THESIS

Embedding Ghost-free Bigravity into Higher-Dimensional Gravity

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abstract

General relativity is the current standard theory of gravity in which massless gravitons are the unique degrees of freedom that describe the dynamical nature of gravity. There are many other modified gravity theories. But gravitons are still massless in almost all of those theories. As one of the possibilities to challenge this fundamental nature of gravity, bigravity, *i.e.*, a gravitational theory that contains two sets of gravitons, has been investigated. It is known that the model suffers from a ghost instability if one allows a general form of the interactions between the two graviton sectors, *i.e.*, there appears the so-called Boulware-Deser (BD) ghost. Imposing BD-ghost-freeness restricts the interaction form to the specific mass interactions proposed by de Rham, Gabadadze and Tolley.

The ghost-free interactions are derived artificially by the ghost-free condition, and hence we have no idea about the mechanism that tunes the interaction to be ghost-free. In order to address this issue, we propose to embed this ghost-free bigravity into higher-dimensional gravity. We focus on the Dvai-Gabadadze-Porrati 2-brane model, *i.e.*, the 5-dimensional braneworld model sandwiched by two branes equipped with brane-localized gravity. If the brane-localized gravity on two branes is sufficiently strong compared with the bulk gravity, only the lowest two states in the graviton spectrum will be trapped on the branes by the brane-localized gravity, while the other Kaluza-Klein gravitons are driven away to high energy and become ignorable in the low-energy effective theory. In this setup, we confirm that the graviton's mass spectrum recovers the one in the ghost-free bigravity at low energies. We then derive a part of the ghost-free bigravity action as a low-energy effective action of the DGP 2-brane model. Furthermore, we introduce a stabilization scalar field to fix the brane separation. We investigate the potential of the stabilization scalar which enables to fix the brane separation stably in the setup that reproduces the ghost-free bigravity as a low-energy effective theory.

Also, we study instabilities in both the DGP 2-brane model and the ghost-free bigravity. We show that the stabilization scalar mode in the DGP 2-brane model becomes a tachyon when the energy densities localized on the branes are increased. This energy scale is relatively low when we consider the setup that reproduces the ghost-free bigravity. This tachyonic instability leads to the breakdown of the stabilization, and consequently we cannot sustain the setup necessary to derive the ghost-free bigravity from the DGP 2-brane model. Therefore, unfortunately the correspondence between the DGP 2-brane model and the ghost-free bigravity breaks down in a relatively low energy regime.

Contents

Chapter 1

Intorduction

General relativity is the current standard theory of gravity. General relativity describes gravity in terms of metric, *i.e.*, a tensor that represents the structure of our spacetime. General relativity has a unique graviton corresponding to the metric, and the graviton is massless: the number of degrees of freedom is 2. As a challenge to this nature of the graviton, there exists an attempt to introduce another dynamical graviton, so-called bigravity [1]. If bigravity can describe the real universe and observations in strong gravity regime exhibit bigravity-like features, then the nature of graviton should be modified from the one in general relativity, namely, the gravity should be written in terms of two metrics interacting with each other. However, the idea of bigravity has an obstacle to describe the real universe, since the interaction between two metrics in general yields an extra degree of freedom whose kinetic term has a wrong sign, in addition to two gravitons. Such mode is called ghost mode, by which the vacuum energy becomes unbounded from below and its quantum effective theory becomes unstable by infinitely fast pair creation of particles. Especially, we call the ghost that appears in bigravity Boulware-Deser(BD) ghost [2]. In order to obtain healthy bigravity, we must tune the interaction term so that the extra ghost mode is erased by constraints. One candidate of covariant ghost-free bigravity is the one with the non-derivative interactions proposed by de Rham, Gabadadze and Tolley (dRGT) [3, 4] in the context of massive gravity, which is called the ghost-free bigravity. This dRGT interaction form is constructed by imposing BD-ghost-free conditions, and shown to contain seven degrees of freedom, corresponding to one massless graviton (2 degrees of freedom) and one massive graviton (5 degrees of freedom), and hence no BD ghost [5–8]. Using the ghost-free bigravity, a lot of works have done about fundamental solutions such as cosmological solutions [9–15], spherically symmetric solutions [16–19], gravitational waves [20, 21]. However, it is difficult to obtain physical intuition of the nature of these solutions. This is because the interaction form is derived artificially by imposing that the system is free from BD ghost. Furthermore, although the interaction between two metrics different from this artificial dRGT form must be suppressed to realize a viable model, we do not know the mechanism that realizes such fine tuning.

In order to improve the understanding of the ghost-free bigravity and address the fine-tuning problem, we attempt to reproduce the ghost-free bigravity as a low-energy effective theory of braneworld models. Braneworld models are higher-dimensional theories with an object called brane on which the open strings are confined, inspired by string theory. The graviton propagates to the extra dimension, so-called bulk, while the matter fields of the standard model are localized on the brane. The extra dimension should be compactified to an invisibly small scale, since we do not observe the existence of the extra dimension. Especially, the simplified braneworld models have been studied in which the spacetime is 5-dimensional and as the boundary condition on the brane Z_2 symmetry across the brane is imposed [22, 23]. In 4-dimensional effective theory on the brane, the 4-dimensional graviton has a mass corresponding to the leak of the graviton's momentum into the bulk, whose mass spectrum becomes discretized and has infinite number of massive gravitons determined by the size of extra dimension when the extra dimension is compactified. These 4-dimensional massive gravitons are called Kaluza-Klein(KK) gravitons. If one can give a hierarchy between the lowest massive KK mode and the other KK modes, its low-energy effective theory becomes bigravity. When it is possible to truncate these massive KK modes expect for the lowest one ghost-freely, the obtained low-energy effective theory should be free from ghost, and consequently the bigravity obtained in this manner should coincide with the ghost-free bigravity. Namely, we expect that the ghost-free bigravity may be reproduced as a low-energy effective theory of an appropriate braneworld model. Considering a discretized extra dimension has been proposed as one way to obtain the ghost-free bigravity [24].

