

A G -family of quandles and handlebody-knots

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We introduce the notion of a G -family of quandles and use it to construct invariants for handlebody-knots. Our invariant can detect the chiralities of some handlebody-knots including unknown ones. This is a joint work with Atsushi Ishii, Yeonhee Jang and Kanako Oshiro ([8]).

1 Handlebody-links

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3 . Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *spatial graph* is a finite graph embedded in S^3 . Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. When a handlebody-link H is a regular neighborhood of a spatial graph K , we say that K *represents* H , or H *is represented by* K . In this paper, a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link.

An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a 3-ball embedded in S^3 . Then we have the following theorem.

Theorem 1.1 ([6]). *For spatial trivalent graphs K_1 and K_2 , the following are equivalent.*

- K_1 and K_2 represent an equivalent handlebody-link.
- K_1 and K_2 are related by a finite sequence of IH-moves.
- Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.

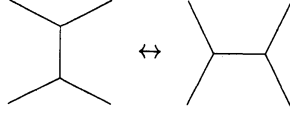


Figure 1:

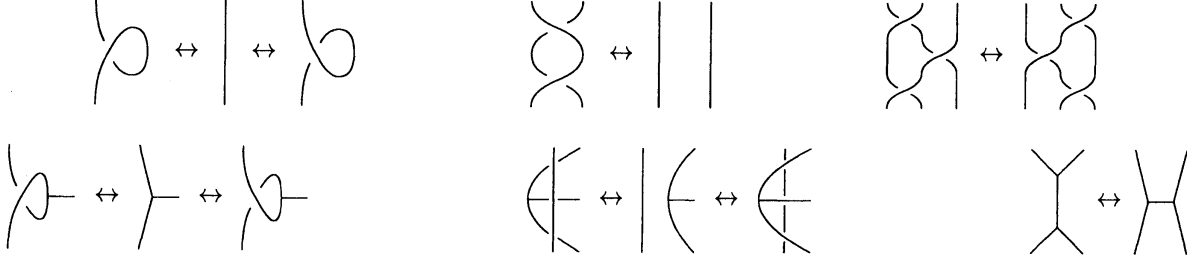


Figure 2:

2 A G -family of quandles

A *quandle* [12, 16] is a non-empty set X with a binary operation $*$: $X \times X \rightarrow X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.
- For any $x \in X$, the map $S_x : X \rightarrow X$ defined by $S_x(y) = y * x$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

When we specify the binary operation $*$ of a quandle X , we denote the quandle by the pair $(X, *)$. An *Alexander quandle* $(M, *)$ is a Λ -module M with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A *conjugation quandle* $(G, *)$ is a group G with the binary operation defined by $x * y = y^{-1}xy$.

Let G be a group with identity element e . A G -family of quandles is a non-empty set X with a family of binary operations $*^g : X \times X \rightarrow X$ ($g \in G$) satisfying the following axioms.

- For any $x \in X$ and any $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and any $g, h \in G$,

$$x *^{gh} y = (x *^g y) *^h y \text{ and } x *^e y = x.$$

- For any $x, y, z \in X$ and any $g, h \in G$,

$$(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z).$$

When we specify the family of binary operations $*^g : X \times X \rightarrow X$ ($g \in G$) of a G -family of quandles, we denote the G -family of quandles by the pair $(X, \{*\}^g_{g \in G})$.

Proposition 2.1. *Let G be a group. Let $(X, \{*\}^g_{g \in G})$ be a G -family of quandles.*

(1) *For each $g \in G$, the pair $(X, *^g)$ is a quandle.*

(2) *We define a binary operation $\triangleright : (X \times G) \times (X \times G) \rightarrow X \times G$ by*

$$(x, g) \triangleright (y, h) = (x *^h y, h^{-1}gh).$$

Then $(X \times G, \triangleright)$ is a quandle.

We call the quandle $(X \times G, \triangleright)$ in Proposition 2.1 the *associated quandle* of X .

Example 2.2. (1) Let $(X, *)$ be a quandle. Let $S_x : X \rightarrow X$ be the bijection defined by $S_x(y) = y * x$. Let m be a positive integer such that $S_x^m = \text{id}_X$ for any $x \in X$ if such an integer exists. We define the binary operation $*^i : X \times X \rightarrow X$ by $x *^i y = S_y^i(x)$. Then X is a $\mathbb{Z}$ -family of quandles and a $\mathbb{Z}_m$ -family of quandles, where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. (2) Let R be a ring, and G a group with identity element e . Let X be a right $R[G]$ -module, where $R[G]$ is the group ring of G over R . We define the binary operation $*^g : X \times X \rightarrow X$ by $x *^g y = xg + y(e - g)$. Then X is a G -family of quandles.

