-b)}$.
If $p(z) \in \mathcal{P}$ *and*

$$(1) \qquad \lambda(p(z))^2 + \mu p(z) + zp'(z) \prec h(z),$$

where

$$(2) \qquad h(z) = \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a+b) + a - b)z + \lambda + \mu}{(1+bz)^2}$$

then $p(z) \prec \frac{1+az}{1+bz}$ *and* $\frac{1+az}{1+bz}$ *is the best dominant of* (1) .

Proof. Set

$$(3) \qquad g(z) = \frac{1+az}{1+bz}, \theta(w) = \lambda w^2 + \mu w, \phi(w) = 1.$$

Then $g(z)$ is analytic and univalent in $\mathbb{E}$, $g(0) = p(0) = 1$, $\theta(w)$ and $\phi(w)$ are analytic with $\phi(w) \neq 0$ in the w-plane . The function

$$(4) \qquad Q(z) = zg'(z)\phi(g(z)) = \frac{(a-b)z}{(1+bz)^2}$$

is univalent and starlike in $\mathbb{E}$ because

$$\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = \operatorname{Re}\left(\frac{1-bz}{1+bz}\right) > 0 \quad (z \in \mathbb{E}).$$

Further , we have

$$\theta(g(z)) + Q(z) = \lambda\left(\frac{1+az}{1+bz}\right)^2 + \mu\frac{1+az}{1+bz} + \frac{(a-b)z}{(1+bz)^2} \tag{5}$$

$$= \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a+b) + a - b)z + \lambda + \mu}{(1+bz)^2} = h(z)$$

and

$$\frac{zh'(z)}{Q(z)} = 2\lambda\frac{1+az}{1+bz} + \mu + \frac{1-bz}{1+bz}. \tag{6}$$

Therefore

(i) For $0 < a = -b \leqq 1$, $\lambda > -\frac{1}{2}$, and $\operatorname{Re}(\mu) \geqq 0$, it follows from (6) that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = (2\lambda+1)\operatorname{Re}\left(\frac{1-bz}{1+bz}\right) + \operatorname{Re}(\mu) > 0 \quad (z \in \mathbb{E}).$$

(ii) For $-1 \leqq b < a \leqq 1$, $\lambda \geq 0$, and $\operatorname{Re}(\mu) \geqq -2\lambda\frac{(1-a)}{(1-b)}$, from (6) we get

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 2\lambda\frac{1-a}{1-b} + \operatorname{Re}(\mu) \geqq 0 \quad (z \in \mathbb{E}).$$

Thus the function $h(z)$ in (5) is close-to-convex and univalent in $\mathbb{E}$. From (1) to (5) , we see that

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(g(z)) + zg'(z)\phi(g(z)) = h(z).$$

Therefore , by applying the lemma , we conclude that $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (1) . The proof of the theorem is complete .

□

Remark 1. For $a = -b = 1$, $\lambda = \frac{\alpha}{\beta}$, $\beta > 0$, $\alpha > -\frac{\beta}{2}$ and $\mu = 0$, Theorem 1 (i) coincides with Theorem A by Nunokawa et al [3].

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{(a-b)z}{(1+bz)^2} (z \in \mathbb{E}) \tag{7}$$

for some a and b $(-1 \leqq b < a \leqq 1)$, then $f(z) \in \mathcal{S}^*(a,b)$.

Proof. Let $p(z) = \dfrac{zf'(z)}{f(z)}$. Then $p(z) \in \mathcal{P}$ and (7) can be written as

$$zp'(z) \prec \frac{(a-b)z}{(1+bz)^2} \tag{8}$$

Putting $\lambda = \mu = 0$ in Theorem 1 (ii) and using (8) ,
the desired result follows at once .

□

Remark 2. Corollary 1 with $a = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ and $b = -1$ implies that if $f(z) \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right) \prec 2(1-\alpha)\frac{z}{(1-z)^2},$$

then $f(z) \in \mathcal{S}^*(\alpha)$ and the order α is sharp for $f(z) = \dfrac{z}{(1-z)^{2(1-\alpha)}}$. When $\alpha = 0$, this result improves Theorem C by Owa and Obradović [4] .

Corollary 2. *If $f(z) \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1)\left(\frac{zf'(z)}{f(z)}\right)^2 \prec \frac{\lambda + z}{(1-z)^2} \qquad (z \in \mathbb{E}) \tag{9}$$

for some λ $(\lambda \geqq 0)$, then $f(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$ and the order $\dfrac{1}{2}$ is sharp .

