

PSL(2, $\mathbb{C}$) and its Cayley transforms

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Let $\mathbb{H}^n$ be the upper half space model of the hyperbolic n -space and $\mathbb{H} = \{h = h_0 + ih_1 + jh_2 + kh_3; h_m \in \mathbb{R} \ (m=0, \dots, 3)\}$ be the set of quaternions. It is well-known (see [B], for example) that, under the identification of $\mathbb{H}^3$ with $\mathbb{C} \times j \cdot \mathbb{R}_{>0}$ and via the embedding $\mathbb{H} \ni h \mapsto \begin{pmatrix} h \\ 1 \end{pmatrix} \in \mathbb{HP}^1$, the orientation preserving isometry group $\text{PSL}(2, \mathbb{C})$ ($\subset \text{PGL}(2, \mathbb{H})$) acts on $\mathbb{H}^3$ as fractional linear transformations (here $\mathbb{H}^\times$ acts on $\mathbb{H}^2 \setminus \{0\}$ from right and $\text{PGL}(2, \mathbb{H})$ acts on $\mathbb{HP}^1 = (\mathbb{H}^2 \setminus \{0\})/\mathbb{H}^\times$ from left). We will construct a family of "Cayley transformations" for this pair $(\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)$.

Suppose that $h \notin (-\mathbb{H}^3) \cup \{0\}$. Then $T_h(x) = (x-h)(x+h)^{-1}$ is well-defined for $x \in \mathbb{H}^3$ and has its inverse $T_h^{-1} : y \mapsto (1-y)^{-1}(1+y)h$; thus T_h is a diffeomorphism onto its image. We will describe the image of $\mathbb{H}^3$ under T_h :

THEOREM. For $h \in \mathbb{H} \setminus ((-\mathbb{H}^3) \cup \{0\})$ and the map $T_h : \mathbb{H}^3 \ni x \mapsto (x-h)(x+h)^{-1}$, we have

(1). The image $T_h(\mathbb{H}^3)$ is given by $\{y \in \mathbb{H}; y \text{ satisfies } 2y_3h_0 + (1-|y|^2)h_3 + 2y_1h_2 - 2y_2h_1 = 0 \text{ and } 2y_2h_0 + (1-|y|^2)h_2 + 2y_3h_1 - 2y_1h_3 > 0\}$.

(2). The pull-back of the Poincaré metric $g_P = \sum_{m=0}^2 (dx_m)^2/x_2^2$ by T_h^{-1} is written as

$$(T_h^{-1})^*g_P = \iota^* \left(4|h|^2 \sum_{m=0}^3 (dy_m)^2 / (2y_2h_0 + (1-|y|^2)h_2 + 2y_3h_1 - 2y_1h_3)^2 \right),$$

where $\iota : T_h(\mathbb{H}^3) \hookrightarrow \mathbb{H}$ is the inclusion.

(3). The orientation preserving isometry group $T_h \circ \text{PSL}(2, \mathbb{C}) \circ T_h^{-1}$ on $T_h(\mathbb{H}^3)$ is given by

$$\left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{PGL}(2, \mathbb{H}) ; \frac{1}{2} \begin{bmatrix} \alpha + \beta + \gamma + \delta & (-\alpha + \beta - \gamma + \delta)h \\ h^{-1}(-\alpha - \beta + \gamma + \delta) & h^{-1}(\alpha - \beta - \gamma + \delta)h \end{bmatrix} \in \text{PSL}(2, \mathbb{C}) \right\}.$$

Moreover if $h \in j \cdot \mathbb{R} \times k \cdot \mathbb{R}$ or $h \in \mathbb{C}$, this group has quite simple description:

(3.1). For $h = q \in (j \cdot \mathbb{R} \times k \cdot \mathbb{R}) \cap (\mathbb{H} \setminus ((-\mathbb{H}^3) \cup \{0\}))$, we have

$$T_q \text{ PSL}(2, \mathbb{C}) T_q^{-1} = \left\langle \begin{bmatrix} \alpha_1 + q \alpha_2 & \beta_1 + q \beta_2 \\ \beta_1 - q \beta_2 & \alpha_1 - q \alpha_2 \end{bmatrix} \in \text{PGL}(2, \mathbb{H}) ; \alpha_n, \beta_n \in \mathbb{C} \ (n=1,2), \right. \\ \left. (\alpha_1 + \beta_1)(\bar{\alpha}_1 - \bar{\beta}_1) + |q|^2 (\alpha_2 + \beta_2)(\bar{\alpha}_2 - \bar{\beta}_2) = 1 \right\rangle.$$