As a candidate of braneworld model that can reproduce bigravity, we propose the Dvai-Gabadadze-Porrati (DGP) 2-brane model [25] in which the bulk spacetime is sandwiched by two branes with induced gravity terms localized on the branes [26, 27]. In braneworld models, the eigenvalue problem that determines the 4-dimensional graviton's mass is analogous to the one in the quantum mechanics of a particle propagating in a one-dimensional potential. The DGP 2-brane model has two induced gravity terms on branes, *i.e.*, two gravitational potential wells at the boundaries of the bulk. If these potential wells are sufficiently deep, namely, the coupling constants of the induced gravity terms are large, only two modes trapped in the potential wells are kept at low energies, while the other modes can be driven away to the high energies by making the potential well to be steep. The superpositions of two trapped modes yield one massless and one massive mode. In the DGP 2-brane model, the length parameters that correspond to the depth of the potential wells are given as $r_c^{(\pm)} := M_{\rm Pl}^{(\pm)\,2}/2M_5^3$ where $M_{\rm Pl}^{(\pm)}$ are the 4-dimensional Planck masses for the induced gravity terms on the respective $(\pm)$-branes and M_5 is the 5-dimensional Planck mass. Therefore, we expect that one can achieve the objective hierarchy in graviton's mass spectrum by tuning $\Delta y \ll r_c^{(\pm)}$ for the brane separation Δy.

Reproducing bigravity from the DGP 2-brane model has been already investigated by Padilla [28] before the invention of the ghost-free bigravity. However, the model studied in Ref. [28] contains a radion scalar corresponding to the fluctuation of the brane separation, which is absent in the ghost-free bigravity. As we should erase this degree of freedom to reproduce the ghost-free bigravity, we introduce a stabilization mechanism with which one can fix the brane separation. With an appropriate stabilization, we can naturally choose the setup with the small brane separation necessary for the hierarchy among the KK gravitons. As a concrete stabilization model, we consider Goldberger-Wise stabilization model [29] which stabilizes the brane separation by introducing a bulk scalar field with localized potentials on the branes.

Considering linear perturbations around de Sitter background in the DGP 2-brane model with a scalar stabilization, we analyze the mass eigenvalue problem to find that it is possible to achieve the hierarchy between the lowest massive KK graviton and the other gravitons when $\Delta y \ll r_c^{(\pm)}$. Also, we show that we can stabilize the two branes that locate closely when the energy densities on the branes are sufficiently small and the geometries on the branes are almost Minkowski. Furthermore, we investigate ghost instabilities in the DGP 2-brane model, since if the system contain ghost instablities, the obtained effective bigravity should also be contaminated by ghost modes and may become different from the ghost-free bigravity. It is known that there exist two branches in the DGP model [30–34]: one is the self-accelerating branch where either the helicity-0 modes of massive gravitons or spin-0 mode which corresponds to the radion scalar field becomes ghost inevitably, and the other one is the normal branch free from ghost. We show that the conditions for the ghost-free normal branch are given as $1 - 2r_c^{(\pm)}(n^\mu_{(\pm)}\partial_\mu \log a) > 0$ on the respective branes, where a is the background bulk warp factor and $n^\mu_{(\pm)}$ are the unit normal vectors pointing toward the bulk on the respective branes. Assuming that the values of the extrinsic curvature on two branes are approximately the same, *i.e.*, $K \approx K_+ \approx K_- \approx \mp 4(n^\mu_{(\pm)}\partial_\mu \log a)$, the normal-branch conditions require $|K| \lesssim 1/r_c^{(\pm)}$. If the normal branch conditions are satisfied, one can stably fix the brane separation and when $\Delta y \ll r_c^{(\pm)}$ one obtain a low-energy effective theory whose equations of motion are identical to the ones in the ghost-free bigravity. The solution with Minkowski branes easily satisfies both conditions $|K| \lesssim 1/r_c^{(\pm)}$ and $\Delta y \ll r_c^{(\pm)}$. If one consider to add energy densities on these Minkowski branes, however, $|K|$ becomes larger and soon it becomes difficult to keep satisfying these conditions. We show that if we increase the energy densities on the branes, the mass of the spin-0 mode gets lower, and at the relatively low energies before crossing the critical point where ghost instabilities appear, the mass squared of the spin-0 mode becomes less than zero, namely, the spin-0 mode becomes tachyon. Because of the spin-0 tachyon, which is absent in the ghost-free bigravity, it becomes impossible to sustain the setup with closely-located two branes, and consequently one cannot reproduce bigravity from the DGP 2-brane model in a slightly high energy regime.

Next, we attempt to derive the ghost-free bigravity action from the DGP 2-brane model. In this attempt, we do not introduce any radion stabilization mechanism to make the analysis simpler, and hence the low-energy effective action should consist of two gravitons and one scalar radion. In the normal branch, the radion ghost-freely couples to both induced metrics on branes, although it is known that in general BD ghost reappears in the ghost-free bigravity with a field that couples to both metric, so-called doubly-coupled matter. Therefore, the low-energy effective theory derived from the DGP 2-brane model possibly give a non-trivial ghost-free doubly-coupled matter model different from the previously proposed ones [35–41]. As is discussed above, we need to set the model parameters to satisfy the conditions $|K| \lesssim 1/r_c^{(\pm)}$ and $\Delta y/r_c^{(\pm)} \ll 1$ so as to reproduce the ghost-free bigravity from the DGP 2-brane model. Combining these conditions, the region of the model parameters of our interest is restricted to

$$|K\Delta y| \ll 1\,. \tag{1.0.1}$$

Therefore, we consider the gradient expansion [42], in which we expand the variables with respect to the small quantity $K\Delta y$ to construct the bulk solution. We can systematically expand the action in this manner so that the higher-order terms in the gradient expansion cannot cancel the lower-order ones. Hence, we expect that the action at the leading order of the gradient expansion reproduces the one of the ghost-free bigravity. In order to derive the 4-dimensional effective action written in terms of the metrics on the $(\pm)$-branes, $g^{(\pm)}_{\mu\nu}$, we solve the bulk equations for given boundary metrics $g^{(\pm)}_{\mu\nu}$ at the lowest order of the gradient expansion, and then we integrate out the bulk degrees of freedom by substituting back the obtained bulk solution into the action and performing the integration along the extra dimension. The higher order corrections in the gradient expansion would naturally yield higher-derivative terms in the effective action which seem to introduce extra degrees of freedom including ghost modes in addition to two gravitons and one scalar radion. However, the appearance of extra ghost modes itself will not directly indicate the breakdown of the gradient expansion, since it should be because of the contamination by the massive KK gravitons with the mass squared of $O(\Delta y^{-1})$ or higher.