3 Colorings

Let D be a diagram of a handlebody-link H . We set an orientation for each edge in D . Then D is a diagram of an oriented spatial trivalent graph K . We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $\mathcal{A}(D)$ the set of arcs of D , where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. For an arc α incident to a vertex ω , we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$\epsilon(\alpha; \omega) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

Let X be a G -family of quandles, and Q the associated quandle of X . Let p_X (resp. p_G) be the projection from Q to X (resp. G). An X -coloring of D is a map $C : \mathcal{A}(D) \rightarrow Q$ satisfying the following conditions at each crossing χ and each vertex ω of D (see Figure 3).

- Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ

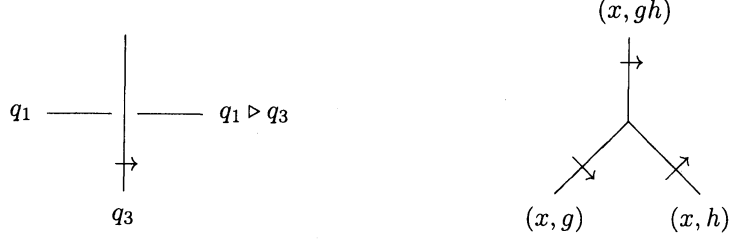


Figure 3:

such that the normal orientation of χ_3 points from χ_1 to χ_2 . Then

$$C(\chi_2) = C(\chi_1) \triangleright C(\chi_3).$$

- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex ω arranged clockwise around ω . Then

$$\begin{aligned} (p_X \circ C)(\omega_1) &= (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3), \\ (p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)} (p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)} (p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} &= e. \end{aligned}$$

We denote by $\text{Col}_X(D)$ the set of X -colorings of D . For two diagrams D and E which locally differ, we denote by $\mathcal{A}(D, E)$ the set of arcs that D and E share.

Lemma 3.1. *Let X be a G -family of quandles. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)$, there is a unique X -coloring $C_{D,E} \in \text{Col}_X(E)$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$.*

Remark 3.2. Let X be a $\mathbb{Z}$ -family of quandles or a $\mathbb{Z}_m$ -family of quandles defined as in Example 2.2 (2). Then an X -coloring be regarded as an X -coloring defined in [7].

Let X be a G -family of quandles, and Q the associated quandle of X . An X -set is a non-empty set Y with a family of maps $*^g : Y \times X \rightarrow Y$ satisfying the following axioms, where we note that we use the same symbol $*^g$ as the binary operation of the G -family of quandles.

- For any $y \in Y$, $x \in X$, and any $g, h \in G$,

$$y *^{gh} x = (y *^g x) *^h x \text{ and } y *^e x = y.$$

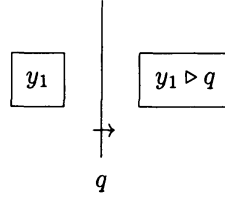


Figure 4:

- For any $y \in Y$, $x_1, x_2 \in X$, and any $g, h \in G$,

$$(y *^g x_1) *^h x_2 = (y *^h x_2) *^{h^{-1}gh} (x_1 *^h x_2).$$

Put $y \triangleright (x, g) := y *^g x$ for $y \in Y$, $(x, g) \in Q$. Then the second axiom implies that $(y \triangleright q_1) \triangleright q_2 = (y \triangleright q_2) \triangleright (q_1 \triangleright q_2)$ for $q_1, q_2 \in Q$. Any G -family of quandles $(X, \{ *^g \}_{g \in G})$ itself is an X -set with its binary operations. Any singleton set $\{y\}$ is also an X -set with the maps $*^g$ defined by $y *^g x = y$ for $x \in X$ and $g \in G$, which is a trivial X -set.

Let D be a diagram of an oriented spatial trivalent graph. We denote by $\mathcal{R}(D)$ the set of complementary regions of D . Let X be a G -family of quandles, and Y an X -set. Let Q be the associated quandle of X . An X_Y -coloring of D is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$ satisfying the following conditions.

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- The restriction $C|_{\mathcal{A}(D)}$ of C on $\mathcal{A}(D)$ is an X -coloring of D .
- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$C(\alpha_1) \triangleright C(\alpha) = C(\alpha_2),$$

where α_1, α_2 are the regions facing the arc α so that the normal orientation of α points from α_1 to α_2 (see Figure 4).