Proof. If we let $p(z) = \dfrac{zf'(z)}{f(z)}$, then $p(z) \in \mathcal{P}$ and it follows from (9) that

$$\begin{aligned} \lambda(p(z))^2 + zp'(z) &= \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1)\left(\frac{zf'(z)}{f(z)}\right)^2 \\ &\prec \frac{\lambda + z}{(1-z)^2}. \end{aligned} \tag{10}$$

Taking $a = 0$, $b = -1$, $\lambda \geqq 0$ and $\mu = 0$ in Theorem 1 (ii) and using (10) , we know that $f(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$.
For $f(z) = \dfrac{z}{(1-z)}$, we have

$$\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1)\left(\frac{zf'(z)}{f(z)}\right)^2 = \frac{\lambda + z}{(1-z)^2}$$

and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \to \frac{1}{2} \quad as \quad z \to -1.$$

Hence the corollary is proved .

□

Remark 3. If we put $h(z) = \dfrac{\lambda + z}{(1-z)^2}$ $(\lambda > 0)$, then

$$h(e^{i\theta}) = -\frac{1 + \lambda cos\theta - i\lambda sin\theta}{2(1 - cos\theta)} \qquad (0 < \theta < 2\pi)$$

and hence

$$h(\mathbb{E}) = \left\{ w = u + iv : v^2 > -\frac{\lambda^2}{1+\lambda}\left(u - \frac{\lambda - 1}{4}\right)\right\},$$

which properly contains the half plane $\mathrm{Re}(w) > \dfrac{\lambda - 1}{4}$. Thus Corollary 2 with $\lambda = 1$ improves Theorem D by Li and Owa [1].

Corollary 3. *Let* $-1 \leqq b < a \leqq 1$ *and* $\mathrm{Re}(\mu) \geqq 0$. *If* $f(z) \in \mathcal{A}$ *satisfies* $f'(z) \neq 0$ *and*

$$(11) \qquad (1-\mu)\frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} \prec h(z) \qquad (z \in \mathbb{E}),$$

where

$$(12) \qquad h(z) = \frac{b(b - \mu a)z^2 + (3b - a - \mu(a+b))z + 1 - \mu}{(1+bz)^2},$$

then $f(z) \in \mathcal{S}^*(b, a)$.

Proof. Let us define $p(z)$ in $\mathbb{E}$ by

$$(13) \qquad p(z) = \frac{f(z)}{zf'(z)}.$$

Then $p(z) \in \mathcal{P}$ and it follows from (11) , (12) and (13) that

$$\begin{aligned} \mu p(z) + zp'(z) &= 1 + (\mu - 1)\frac{f(z)}{zf'(z)} - \frac{f(z)f''(z)}{(f'(z))^2} \\ &\prec \frac{\mu abz^2 + (\mu(a+b) + a - b)z + \mu}{(1+bz)^2} \qquad (z \in \mathbb{E}). \end{aligned}$$

Therefore , by applying Theorem 1 (ii) with $\lambda = 0$ and $\mathrm{Re}(\mu) \geqq 0$, we have

$$p(z) = \frac{f(z)}{zf'(z)} \prec \frac{1+az}{1+bz}.$$

This implies that $f(z) \in \mathcal{S}^*(b, a)$. □

Remark 4. Letting $a = 1$, $b = -1$ and $\mu = 1$ in Corollary 3 , we get Theorem E by Tuneski [5].

For $a = 1$, $b = 0$ and $\mu = 1$, Corollary 3 lead to

Corollary 4. *If* $f(z) \in \mathcal{A}$ *satisfies* $f'(z) \neq 0$ *and*

$$\left|\frac{f(z)f''(z)}{(f'(z))^2}\right| < 2 \quad (z \in \mathbb{E}),$$

then $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *and the order* $\frac{1}{2}$ *is sharp for the function* $f(z) = \frac{z}{1-z}$.

Remark 5. Corollary 4 refines Theorem F by Tuneski [5].

Taking $a = 0$, $b = -c$ and $\mu = 1$ in Corollary 3 , we have

Corollary 5. *If* $f(z) \in \mathcal{A}$ *satisfies* $f'(z) \neq 0$ *and*

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec 1 - \frac{1}{(1-cz)^2} \qquad (z \in \mathbb{E})$$

for some c $(0 < c \leqq 1)$, *then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < c \qquad (z \in \mathbb{E}). \tag{14}$$

The bound c *in* (14) *is sharp for the function* $f(z) = ze^{-cz}$.