This paper is organized as follows. In Chapter. 2, we review massive gravity and the ghost-free bigravity, and also the nature of its de Sitter solution. In Chapter. 3, we briefly review braneworld models: representative brane models and their basic ideas. Chapter. 4 is devoted to analyze the mass spectrum of the DGP 2-brane model with scalar stabilization. Here, we show that the mass spectrum of the DGP 2-brane model becomes identical to the one of the ghost-free bigravity at low energies by choosing appropriate parameters. In Chapter. 5, we discuss whether the equations of motion and instabilities of the bigravity as a low-energy effective theory of the DGP 2-brane model coincide with the ghost-free bigravity. In this chapter, we show that the correspondence between two models breaks down in slightly high energy regime because of the tachyonic radion. In Chapter. 6, we derive the effective bigravity action from the DGP 2-brane model without introducing radion stabilization for simplicity. We finally summarize this paper and present the conclusion in Chapter. 7.

In this paper, we use natural units $c = \hbar = 1$ and the signature of metrics as $(-,+,+,+)$.

Chapter 2

Massive gravity and bigravity

In this chapter, we review how one can construct healthy massive gravity and bigravity, in which the gravitons becomes massive and there exist two gravitons interacting with each other ghost-freely, respectively. Also, we investigate the de Sitter solution and its nature in the ghost-free bigravity.

2.1 General Relativity

First, we briefly review general relativity and its degrees of freedom. The action of general relativity is

$$S = \int d^4x\sqrt{-g}L = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}R\,, \tag{2.1.1}$$

where $M_{\rm Pl}$ is the 4-dimensional Planck mass and R is the Ricci scalar defined by the metric $g_{\mu\nu}$. In order to investigate the number of degrees of freedom, we analyze the system with the action (2.1.1) using the Hamiltonian formulation. Using the ADM decomposition in Appendix. A, we foliate the spacetime by t-constant hypersurfaces for the time coordinate t. Using the lapse function N, the shift vector N_i, and the 3-dimensional metric induced on the t-constant hypersurface γ_{ij}, The action becomes

$$2M_{\rm Pl}^{-2}S = \int dt \int d^3xN\sqrt{\gamma}\left({}^3R + K_{ij}K^{ij} - K^2\right)\,, \tag{2.1.2}$$

where we use Eq. (A.2.13). Here we define ${}^3R$ as the Ricci scalar evaluated from the 3-dimensional induced metric γ_{ij}, K_{ij} as the extrinsic curvature on the t-constant hypersurface, and K as the trace of the extrinsic curvature. We introduce the momentum π^{ij} conjugate to γ_{ij}:

$$2M_{\rm Pl}^{-2}\pi^{ij} := \sqrt{\gamma}\left(K^{ij} - \gamma^{ij}K\right)\,, \tag{2.1.3}$$

then, the Hamiltonian $H = \int d^4x\, (\pi^{ij}\dot{\gamma}_{ij} - L)$ is given as

$$2M_{\rm Pl}^{-2}H = -\int d^4x \left(NR_0 + N_i R^i\right) , \tag{2.1.4}$$

$$R_0 = \sqrt{\gamma}\,{}^3R + \frac{1}{\sqrt{\gamma}}\left(\frac{1}{2}\pi^i_i\pi^j_j - \pi_{ij}\pi^{ij}\right) , \tag{2.1.5}$$

$$R^i = 2\sqrt{\gamma}\nabla_j\left(\frac{\pi^{ij}}{\sqrt{\gamma}}\right) . \tag{2.1.6}$$

This Hamiltonian becomes linear in the lapse function N and the shift vector N_i, and hence four primary constraints $R_0 = 0$ and $R^i = 0$ are imposed by the variation with respect to the Lagrange multipliers N and N_i, respectively. In the Hamiltonian formalism, it is nontrivial whether constraints are preserved along the time evolution, and hence we should impose that the time evolution of the constraints are also zero, so-called the consistency relations. In order to investigate the time evolution of these constraints, we need to compute the Poisson brackets of R_0 and R_i:

$$M_g^2\{R_0(x)\,,\,R_0(y)\} = -\left[R^i(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R^i(y)\frac{\partial}{\partial y^i}\delta^3(x-y)\right] , \tag{2.1.7}$$

$$M_g^2\{R_0(x)\,,\,R_i(y)\} = -R^0(y)\frac{\partial}{\partial x^i}\delta^3(x-y)\,, \tag{2.1.8}$$

$$M_g^2\{R_i(x)\,,\,R_j(y)\} = -\left[R_j(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R_i(y)\frac{\partial}{\partial y^j}\delta^3(x-y)\right] . \tag{2.1.9}$$

Using these Poisson brackets, the time evolution of the primary constraints $\{R_0,\, H\}$, $\{R^i,\, H\}$ is shown to be zero. Therefore, the four primary constraints are automatically satisfied under the time evolution and the Lagrange multipliers N and N_i remains undetermined, namely, four gauge degrees of freedom exist. We can fix the gauge degrees of freedom by imposing four constraints among γ_{ij} and π^{ij} whose consistency relations determine the Lagrange multipliers N and N_i. By four primary constraints and four gauge fixing conditions, the 12 variables γ_{ij}, π^{ij} are reduced to 4 degrees of freedom. Then, general relativity has 2 degrees of freedom in the configuration space which correspond to the massless spin-2 graviton.

2.2 massive gravity

General relativity is described by a unique massless graviton. In order to test whether the graviton is massless or not, there exist attempts to give the graviton a mass: massive gravity. In the following, we focus on the attempt to consider non-derivative covariant mass terms, and show that in general the massive graviton contains a ghost mode and that we need to tune the form of the mass terms so as to avoid the ghost instability.