We denote by $\text{Col}_X(D)_Y$ the set of X_Y -colorings of D .

For two diagrams D and E which locally differ, we denote by $\mathcal{R}(D, E)$ the set of regions that D and E share.

Lemma 3.3. *Let X be a G -family of quandles, Y an X -set. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)_Y$, there is a unique X_Y -coloring $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$.*

4 A homology

Let X be a G -family of quandles, and Y an X -set. Let $(Q, \triangleright)$ be the associated quandle of X . Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \dots, q_i) \triangleright q, q_{i+1}, \dots, q_n) := (y \triangleright q, q_1 \triangleright q, \dots, q_i \triangleright q, q_{i+1}, \dots, q_n)$$

for $y \in Y$ and $q, q_1, \dots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \rightarrow B_{n-1}(X)_Y$ by

$$\begin{aligned} \partial_n(y, q_1, \dots, q_n) &= \sum_{i=1}^n (-1)^i (y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) \\ &\quad - \sum_{i=1}^n (-1)^i ((y, q_1, \dots, q_{i-1}) \triangleright q_i, q_{i+1}, \dots, q_n) \end{aligned}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 4, 5]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) \mid \begin{array}{l} y \in Y, x \in X, g, h \in G \\ q_1, \dots, q_n \in Q \end{array} \right\}$$

and

$$\bigcup_{i=1}^n \left\{ \begin{array}{l} (y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n) \\ -(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ -((y, q_1, \dots, q_{i-1}) \triangleright (x, g), (x, h), q_{i+1}, \dots, q_n) \end{array} \mid \begin{array}{l} y \in Y, x \in X, \\ g, h \in G, \\ q_1, \dots, q_n \in Q \end{array} \right\}.$$

We remark that

$$(y, q_1, \dots, q_{i-1}, (x, e), q_{i+1}, \dots, q_n)$$

and

$$\begin{aligned} &(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ &+ ((y, q_1, \dots, q_{i-1}) \triangleright (x, g), (x, g^{-1}), q_{i+1}, \dots, q_n) \end{aligned}$$

belong to $D_n(X)_Y$.

Lemma 4.1. *For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.*

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group A , we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the n th homology group of $C_*(X)_Y$.

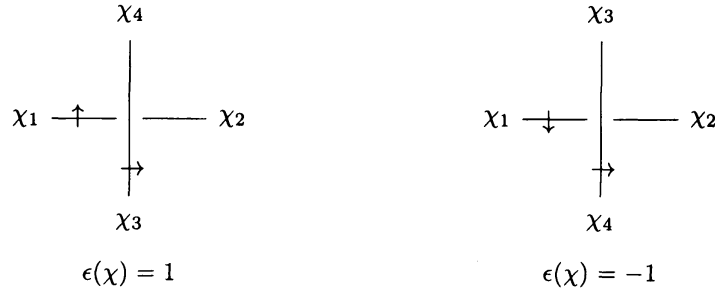


Figure 5:

5 Cocycle invariants

Let X be a G -family of quandles, and Y an X -set. Let D be a diagram of an oriented spatial trivalent graph. For an X_Y -coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing χ of D as follows. Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2 . Let R_χ be the region facing χ_1 and χ_3 such that the normal orientations χ_1 and χ_3 point from R_χ to the opposite regions with respect to χ_1 and χ_3 , respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing χ . We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_{\chi} w(\chi; C),$$

where χ runs over all crossings of D .

Lemma 5.1. *The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.*

Lemma 5.2. *Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.*

We denote by G_H (resp. G_K) the fundamental group of the exterior of a handlebody-link H (resp. a spatial graph K). When H is represented by K , the groups G_H and G_K are identical. Let D be a diagram of an oriented spatial trivalent graph K . By the definition

of an X_Y -coloring C of D , the map $p_G \circ C|_{\mathcal{A}(D)}$ represents a homomorphism from G_K to G , which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle θ of $C^*(X; A)_Y$, we define

$$\begin{aligned}\mathcal{H}(D) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\}, \\ \Phi_\theta(D) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\}, \\ \mathcal{H}(D; \rho) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\}, \\ \Phi_\theta(D; \rho) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\}\end{aligned}$$

as multisets.