(3.2). For $h = p \in \mathbb{C} \setminus \{0\}$, we have $T_p \circ \text{PSL}(2, \mathbb{C}) \circ T_p^{-1} = \text{PSL}(2, \mathbb{C})$. $\square$

EXAMPLES. (1). For $h = j$, we have $T_j(\mathbb{H}^3) = \{y = y_0 + jy_2 + ky_3 ; |y| < 1\}$ (which we denote by $\mathbb{D}^3$), $(T_j^{-1})^* g_P = 4 \langle (dy_0)^2 + (dy_2)^2 + (dy_3)^2 \rangle / (1 - |y|^2)^2$ and $\text{Isom}_+(\mathbb{D}^3) =$

$$\left\langle \begin{bmatrix} \alpha_1 + j \alpha_2 & \beta_1 + j \beta_2 \\ \beta_1 - j \beta_2 & \alpha_1 - j \alpha_2 \end{bmatrix} \in \text{PGL}(2, \mathbb{H}) ; \alpha_n, \beta_n \in \mathbb{C} \ (n=1,2), \right. \\ \left. (\alpha_1 + \beta_1)(\bar{\alpha}_1 - \bar{\beta}_1) + (\alpha_2 + \beta_2)(\bar{\alpha}_2 - \bar{\beta}_2) = 1 \right\rangle$$

(Poincaré disk model). (Remark: This case, together with more higher dimensional ones, is already known. See [B], [W].)

(2). For $h = k$, we have $T_k(\mathbb{H}^3) = \{y \in \mathbb{H} ; |y| = 1, y_1 < 0\}$ (the lower unit hemi-

sphere with respect to the i -axis) and $(T_k^{-1})^* g_P = 2^* \left(\sum_{m=0}^3 (dy_m)^2 / y_1^2 \right)$ (the

restriction of the hyperbolic metric on $\{y \in \mathbb{H} ; y_1 < 0\} (\cong \mathbb{H}^4)$ to the totally geodesic hemi-sphere $T_k(\mathbb{H}^3)$).

(3). For $h = i$, we have $T_i(\mathbb{H}^3) = \mathbb{C} \times k \cdot \mathbb{R}_{>0}$ and $(T_i^{-1})^* g_P = \langle (dy_0)^2 + (dy_1)^2 + (dy_3)^2 \rangle / y_3^2$.

Thus we obtain a "natural" homotopy of models of the hyperbolic 3-space, connecting the above three well-known models.

Let us focus on the case " $h=j$ ". Suppose that $\mathbb{H}$ is oriented by the ordered set $\langle 1, i, j, k \rangle = \langle j, k, 1, i \rangle$ of generators over $\mathbb{R}$, and introduce an orientation on $\mathbb{D}^3$ by $\langle 1, j, k \rangle = \langle j, k, 1 \rangle$.

EXAMPLES of elements of $\text{Isom}_+(\mathbb{D}^3)$. (0). $eh(\ell) = \begin{bmatrix} \text{ch}(\ell/2) & \text{sh}(\ell/2) \\ \text{sh}(\ell/2) & \text{ch}(\ell/2) \end{bmatrix}$ ($\ell \in \mathbb{R}$) (hyperbolic displacement of length ℓ along the real axis).

(1). $e_i(\varphi) = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix}$ ($\varphi \in \mathbb{R}/2\pi\mathbb{Z}$) (rotation of angle φ on the oriented plane $\langle j, k \rangle_{\mathbb{R}}$).

(2). $e_j(\theta) = \begin{bmatrix} e^{j\theta/2} & 0 \\ 0 & e^{-j\theta/2} \end{bmatrix}$ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$) (rotation of angle θ on $\langle 1, j \rangle_{\mathbb{R}}$).

(3). $e_k(\psi) = \begin{bmatrix} e^{-k\psi/2} & 0 \\ 0 & e^{k\psi/2} \end{bmatrix}$ ($\psi \in \mathbb{R}/2\pi\mathbb{Z}$) (rotation of angle ψ on $\langle k, 1 \rangle_{\mathbb{R}}$).

Now a combination of the method [O, Lemma] of reading the holonomy representations of Riemann surfaces (the $\text{PSU}(1,1)$ -case) and the well-known argument of Euler's angles for $\text{SO}(3)$ yields the following

PROPOSITION. $\text{Isom}_+(\mathbb{D}^3)$ is generated by these $eh(\ell)$, $e_i(\varphi)$, $e_j(\theta)$ and $e_k(\psi)$.

□

References.

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