In order to invent non-derivative mass terms, we must introduce a so-called fiducial metric $f_{\mu\nu}$, since if we try to construct mass terms only with one physical metric $g_{\mu\nu}$, the mass terms become just a cosmological constant. The general action of massive gravity is written as

$$S=\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2\left(\frac{1}{2}R-m^2U(g,\,f)\right)\right]\,, \tag{2.2.1}$$

where m is a mass parameter and $U(g,\,f)$ is an arbitrary scalar function of $g_{\mu\nu}$ and $f_{\mu\nu}$, not of their derivatives. Performing the Hamiltonian analysis for this action as in Sec. 2.1, because of the presence of the mass term $U(g,\,f)$, variations of the action with respect to N and N^i yield the equations that determine N and N^i, respectively, which means that N and N^i are not Lagrange multipliers, and that γ_{ij} and π^{ij} are not constrained. This also means that there are no gauge degrees of freedom, as is expected because the existence of the fiducial metric $f_{\mu\nu}$ should break the invariance under the coordinates transformations. Hence, the system contains 12 degrees of freedom corresponding to γ_{ij} and π^{ij} in the phase space, and equivalently 6 degrees of freedom in the configuration space corresponding to 2 helicity-2 modes and 2 helicity-1 modes and 2 helicity-0 modes. As is shown later in this section, one of the helicity-0 modes has wrong-signed kinetic term which yields a quantum instability of vacuum, so-called ghost instability. In order to construct healthy massive gravity, therefore, we should tune the mass term so that the helicity-0 ghost mode is erased.

2.2.1 Fierz-Pauli massive gravity

As the first step, we show how to construct healthy massive gravity at linear level of the metric perturbation around flat background. Here, we refer to Refs. [43, 44]. We introduce the deviation $h_{\mu\nu}$ from Minkowski metric: $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. We treat $h_{\mu\nu}$ as a perturbation and consider the action quadratic in $h_{\mu\nu}$. Choosing Minkowski metric $\eta_{\mu\nu}$ as the fiducial metric $f_{\mu\nu}$, the mass term U is in general written as $U=h_{\mu\nu}h^{\mu\nu}+ah^2$ up to the quadratic order of $h_{\mu\nu}$, where indices are raised by means of $\eta_{\mu\nu}$.

First, we show that for general parameter a the mass term brings ghost instability. For this purpose, we introduce four Stückelberg scalar fields Y^a. The index a represents the kind among the four scalar fields, not the vector component. Introducing the Stückelberg fields $Y^a(x^\mu)$, we can apparently recover the gauge invariance as follows. We define a new covariant tensor using Y^a as

$$H_{\mu\nu}=g_{\mu\nu}-f_{ab}\partial_\mu Y^a(x)\partial_\nu Y^b(x)\,, \tag{2.2.2}$$

whose second term in fact transforms covariantly under the coordinates transformation $x^\mu\to f^\mu(x)$ as

$$f_{ab}\partial_\mu Y^a(x)\partial_\nu Y^b(x)\to\partial_\mu f^\alpha\partial_\nu f^\beta\left(f_{ab}\frac{\partial Y^a\left(f(x)\right)}{\partial f^\alpha}\frac{\partial Y^b\left(f(x)\right)}{\partial f^\beta}\right)\,. \tag{2.2.3}$$

We use this tensor $H_{\mu\nu}$ instead of $h_{\mu\nu}$ to construct an apparently covariant mass term

$$U=-\frac{1}{4}\left(H^{\mu\nu}H_{\mu\nu}+a(H^\mu_\mu)^2\right)\,. \tag{2.2.4}$$

Notice that the covariance is recovered by the Stückelberg fields just apparently, since we should fix the Stückelberg field, namely, the gauge degrees of freedom recovered by the Stückelberg fields are to be fixed. Fixing the gauge as $Y^a = x^a$ yields $H_{\mu\nu} = h_{\mu\nu}$, and consequently the original mass term $h_{\mu\nu}h^{\mu\nu} + ah^2$. Considering the perturbation around Minkowski background, we set the Stückelberg fields to be $Y^a = x^a - \xi^a$ and treat ξ^a perturbatively. Then, $H_{\mu\nu}$ becomes

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\mu \xi_\nu \,. \tag{2.2.5}$$

We consider the mass term Eq. (2.2.4) with this $H_{\mu\nu}$ and analyze ghost instabilities. Since ξ_μ corresponds to the apparent gauge degrees of freedom, these modes do not contribute to the gauge-invariant Einstein-Hilbert term. The kinetic term of $h_{\mu\nu}$ appears only from the Einstein-Hilbert term, and hence $h_{\mu\nu}$ does not exhibit ghost instabilities. The mass term yields the kinetic term of ξ_μ as

$$-\frac{1}{4}m^2 \left(\frac{1}{2}(\partial_\mu \xi_\nu)^2 + \left(\frac{1}{2} + a \right) (\partial_\mu \xi^\mu)^2 \right) \,. \tag{2.2.6}$$

Here, we again introduce a Stückelberg field ϕ for ξ_μ as $\xi_\mu \to \xi_\mu + \partial_\mu \pi$ to focus on the scalar modes. Then, the kinetic terms of the spin-0 mode π contains a higher derivative term

$$-\frac{1}{4}m^2(1+a)(\Box \pi)^2 \,. \tag{2.2.7}$$

The kinetic terms for π also contain the terms such as $h\Box\pi$, but these are all second-derivative terms and hence do not yield ghost instabilities. Introducing an auxiliary field ψ, the Lagrangian $L = -(\Box\pi)^2$ can be rewritten as $L = -\psi\Box\pi + \psi^2/4$, and furthermore we redefine the scalar fields π and ψ as $\pi := \sigma + \chi$, $\psi := \sigma - \chi$ to obtain

$$L = \partial_\mu \sigma \partial^\mu \sigma - \partial_\mu \chi \partial^\mu \chi + \frac{1}{4}\left(\sigma^2 + \chi^2 - 2\sigma\chi \right) \,. \tag{2.2.8}$$

This shows that one of the scalar fields becomes ghost whatever the sign of the coupling of this Lagrangian is. However, this ghost mode can be removed by tuning of the parameter a as $a = -1$. When $a = -1$, there is no higher-derivative terms for π, and hence there exist only one scalar degree of freedom. On the other hand, the kinetic term of the spin-1 mode ξ_μ becomes $F_{\mu\nu}F^{\mu\nu}$ for $F_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, which is healthy. Therefore, the linear massive gravity with $a = -1$ has 2 helicity-2 modes, 2 helicity-1 modes and 1 helicity-0 mode corresponding to one massive graviton, and all of them are healthy around Minkowski background. This ghost-free mass term $U = -\frac{1}{4}\left(H^{\mu\nu}H_{\mu\nu} - (H^\mu_\mu)^2 \right)$ is called Fierz-Pauli mass term [45].