Lemma 5.3. *Let D be a diagram of an oriented spatial trivalent graph K . For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that ρ and ρ' are conjugate, we have $\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho')$ and $\Phi_\theta(D; \rho) = \Phi_\theta(D; \rho')$.*

We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from G_K to G . By Lemma 5.3, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

Lemma 5.4. *Let D be a diagram of an oriented spatial trivalent graph K . Let E be a diagram obtained from D by reversing the orientation of an edge e . For $\rho \in \text{Hom}(G_K, G)$, we have $\mathcal{H}(D) = \mathcal{H}(E)$, $\Phi_\theta(D) = \Phi_\theta(E)$, $\mathcal{H}(D; \rho) = \mathcal{H}(E; \rho)$ and $\Phi_\theta(D; \rho) = \Phi_\theta(E; \rho)$.*

By Lemma 5.4, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for a diagram D of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram D of a handlebody-link H , we define

$$\begin{aligned}\mathcal{H}^{\text{hom}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \Phi_\theta^{\text{hom}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \mathcal{H}^{\text{conj}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}, \\ \Phi_\theta^{\text{conj}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}\end{aligned}$$

as “multisets of multisets”. We remark that, for X_Y -colorings C and $C_{D,E}$ in Lemma 5.2, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 5.1–5.4, we have the following theorem.

Theorem 5.5. *Let X be a G -family of quandles, Y an X -set. Let θ be a 2-cocycle of $C^*(X; A)_Y$. Let H be a handlebody-link represented by a diagram D . Then the following are invariants of a handlebody-link H .*

$$\mathcal{H}(D), \quad \Phi_\theta(D), \quad \mathcal{H}^{\text{hom}}(D), \quad \Phi_\theta^{\text{hom}}(D), \quad \mathcal{H}^{\text{conj}}(D), \quad \Phi_\theta^{\text{conj}}(D).$$

We denote the invariants of H given in Theorem 5.5 by

$$\mathcal{H}(H), \quad \Phi_\theta(H), \quad \mathcal{H}^{\text{hom}}(H), \quad \Phi_\theta^{\text{hom}}(H), \quad \mathcal{H}^{\text{conj}}(H), \quad \Phi_\theta^{\text{conj}}(H),$$

respectively.

We denote by H^* the mirror image of a handlebody-link H . Then we have the following theorem.

Theorem 5.6. *For a handlebody-link H , we have*

$$\begin{aligned} \mathcal{H}(H^*) &= -\mathcal{H}(H), & \Phi_\theta(H^*) &= -\Phi_\theta(H), \\ \mathcal{H}^{\text{hom}}(H^*) &= -\mathcal{H}^{\text{hom}}(H), & \Phi_\theta^{\text{hom}}(H^*) &= -\Phi_\theta^{\text{hom}}(H), \\ \mathcal{H}^{\text{conj}}(H^*) &= -\mathcal{H}^{\text{conj}}(H), & \Phi_\theta^{\text{conj}}(H^*) &= -\Phi_\theta^{\text{conj}}(H), \end{aligned}$$

where $-S = \{-a \mid a \in S\}$ for a multiset S .

6 Applications

In this section, we calculate cocycle invariants defined in the previous section for the handlebody-knots $0_1, \dots, 6_{16}$ in the table given in [9], by using a 2-cocycle given by Nosaka [18]. This calculation enables us to distinguish some of handlebody-knots from their mirror images, and a pair of handlebody-knots whose complements have isomorphic fundamental groups.

Let $G = SL(2; \mathbb{Z}_3)$ and $X = (\mathbb{Z}_3)^2$. Then X is a G -family of quandles with the proper binary operation as given in Proposition 2.2 (2). Let Y be the trivial X -set $\{y\}$. We define a map $\theta : Y \times (X \times G)^2 \rightarrow \mathbb{Z}_3$ by

$$\theta(y, (x_1, g_1), (x_2, g_2)) := \lambda(g_1) \det(x_1 - x_2, x_2(1 - g_2^{-1})),$$

