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Strang-Fix Theory for Approximation Order in Weighted L^p-spaces

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We consider the Strang-Fix theory for approximation order in the weighted L^p-spaces. Let φ be an element of $C_c(\mathbb{R}^n)$. For a sequence c on $\mathbb{Z}^n$, the semi-discrete convolution product $\varphi *' c$ is the function defined by

$$\varphi *' c = \sum_{\nu\in\mathbb{Z}^n} \varphi(\cdot - \nu)c(\nu).$$

The collection $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ of $C_c(\mathbb{R}^n)$ is said to satisfy the Strang-Fix condition of order k if there exist finitely supported sequences b_j ($j = 1, \cdots, N$) such that the function $\varphi = \sum_{j=1}^{N} \varphi_j *' b_j$ satisfies

$$\hat{\varphi}(0) \neq 0$$

and

$$(\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \qquad (|\alpha| < k, \quad \nu \in \mathbb{Z}^n \setminus \{0\}),$$

where $\hat{\varphi}$ denotes the Fourier transform of φ. For a positive integer k, $L_k^p(\mathbb{R}^n)$ denotes the Sobolev space. For $f \in L_k^p(\mathbb{R}^n)$, we define semi-norms by

$$|f|_{k,p} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

For $h > 0$, σ_h is the scaling operator defined by

$$\sigma_h f(x) = f(hx) \qquad (x \in \mathbb{R}^n).$$

We say that $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ provides local L^p-approximation of order k if there exist constants C and r such that for each $f \in L_k^p(\mathbb{R}^n)$ there exist sequences c_j^h ($h > 0$, $j = 1, \cdots, N$) so that

$$\text{(i)} \quad \|f - \sigma_{1/h}(\sum_{j=1}^{N} \varphi_j *' c_j^h)\|_{L^p(\mathbb{R}^n)} \leq Ch^k |f|_{k,p},$$

(ii) $c_j^h(\nu) = 0$ $(j = 1, \cdots N)$ whenever $\mathrm{dist}(h\nu,\ \mathrm{supp}\ f) > r$.

Boor and Jia [1] proved that Φ satisfies the Strang-Fix condition of order k if and only if Φ provides local L^p-approximation of order k.

We give the definition of A_p in $\mathbb{R}^n$. A weight $w \geq 0$ is said to belong to A_p for $1 < p < \infty$ if

$$A_p(w) = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where Q is a cube in $\mathbb{R}^n$ and p' is a conjugate exponent of p. $A_p(w)$ is called the A_p-constant of w. The class A_1 is defined by

$$A_1(w) = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\|w^{-1}\|_{L^\infty(Q,\ dx)} < \infty,$$

where $\|w^{-1}\|_{L^\infty(Q,\ dx)} = \mathrm{ess\ sup}_{x\in Q} w(x)^{-1}$. The class A_∞ is the union of the classes of A_p, $1 \leq p < \infty$. These classes were introduced by Muckenhoupt in [3]. Let $1 \leq p \leq \infty$ and $w \in A_p$. Then the weighted L^p-space $L^p(w)$ consists of all measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty,$$

with the usual modifications when $p = \infty$. We define the weighted Sobolev spaces $L_k^p(w)$, where $1 \leq p \leq \infty$, w is an A_p-weight and k is a positive integer. A function f belongs to $L_k^p(w)$ if $f \in L^p(w)$ and the partial derivatives $\partial^\alpha f$, taken in the sense of distributions, belong to $L^p(w)$, whenever $0 \leq |\alpha| \leq k$. The norm in $L_k^p(w)$ is given by

$$\|f\| = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(w)}.$$

In the weighted case, we use the following notation

$$|f|_{k,p,w} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)}$$

and say that $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ provides local $L^p(w)$-approximation of order k if there exist constants C and r such that for each $f \in L_k^p(w)$ there exist sequences c_j^h $(h > 0,\ j = 1, \cdots, N)$ so that (ii) and the following condition (iii) are satisfied

$$\text{(iii)}\quad \|f - \sigma_{1/h}(\sum_{j=1}^N \varphi_j *' c_j^h)\|_{L^p(w)} \leq Ch^k|f|_{k,p,w}.$$

Based on [2], using boundedness of the Hardy-Littlewood maximal operator on $L^p(w)$, we prove the following theorem.

Theorem. *Let $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ be a finite collection of $C_c(\mathbb{R}^n)$. Then the following statements are equivalent.*

(i)' *Φ satisfies the Strang-Fix condition of order k.*

(ii)' *For all $p \in [1, \infty]$ and $w \in A_p$, Φ provides local $L^p(w)$-approximation of order k.*

(iii)' *For some $p \in [1, \infty]$ and $w \in A_p$, Φ provides local $L^p(w)$-approximation of order k.*

Lastly we introduce the main lemma to prove the above theorem.

Lemma. *Let $1 \leq p < \infty$ and $w \in A_p$. Suppose that φ is a function on $\mathbb{R}^n$ which is non-negative, radial, decreasing and integrable. Then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x + \alpha y)| \varphi(y) dy \right)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

for all $f \in L^p(w)$ and $\alpha \in \mathbb{R}$.

N. Tomita proved the above lemma when $1 < p < \infty$, using Calderón-Zygmund Operator. Then Professer E. Nakai provided the simple proof when $1 < p < \infty$, using Hardy-Littlewood maximal operator. Then Professer K. Yabuta proved the case $p = 1$.

References

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[3] B. Muckenhoupt, Weighted Norm Inequalities for the Hardy Maximal Function, Tran. Amer. Math. Soc., 165, (1972), 207-226.