However, the above ghost-free proof is valid only in the linear level and the scalar ghost reappears when the deviation $h_{\mu\nu}$ from the Minkowski metric becomes large. In order to see this, let us consider the third order terms in $H_{\mu\nu}$:

$$U_3 = m^2 \left[\lambda_1 H^3 + \lambda_2 H H_{\mu\nu} H^{\mu\nu} + \lambda_3 H_{\mu\nu} H^\nu_\rho H^{\rho\mu} \right] \,, \tag{2.2.9}$$

where we introduce parameters λ_n. Decomposing the deviation $h_{\mu\nu}$ from the Minkowski background into the background one $h^{(b)}_{\mu\nu}$ and the perturbative one $\bar{h}_{\mu\nu}$, we rewrite the tensor $H_{\mu\nu} = g_{\mu\nu} - f_{ab}\partial_\mu Y^a(x)\partial_\nu Y^b(x)$ as

$$H_{\mu\nu} = h^{(b)}_{\mu\nu} + \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\mu \xi_\nu \, . \tag{2.2.10}$$

As in the Fierz-Pauli massive gravity case, we focus on the spin-0 kinetic term that comes from U_3, introducing perturbative Stückelberg modes ξ^a as $Y^a = x^a - \xi^a$ and another Stückelberg scalar π as $\xi_\mu \to \xi_\mu + \partial_\mu \pi$. The kinetic term up to the quadratic order in π becomes

$$m^2 \left[\tilde{\lambda}_1 h^{(b)}_{\mu\nu} \partial^\mu \partial^\nu \pi \Box \pi + \tilde{\lambda}_2 h^{(b)} (\Box \pi)^2 \right] , \tag{2.2.11}$$

where we redefine the parameters $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ using λ_1, λ_2, λ_3. For simplicity, let us consider the background $h^{(b)}_{\mu\nu} = \omega(x)\eta_{\mu\nu}$. Then, the kinetic terms Eq. (2.2.11) are rewritten to the one proportional to $(\Box\pi)^2$, which yields a scalar ghost as in the case of Minkowski background. This ghost in the non-linear massive gravity is called Boulware-Deser(BD) ghost [2]. In order to obtain nonlinear ghost-free massive gravity, we need further tuning so as to erase the BD ghost, which will be discussed in the next subsection.

2.2.2 dRGT massive gravity

Next, we consider the extension of Fierz-Pauli massive gravity to the non-linear theory. In order to construct ghost-free massive gravity, we necessarily erase the higher derivative terms of the spin-0 mode. Refs. [3,4] derive the mass terms in which the higher derivative terms of the spin-0 mode disappear as total derivatives by introducing Stückelberg fields. We consider the fiducial metric as $f_{\mu\nu} = \eta_{\mu\nu}$ and focus only on the spin-0 mode of the Stückelberg fields by setting $Y^a = x^a + \partial^a \pi$. Then, $H_{\mu\nu}$ becomes

$$H_{\mu\nu} = h_{\mu\nu} + 2\partial_\mu \partial_\nu \pi - \partial_\mu \partial^\alpha \pi \, \partial_\nu \partial_\alpha \pi \, . \tag{2.2.12}$$

Using this $H_{\mu\nu}$, we want to construct the mass term without higher derivatives of π. For this purpose, we derive the total derivative terms written in terms of $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$. Such total derivative terms are uniquely determined as

$$V_n(\Pi_{\mu\nu}) = \sum_{n=0}^{d} \epsilon^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} \Pi^{\nu_1}_{\mu_1} \cdots \Pi^{\nu_n}_{\mu_n} \tag{2.2.13}$$

$$\begin{aligned} &= \sum_{n=0}^{d} \epsilon^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} \partial^{\nu_1} \left(\partial_{\mu_1} \pi \cdots \partial_{\mu_n} \partial^{\nu_n} \pi \right) \\ &\quad - \sum_{n=0}^{d} \epsilon^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} \partial_{\mu_1} \pi \left(\partial_{\mu_2} \partial^{\nu_1} \partial^{\nu_2} \pi \cdots \partial_{\mu_n} \partial^{\nu_n} \pi + \partial_{\mu_2} \partial^{\nu_2} \pi \partial_{\mu_3} \partial^{\nu_1} \partial^{\nu_3} \pi \cdots \partial_{\mu_n} \partial^{\nu_n} \pi + \cdots \right) . \end{aligned} \tag{2.2.14}$$

Because of the antisymmetric tensor ϵ, the terms with $n > d$ disappear and the second and the following terms in the second line of Eq. (2.2.13) become zero, which reduces V_n to a total derivative. Let us define such a tensor $Y_{\mu\nu}$ that coincides with $\Pi_{\mu\nu}$ when $h_{\mu\nu} = 0$ as

$$Y^\mu_\nu = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu}\,, \tag{2.2.15}$$

and construct the mass terms in the same matter as in Eq. (2.2.13) using $Y_{\mu\nu}$ instead of $\Pi_{\mu\nu}$. These mass terms are expected to avoid helicity-0 ghost instability since they do not include higher derivatives of spin-0 mode in $h_{\mu\nu} \to 0$ limit. We define the mass terms using parameters α_n as

$$U = \alpha_0 + \alpha_1 V_1(Y^\mu_\nu) + \alpha_2 V_2(Y^\mu_\nu) + \alpha_3 V_3(Y^\mu_\nu) + \alpha_4 V_4(Y^\mu_\nu)\,. \tag{2.2.16}$$

These mass terms are proved to be ghost-free and contains five degrees of freedom corresponding to a massive graviton [5, 7, 8]. We will show a brief review of this proof in the next section, extending the model to bigravity where the fiducial metric becomes dynamical.

2.3 ghost-free bigravity

The invention of non-linear massive gravity also enable us to extend massive gravity to the multi-gravity, *i.e.*, the gravitational theory with multiple dynamical gravitons interacting with each other. The multi-gravity modify the number of gravitons from General Relativity, but, in general the interaction between the gravitons bring the BD ghost and so the interaction between the gravitons must be restricted to some specific form, for example the dRGT mass interaction. In the following, we focus on the bigravity where two gravitons interacts through the dRGT mass interaction and call this the ghost-free bigravity.