where the abelianization $\lambda : G \rightarrow \mathbb{Z}_3$ is given by

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d)(b - c)(1 - bc).$$

	$\Phi_\theta(H)$
0 ₁	$\{\{0_9\}_{76}\}$
4 ₁	$\{\{0_9\}_{83}, \{0_{27}\}_{22}, \{0_{81}\}_{3}\}$
5 ₁	$\{\{0_9\}_{76}\}$
5 ₂	$\{\{0_9\}_{95}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{4}, \{0_{27}, 1_{54}\}_{2}\}$
5 ₃	$\{\{0_9\}_{102}, \{0_{27}\}_{4}, \{0_{27}, 2_{54}\}_{2}\}$
5 ₄	$\{\{0_9\}_{74}, \{0_{81}\}_{2}\}$
6 ₁	$\{\{0_9\}_{91}, \{0_{27}\}_{16}, \{0_{81}\}_{1}\}$
6 ₂	$\{\{0_9\}_{106}, \{0_{45}, 1_{18}, 2_{18}\}_{2}\}$
6 ₃	$\{\{0_9\}_{74}, \{0_{27}\}_{2}\}$
6 ₄	$\{\{0_9\}_{76}\}$
6 ₅	$\{\{0_9\}_{74}, \{0_9, 1_{18}\}_{2}\}$
6 ₆	$\{\{0_9\}_{72}, \{0_{27}\}_{4}\}$
6 ₇	$\{\{0_9\}_{85}, \{0_{27}\}_{16}, \{0_{81}\}_{3}, \{0_{45}, 1_{18}, 2_{18}\}_{4}\}$
6 ₈	$\{\{0_9\}_{76}\}$
6 ₉	$\{\{0_9\}_{91}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{6}, \{0_{27}, 1_{54}\}_{2}, \{0_{27}, 2_{54}\}_{2}\}$
6 ₁₀	$\{\{0_9\}_{76}\}$
6 ₁₁	$\{\{0_9\}_{70}, \{0_9, 1_{18}\}_{6}\}$
6 ₁₂	$\{\{0_9\}_{97}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{8}, \{0_9, 1_{36}, 2_{36}\}_{2}\}$
6 ₁₃	$\{\{0_9\}_{95}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 2_{18}\}_{4}, \{0_{27}, 2_{54}\}_{2}\}$
6 ₁₄	$\{\{0_9\}_{119}, \{0_{27}\}_{6}, \{0_{81}\}_{11}, \{0_9, 1_{18}\}_{12}, \{0_{27}, 1_{54}\}_{24}\}$
6 ₁₅	$\{\{0_9\}_{119}, \{0_{27}\}_{6}, \{0_{81}\}_{11}, \{0_9, 2_{18}\}_{12}, \{0_{27}, 1_{54}\}_{24}\}$
6 ₁₆	$\{\{0_9\}_{44}, \{0_{81}\}_{32}\}$

表 1:

By [18], the map θ is a 2-cocycle of $C^*(X; \mathbb{Z}_3)_Y$. Table 1 lists the invariant $\Phi_\theta^{\text{conj}}(H)$ for the handlebody-knots $0_1, \dots, 6_{16}$. We represent the multiplicity of elements of a multiset by using subscripts. For example, $\{\{0_2, 1_3\}_1, \{0_3\}_2\}$ represents the multiset $\{\{0, 0, 1, 1, 1\}, \{0, 0, 0\}, \{0, 0, 0\}\}$.

From Table 1, we see that our invariant can distinguish the handlebody-knots $6_{14}, 6_{15}$, whose complements have the isomorphic fundamental groups. Together with Theorem 5.6, we also see that handlebody-knots $5_2, 5_3, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15}$ are not equivalent to their mirror images. In particular, the chiralities of $5_3, 6_5, 6_{11}$ and 6_{12} were not known. Table 2 shows us known facts on the chirality of handlebody-knots in [9] so far. In the column of “chirality”, the symbols $\bigcirc$ and $\times$ mean that the handlebody-knot is amphichiral and chiral, respectively, and the symbol $?$ means that it is not known whether the handlebody-knot is amphichiral or chiral. The symbols $\checkmark$ in the right five columns mean that the handlebody-knots can be proved chiral by using the method introduced

	chirality	M	II	LL	IKO	IIJO
0_1	○					
4_1	○					
5_1	×			✓		
5_2	×		✓	✓		✓
5_3	×					✓
5_4	×				✓	
6_1	×	✓				
6_2	?					
6_3	?					
6_4	×			✓		
6_5	×					✓
6_6	○					
6_7	○					
6_8	?					
6_9	×		✓			✓
6_{10}	?					
6_{11}	×					✓
6_{12}	×					✓
6_{13}	×		✓	✓		✓
6_{14}	×				✓	✓
6_{15}	×				✓	✓
6_{16}	○					

表 2:

in the papers corresponding to the columns. Here, M, II, LL, IKO and IIJO denote the papers [17], [7], [15], [10] and this paper, respectively.

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