

SOME OBSERVATIONS OF APPROXIMANTS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract

Let E be a Banach space, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E . In this paper, we study the convergence of the two sequences defined by

$$\begin{aligned}x_1 &= x \in C, x_{n+1} = \alpha_n x + (1 - \alpha_n)QTx_n, \\y_1 &= y \in C, y_{n+1} = Q(\alpha_n y + (1 - \alpha_n)Ty_n), \quad n = 1, 2, \dots,\end{aligned}$$

where $0 \leq \alpha_n \leq 1$, and Q is a sunny nonexpansive retraction from E onto C .

1 Introduction

Let E be a Banach space, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E such that the set $F(T)$ of all fixed points of T is nonempty. In 1998, Takahashi and Kim[8] defined two contraction mappings S_t and U_t the following: For a given $u \in C$ and each $t \in (0, 1)$,

$$S_t x = tu + (1 - t)QTx \quad \text{for all } x \in C \quad (1.1)$$

and

$$U_t x = Q(tu + (1 - t)Tx) \quad \text{for all } x \in C, \quad (1.2)$$

where Q is a sunny nonexpansive retraction from E onto C . Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$ (resp. $y_t \in F(U_t)$), i.e.

$$x_t = tu + (1 - t)QTx_t \quad (1.3)$$

and

$$y_t = Q(tu + (1 - t)Ty_t). \quad (1.4)$$

Also, Takahashi and Kim[8] proved that if E is a reflexive Banach space, C is a nonempty closed convex subset of E which has normal structure, T is a nonexpansive nonself-mapping from C into E satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract. Then $\{x_t\}$ (resp. $\{y_t\}$) defined by (1.3) (resp. (1.4)) converges strongly as $t \rightarrow 0$ to an element of $F(T)$. On the other hand, Shioji and Takahashi[7] studied the convergence of the iteration

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Sx_n \quad \text{for } n \geq 1.$$

where x, x_1 are elements of C , S is a nonexpansive mapping from C into itself such that $F(S)$ is nonempty. They proved $\{x_n\}$ converges strongly to an element of $F(S)$.

In this paper, we deal with the strong convergence to fixed points of nonexpansive nonself-mapping T , which satisfies new boundary condition. At first, We define a new boundary condition and obtain some results with respect to new boundary condition. Further we consider two iteration schemes for T . Then we prove that the iterates converge strongly to fixed points of T .

2 Preliminaries

Throughout this paper, we denote the set of all positive integer by $\mathbb{N}$. Let E be a real Banach space with norm $\|\cdot\|$, E^* a dual space of E . The value of $x^* \in E^*$ at $x \in E$ will be denote by $\langle x, x^* \rangle$. Let C be a closed convex subset of E , and T a nonexpansive nonself-mapping from C into E . We denote the set of all fixed points of T by $F(T)$. Let D be a subset of C . A mapping Q from C into D is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$ whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q from C into D is said to be retraction if $Q^2 = Q$. A subset D of C is said to be a sunny nonexpansive retract if there exists sunny nonexpansive retraction of C onto D . Concerning sunny nonexpansive retractions, The following lemma was proved by Bruck, Jr.[1], Reich[5]:

Lemma 2.1 *Let E be Banach space whose norm Gâteaux differentiable, C a convex subset of E , D a nonempty subset of C , and Q a retraction from C onto D . Then Q is sunny nonexpansive if and only if*

$$\langle x - Qx, J(y - Qx) \rangle \leq 0 \text{ for each } x \in C \text{ and } y \in D.$$

The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y\|^2\}, x \in E.$$

The norm of E is said to be Gâteaux differentiable norm if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.5)$$

exists for each $x, y \in U$. It is also said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit(2.5) is attained uniformly for $x \in U$. It is well known that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm weak star, uniformly continuous on each bounded subset of E . A closed convex subset C of E is said to have normal structure, if for each bounded closed convex subset K of C , which contains at least two points, there exists an element of K which is not a diametral point of K . It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure.

Let μ be a continuous, linear functional on l^∞ and let $(a_1, a_2, \dots) \in l^\infty$. We write $\mu(a_n)$ instead of $\mu((a_1, a_2, \dots))$. A function μ is said to be Banach limit if

$$\|\mu\| = \mu_n(1) = 1 \text{ and } \mu_n(a_{n+1}) = \mu_n(a_n) \text{ for all } (a_1, a_2, \dots) \in l^\infty.$$

We know that if μ is Banach limit then

$$\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for all $a = (a_1, a_2 \dots) \in l^\infty$. The following lemma was proved by Shioji and Takahashi[7].