We introduce the ghost-free bigravity action, where the physical metric $g_{\mu\nu}$ and the hidden metric $\tilde{g}_{\mu\nu}$ interact with each other, as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}(R + V(g,\tilde{g})) + L_m\right] + \frac{\chi M_{\rm Pl}^2}{2}\int d^4x\sqrt{-\tilde{g}}\tilde{R}\,, \tag{2.3.1}$$

$$V(Y^\mu_\nu) = \sum_{n=0}^{4}\alpha_n V_n(Y^\mu_\nu)\,, \tag{2.3.2}$$

$$Y^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\alpha}\tilde{g}_{\alpha\nu}}\,. \tag{2.3.3}$$

where $\tilde{R}$ is the Ricci tensor defined by the hidden metric $\tilde{g}_{\mu\nu}$, and $M_{\rm Pl}$ and $\chi M_{\rm Pl}$ are the 4-dimensional Planck mass for the physical metric $g_{\mu\nu}$ and the hidden metric $\tilde{g}_{\mu\nu}$, respectively. Here, we introduce a tensor Y^μ_ν identical to the one in Eq. (2.2.15) when $f_{\mu\nu} = \tilde{g}_{\mu\nu}$ and $Y^a = x^a$. Also, we assumed that the matter Lagrangian L_m couples only to the physical metric by which our universe are described[1]. The interaction between two metrics $V(Y^\mu_\nu)$ can be rewritten as

$$V(\mathcal{U}^\mu_\nu) = \sum_{n=0}^{4} c_n V_n(\mathcal{U}^\mu_\nu)\,, \tag{2.3.4}$$

in terms of a tensor $\mathcal{U}^\mu_\nu$ defines by

$$\mathcal{U}^\mu_\nu := \sqrt{g^{\mu\rho}\tilde{g}_{\rho\nu}} \tag{2.3.5}$$

and the parameters c_n related to α_n as

$$\begin{aligned} &c_0 = \alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4\,,\ \ c_1 = \alpha_1 - 3\alpha_2 + 3\alpha_3 - \alpha_4\,,\\ &c_2 = \alpha_2 - 2\alpha_3 + \alpha_4\,,\ \ c_3 = \alpha_3 - \alpha_4\,,\ \ c_4 = \alpha_4\,. \end{aligned} \tag{2.3.6}$$

The interaction form (2.3.4) is invariant under the transformation with which both metrics' coordinates rotate in the same way. Hence, the system 4 gauge degrees of freedom corresponding to one set of 4-dimensional coordinates transformation. Using the Hamiltonian formulation, we can show that this system contains two degrees of freedom that corresponds to the massless graviton and five degrees of freedom that corresponds to the massive graviton [6, 7]. Now, we will count the number of degrees of freedom in the following. First, we decompose two metrics $g_{\mu\nu}$, $\tilde{g}_{\mu\nu}$ as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & \gamma_{ij} \end{pmatrix}\,, \qquad \tilde{g}_{\mu\nu} = \begin{pmatrix} -L^2 + L_k L^k & L_j \\ L_i & \tilde{\gamma}_{ij} \end{pmatrix}\,. \tag{2.3.7}$$

We introduce the conjugate momenta π^{ij} and $\tilde{\pi}^{ij}$ for γ_{ij} and $\tilde{\gamma}_{ij}$, respectively. The system does not contain the time derivatives of the lapse functions and the shift vectors N, L, N^i and L^i, and hence their conjugate momenta are zero. The interaction (2.3.4) is nonlinear in term of the lapse functions and the shift vectors and the variations of the action (2.3.1) with respect to the lapse functions and the shift vectors seem to give equations that determine just themselves, but not ones that constrain the dynamical variables γ_{ij}, $\tilde{\gamma}_{ij}$, π^{ij} and $\tilde{\pi}^{ij}$. However, this idea should be incorrect: the action should be essentially linear in terms of two lapse functions and one shift vector because the system is expected to have four gauge degrees of freedom and two constraints on the dynamical variables in the phase space which constrain one extra scalar degree of freedom that corresponds to the BD ghost. Hence, we introduce a new variable n^i instead of N^i so that the Hamiltonian becomes linear in N, L, L^i. For this purpose, we define the new shift vector n^i as

$$N^i - L^i = (L\delta^i_k + ND^i_k)n^k\,, \tag{2.3.8}$$

[1]We can introduce a matter field that couples only to the hidden metric ghost-freely, but we ignore this for simplicity. If we consider a matter field that couples to both metrics, then in general BD ghost reappears as is discussed in Appendix. E.

as is proposed in Ref. [5,6], where the matrix $D = D^i_k$ is determined by the condition

$$\sqrt{x}D = \sqrt{(\gamma^{-1} - Dnn^T D^T)\,\tilde{\gamma}}\,, \tag{2.3.9}$$

where

$$x := 1 - n^k \tilde{\gamma}_{km} n^m\,. \tag{2.3.10}$$

From the definition of D, D does not depend on the lapse functions and the shift vectors N, L, N^i, and L^i. Rewriting the action in terms of the new shift vector n^i instead of N^i, the Lagrangian density $\mathcal{L}$ of the action (2.3.1) becomes

$$\mathcal{L} = M_{\rm Pl}^2 \pi^{ij}\dot{\gamma}_{ij} + \chi M_{\rm Pl}^2 \tilde{\pi}^{ij}\dot{\tilde{\gamma}}_{ij} - \mathcal{H}_0 + N\mathcal{C}\,. \tag{2.3.11}$$

Here, $\mathcal{H}_0$ and $\mathcal{C}$ are defined as

$$\mathcal{H}_0 := -L^i\left(M_{\rm Pl}^2 C_i + \chi M_{\rm Pl}^2 \tilde{C}_i\right) - L\left(\chi M_{\rm Pl}^2 \tilde{C} + M_{\rm Pl}^2 n^i C_i + m^2 M_{\rm Pl}^2 \sqrt{\gamma} F\right)\,, \tag{2.3.12}$$

$$\mathcal{C} := M_{\rm Pl}^2 C + M_{\rm Pl}^2 C_i D^i_j n^j + m^2 M_{\rm Pl}^2 \sqrt{\gamma} G\,. \tag{2.3.13}$$

where we used C and C^i defined as in Eqs. (2.1.5) and (2.1.6) for the physical metric g_{ij}, and $\tilde{C}$ and $\tilde{C}^i$ defined in the same manner for the hidden metric $\tilde{g}_{ij}$. Also, we introduced F and G as

$$\begin{aligned} F :=\,& c_1\sqrt{x} + c_2\left(\sqrt{x}^2 D^i_i + n^i\tilde{\gamma}_{ij} D^j_k n^k\right) \\ &+ c_3\left[\sqrt{x}\left(D^l_l n^i \tilde{\gamma}_{ij} D^j_k n^k - D^i_k n^k \tilde{\gamma}_{ij} D^j_l n^l\right) + \frac{1}{2}\sqrt{x}^3\left((D^i_i)^2 - D^i_j D^j_i\right)\right] + c_4\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}\,, \end{aligned} \tag{2.3.14}$$

$$G := c_0 + c_1\sqrt{x}D^i_i + \frac{1}{2}c_2\sqrt{x}^2\left((D^i_i)^2 - D^i_j D^j_i\right)\frac{1}{6}c_3\sqrt{x}^3\left((D^i_i)^3 - 3D^i_i D^j_k D^k_j + 2D^i_j D^j_k D^k_i\right)\,. \tag{2.3.15}$$