Next we derive

Theorem 2. *Let* $-1 \leqq b < a \leqq 1$, $\lambda \geqq 0$ *and* $\mu \geqq -\frac{1-a}{1-b}$. *If* $p(z) \in \mathcal{P}$ *with* $p(z) \neq -\mu$ $(z \in \mathbb{E})$ *and*

$$\lambda p(z) + \frac{zp'(z)}{p(z)+\mu} \prec h(z) \qquad (z \in \mathbb{E}), \tag{15}$$

where

$$h(z) = \frac{\lambda acz^2 + (\lambda(a+c)+c-b)z+\lambda}{(1+bz)(1+cz)}, c = \frac{a+\mu b}{1+\mu},$$

then $p(z) \prec \frac{1+az}{1+bz}$ *and* $\frac{1+az}{1+bz}$ *is the best dominant of* (15) .

Proof. We choose

$$g(z) = \frac{1+az}{1+bz}, \qquad \theta(w) = \lambda w, \qquad \phi(w) = \frac{1}{w+\mu}$$

and $\mathbb{D} = w : w \neq -\mu$ in the Lemma . Noting that

$$\operatorname{Re}(g(z)) > \frac{1-a}{1-b} \geqq -\mu \qquad (z \in \mathbb{E}), \tag{16}$$

the function $\phi(w)$ is analytic in $\mathbb{D}$ containing $g(\mathbb{E})$. From (16) we see that

$$1+\mu > 0, \qquad -1 \leq b < c = \frac{a+\mu b}{1+\mu} \leq 1.$$

The function

$$Q(z) = zg'(z)\phi(g(z)) = \frac{(c-b)z}{(1+bz)(1+cz)}$$

is univalent and starlike in $\mathbb{E}$ because

$$\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = -1 + \operatorname{Re}\left(\frac{1}{1+bz}\right) + \operatorname{Re}\left(\frac{1}{1+cz}\right)$$

$$> -1 + \frac{1}{1+|b|} + \frac{1}{1+|c|}$$
$$= \frac{1-|bc|}{(1+|b|)(1+|c|)} \geqq 0$$

for $z \in \mathbb{E}$. Further , we have

$$\theta(g(z)) + Q(z) = \lambda \frac{1+az}{1+bz} + \frac{(c-b)z}{(1+bz)(1+cz)} = h(z)$$

and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \lambda(1+\mu)\operatorname{Re}\left(\frac{1+cz}{1+bz}\right) + \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right)$$
$$> \lambda(1+\mu)\frac{1-c}{1-b} \geq 0 \quad (z \in E)$$

for $\lambda \geqq 0$. The other conditions of the lemma are seen to be satisfied . Hence $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (15) . The proof is complete .

$\square$

Remark 6. Note that the univalent function $h(z)$ defined by

$$h(z) = \frac{\alpha z^2 + 2(\alpha+\beta)z + \alpha}{1-z^2} = \alpha\frac{1+z}{1-z} + 2\beta\frac{z}{1-z^2} \quad (\alpha > 0, \beta > 0)$$

maps $\mathbb{E}$ onto the complex plane minus the half-lines

$$l_1 = w = u + iv : u = 0, \qquad v \geqq \sqrt{\beta(2\alpha+\beta)}$$

and

$$l_2 = w = u + iv : u = 0, \qquad v \leqq -\sqrt{\beta(2\alpha+\beta)}.$$

For $a = 1$, $b = -1$, $\lambda = \dfrac{\alpha}{\beta}$, $\alpha > 0$, $\beta > 0$ and $\mu = 0$, Theorem 2 reduces to Theorem B by Nunokawa et al [3] .

Theorem 2 with $\mu = 0$ and $p(z) = \dfrac{zf'(z)}{f(z)}$ leads to the following corollary .

Corollary 6. *Let $-1 \leqq b < a \leqq 1$ and $\lambda \geqq 0$. If $f(z) \in \mathcal{A}$ satisfies $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and*

$$(\lambda-1)\frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)} \prec h(z) \qquad (z \in \mathbb{E}),$$

where

$$h(z) = \frac{\lambda a^2 z^2 + (2\lambda a + a - b)z + \lambda}{(1+az)(1+bz)},$$

then $f(z) \in \mathcal{S}^*(a,b)$.

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