Lemma 2.2 *Let a be a real number, and $(a_1, a_2 \dots) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) \leq 0$. Then $\limsup_{n \rightarrow \infty} a_n \leq a$.*

Next, we introduce several boundary conditions upon the nonself-mapping.

(i) **Rothe's condition:** $T(\partial C) \subset C$, where ∂C is boundary set of C ;

(ii) **inwardness condition:** $Tx \in I_c(x)$ for all $x \in C$, where

$$I_c(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\};$$

(iii) **weak inwardness condition:** $Tx \in \text{cl } I_c(x)$ for all $x \in C$, where cl denotes the norm-closure; and

(iv) **nowhere normal-outward condition:** $Tx \in \{y \in E | y \neq x, Py = x\}^c$ where P is the metric projection from E onto C .

It is easily seen that there hold implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Now, we define a new boundary condition.

Definition 2.1 (condition (C1)) $Tx \in S_x^c$ for all $x \in C$, where Q is a sunny nonexpansive retraction from E onto C , $x \in C$, and $S_x = \{y \in E | y \neq x, Qy = x\}$.

Remark 2.1 *Let H be a Hilbert space, C a nonempty closed convex subset of H , and T a nonexpansive nonself-mapping from C into H . Then T satisfies nowhere normal-outward condition if and only if T satisfies condition (C1).*

By using condition (C1), we obtain two propositions.

Proposition 2.1 *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E , T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract. and T satisfies weak inwardness condition then T satisfies condition (C1).*

Proposition 2.2 *Let E be a Banach space, C a nonempty closed convex subset of E , T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract, and T satisfies condition (C1). Then $F(T) = F(QT)$, where Q is a sunny nonexpansive retraction from E onto C .*

This proposition is very simple, but very useful. By using this proposition, we can extend all fixed point theorems with respect to nonexpansive self-mappings in Banach space, because when C is a sunny nonexpansive retract, T is a nonexpansive nonself-mapping from C into E which satisfies condition (C1), by applying fixed point theorems to QT where Q is a sunny nonexpansive retraction from E onto C , we can obtain results concerned with fixed points of QT , then we have theorems concerned with fixed points of T . On the other hand, we follow the two corollaries, the proof mainly due to Takahashi and Kim[8].

Corollary 2.1 *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E which has normal structure, and T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 0$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point of T .*

Corollary 2.2 *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E which has normal structure, and T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \rightarrow 0$ and in this case, $\{y_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point of T .*

Also, by using Reich[6]’s result, and proposition 2.2, we obtain two corollaries.

Corollary 2.3 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 0$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 0$ to $Q_2u \in F(T)$ where Q_2 is the unique sunny nonexpansive retraction from C onto $F(T)$.*

Corollary 2.4 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E . Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \rightarrow 0$ and in this case, $\{y_t\}$ converges strongly as $t \rightarrow 0$ to $Q_2u \in F(T)$ where Q_2 is the unique sunny nonexpansive retraction from C onto $F(T)$.*

3 Main Results

In this section, we study two type strong convergence of nonexpansive nonself-mappings which satisfies condition (C1). The proof mainly due to Wittmann[10], and Shioji and Takahashi[7].

Theorem 3.1 *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E such that $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1). Let Q_1 be a sunny nonexpansive retraction from E onto C , $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Q_1 T x_n \text{ for } n \geq 1.$$

Then, $\{x_n\}$ converges strongly to $Q_2 x \in F(T)$, where Q_2 is a sunny nonexpansive retraction from C onto $F(T)$.

Theorem 3.2 *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E , and T a nonexpansive nonself-mapping from C into E such that $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E , and T satisfies condition (C1). Let Q_1 be a sunny nonexpansive retraction from E onto C , $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_1 = y \in C$ and*

$$y_{n+1} = Q_1(\alpha_n y + (1 - \alpha_n)Ty_n) \text{ for } n \geq 1.$$

Then, $\{y_n\}$ converges strongly to $Q_2 y \in F(T)$, where Q_2 is a sunny nonexpansive retraction from C onto $F(T)$.

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