Variation of the Lagrangian (2.3.11) with respect to n^i yields a constraint that determines n^i. The Lagrangian appears to be linear in N, L and L^i at a glance, but the Lagrangian can be nonlinear in them after eliminating n^i since n^i may be a function of N, L, and L^i. However, we can show that n^i determined by the constraint is independent of N, L and L^i as follows. Variation of the Lagrangian (2.3.11) with respect to n^i gives

$$\frac{\delta\mathcal{L}}{\delta n^k} = \mathcal{C}_i\left[L\delta^i_k + N\frac{\partial(D^i_j n^j)}{\partial n^k}\right]\,, \tag{2.3.16}$$

where

$$\begin{aligned} \mathcal{C}_i :=\,& M_{\rm Pl}^2 C_i - m^2 M_{\rm Pl}^2\sqrt{\gamma}\frac{n^l\tilde{\gamma}_{lj}}{\sqrt{x}}\left[c_1\delta^j_i + c_2\sqrt{x}(\delta^j_i D^m_m - D^j_i)\right. \\ &\left.+ c_3\sqrt{x}^2\left(\frac{1}{2}\delta^j_i(D^m_m D^n_n - D^m_n D^n_m) + D^j_m D^m_i - D^j_i D^m_m\right)\right]\,. \end{aligned} \tag{2.3.17}$$

The matrix in the square brackets in Eq. (2.3.16) is the Jacobian of the transformation (2.3.8), and hence is invertible in general. Therefore, the equation of motion of n^i becomes

$$\mathcal{C}_i = 0\,, \tag{2.3.18}$$

which is independent of N, L and L^i (and p^{ij}), and by which we can rewrite n^i in terms of $(\gamma_{ij},\, \tilde{\gamma}_{ij},\, \tilde{\pi}^{ij})$. Substituting $n^k(\gamma_{ij},\, \tilde{\gamma}_{ij},\, \tilde{\pi}^{ij})$ to the Lagrangian, we find that the Lagrangian is linear in N, L and L^i. Therefore, variations with respect to N, L and L^i yield five primary constraint which we denote $\mathcal{C} = 0$, $C^L = 0$ and $C^L_i = 0$, respectively. Next, we present the result of the calculation of the Poisson brackets among the primary constraints $\mathcal{C}$, C^L and C^L_i. The detailed calculation is given in Refs. [8, 35]. The momentum constraint C^L_i exchanges all the primary constraints and they are preserved in the time evolution, as is expected because C^L_i is the generator of the spatial translation. Therefore, the Lagrange multiplier L^i becomes gauge degrees of freedom whose gauge fixing conditions yield three constraints. The Poisson brackets $\{\mathcal{C},\, \mathcal{C}\}$ and $\{C^L,\, C^L\}$ become zero, while $\mathcal{C}$ and C^L do not exchange. Hence, the consistency relations for $\mathcal{C} = 0$ and $C^L = 0$ yield an identical secondary constraint whose consistency relation gives one relation that determines one of the lapse functions N and L. The other lapse function is left undetermined and becomes a gauge degree of freedom whose gauge fixing condition gives one constraint. From above, the system is shown to have five primary constraints, one secondary constraint and four constraints from gauge fixing. Since there exist 24 dynamical variables: γ_{ij}, π^{ij}, $\tilde{\gamma}_{ij}$, and $\tilde{\pi}^{ij}$, the system contains 14 degrees of freedom in the phase space, and equivalently 7 degrees of freedom in the configuration space that corresponds to one massless graviton and one massive graviton. In dRGT massive gravity, since $\tilde{g}_{\mu\nu}$ is fixed, the dynamical variables are given as $(\gamma_{ij},\, \pi^{ij})$. These twelve dynamical variables are constrained by one primary condition obtained from variation with respect to N and one secondary constraint obtained from the consistency relation of the primary constraint, which yields 5 degrees of freedom in the configuration space corresponding to one massive graviton.

2.4 de Sitter solution in ghost-free bigravity

In this section, we present the analysis of the perturbation around the de Sitter solution and its stability in the ghost-free bigravity, based on our paper [26]. From the action of the ghost-free bigravity (2.3.1), (2.3.4), the equations of motion of the ghost-free bigravity become

$$G^\mu_\nu + B^\mu_\nu = M_{\rm Pl}^{-2} T^\mu_\nu\,, \tag{2.4.1}$$
$$\kappa \tilde{G}^\mu_\nu + \tilde{B}^\mu_\nu = 0\,, \tag{2.4.2}$$

where B^μ_ν and $\tilde{B}^\mu_\nu$ are the contributions from the mass interaction:

$$B^\mu_\nu = m^2 \left[V\delta^\mu_\nu - (V'\mathcal{U})^\mu_\nu\right] , \tag{2.4.3}$$
$$\tilde{B}^\mu_\nu = m^2 \left(\frac{\det(\tilde{g})}{\det(g)}\right)^{-1/2} (V'\mathcal{U})^\mu_\nu , \tag{2.4.4}$$

where we define $(V')^\mu_\nu := \partial V/\partial \mathcal{U}^\nu_\mu$. Considering the de Sitter background solution, two metrics are related as $\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$ with a constant ω [9]. In order to make the expression simpler, we introduce $f(\omega)$ and $\bar{f}(\omega)$ as

$$\begin{aligned} f(\omega) :=& V|_{\mathcal{U}^\mu_\nu=\omega\delta^\mu_\nu} \\ =& c_0 + 4c_1\omega + 12c_2\omega^2 + 24c_3\omega^3 + 24c_4\omega^4 , \end{aligned} \tag{2.4.5}$$
$$\begin{aligned} \bar{f}(\omega)\delta^\mu_\nu :=& \left[V\delta^\mu_\nu - (V'\mathcal{U})^\mu_\nu\right]_{\mathcal{U}^\mu_\nu=\omega\delta^\mu_\nu} = \left(f - \frac{\omega}{4}f'\right)\delta^\mu_\nu \\ =& \left(c_0 + 3c_1\omega + 6c_2\omega^2 + 6c_3\omega^3\right)\delta^\mu_\nu , \end{aligned} \tag{2.4.6}$$

where we define $f' = df/d\omega$. Then, the background equations of motion are given as

$$3H^2 - m^2\bar{f} = \Lambda , \tag{2.4.7}$$
$$3\kappa\omega H^2 - m^2 f'/4 = 0 , \tag{2.4.8}$$

where H is the Hubble scale of the physical metric and Λ is the cosmological constant that couples to the physical metric. We consider the linear perturbation $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = \omega^2(\gamma_{\mu\nu} + \tilde{h}_{\mu\nu})$ around the de Sitter background metric $\gamma_{\mu\nu}$. From Eqs. (2.4.1) and (2.4.2), we obtain the equations of motion linear in the perturbative metrics as

$$\mathcal{E}^{\alpha\beta}{}_{\mu\nu}h_{\alpha\beta} + 3H^2 h_{\mu\nu} + \frac{m^2}{2}\Gamma\left(h^{(m)}_{\mu\nu} - h^{(m)}\gamma_{\mu\nu}\right) = M_{\rm Pl}^{-2}T_{\mu\nu} , \tag{2.4.9}$$
$$\mathcal{E}^{\alpha\beta}{}_{\mu\nu}\tilde{h}_{\alpha\beta} + 3H^2 \tilde{h}_{\mu\nu} - \frac{m^2}{2\kappa\omega^2}\Gamma\left(h^{(m)}_{\mu\nu} - h^{(m)}\gamma_{\mu\nu}\right) = 0 . \tag{2.4.10}$$

Here, we define $\Gamma := -\omega\bar{f}'/3 = c_1\omega + 4c_2\omega^2 + 6c_3\omega^3$, $h^{(m)}_{\mu\nu} = h_{\mu\nu} - \tilde{h}_{\mu\nu}$, and

$$\begin{aligned} \mathcal{E}^{\alpha\beta}{}_{\mu\nu}h_{\alpha\beta} := -\frac{1}{2}\big(&\Box h_{\mu\nu} + \nabla_\mu\nabla_\nu h - \nabla_\nu\nabla_\sigma h^\sigma_\mu - \nabla_\mu\nabla_\sigma h^\sigma_\nu \\ &-\gamma_{\mu\nu}\Box h + \gamma_{\mu\nu}\nabla_\alpha\nabla_\beta h^{\alpha\beta} + 4H^2 h_{\mu\nu} - H^2 h\gamma_{\mu\nu}\big) , \end{aligned} \tag{2.4.11}$$

where ∇_μ is the covariant differentiation associated with $\gamma_{\mu\nu}$ and $\Box = \gamma^{\mu\nu}\nabla_\mu\nabla_\nu$. From the conservation law of the energy-momentum tensor $\nabla_\mu T^\mu_\nu = 0$ and the Bianchi identities $\nabla_\mu G^\mu_\nu = 0$ and $\nabla_\mu \tilde{G}^\mu_\nu = 0$, we obtain $\nabla_\mu B^\mu_\nu = 0$ and $\nabla_\mu \tilde{B}^\mu_\nu = 0$, which yield an identical equation as

$$\nabla_\mu h^{\mu(m)}_\nu - \nabla_\nu h^{(m)} = 0 . \tag{2.4.12}$$

Combining Eqs. (2.4.9) and (2.4.10), $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ are decomposed into a massless mode $h^{(0)}_{\mu\nu} = h_{\mu\nu} + \kappa\omega^2\tilde{h}_{\mu\nu}$ and a massive mode $h^{(m)}_{\mu\nu}$. In terms of the massless mode $h^{(0)}_{\mu\nu}$ and the massive mode $h^{(m)}_{\mu\nu}$, the physical metric and the hidden metric are given as

$$\begin{aligned} h_{\mu\nu} &= \frac{1}{1+\kappa\omega^2}\left(h^{(0)}_{\mu\nu} + \kappa\omega^2 h^{(m)}_{\mu\nu}\right), \\ \tilde{h}_{\mu\nu} &= \frac{1}{1+\kappa\omega^2}\left(h^{(0)}_{\mu\nu} - h^{(m)}_{\mu\nu}\right). \end{aligned} \tag{2.4.13}$$

Using Eqs. (2.4.9), (2.4.10), (2.4.11) and (2.4.12), the equations of motion for the massive mode become

$$\Box h^{(m)}_{\mu\nu} - \nabla_\mu\nabla_\nu h^{(m)} - H^2\left(2h^{(m)}_{\mu\nu} + \gamma_{\mu\nu}h^{(m)}\right) - m^2_{\text{eff}}\left(h^{(m)}_{\mu\nu} - \gamma_{\mu\nu}h^{(m)}\right) = -2M_{\text{Pl}}^{-2}T_{\mu\nu}\,, \tag{2.4.14}$$

where we define

$$m^2_{\text{eff}} = m^2(1+(\kappa\omega^2)^{-1})\Gamma\,. \tag{2.4.15}$$

The trace of Eq. (2.4.14) yields

$$h^{(m)} = -\frac{2M_{\text{Pl}}^{-2}}{3\left(m^2_{\text{eff}} - 2H^2\right)}T\,, \tag{2.4.16}$$

and the traceless part of Eq. (2.4.14) gives

$$\begin{aligned} h^{(m)}_{\mu\nu} - \frac{1}{4}h^{(m)}\gamma_{\mu\nu} =& \frac{-2M_{\text{Pl}}^{-2}}{\Box - 2H^2 - m^2_{\text{eff}}}\left(T_{\mu\nu} - \frac{1}{4}\gamma_{\mu\nu}T\right) \\ &+ \left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}\Box\right)\frac{1}{\Box + 6H^2 - m^2_{\text{eff}}}h^{(m)}\,. \end{aligned} \tag{2.4.17}$$

Here, we use an identity that holds for an arbitrary scalar S:

$$\frac{1}{\Box - 2H^2 - m^2_{\text{eff}}}\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}\Box\right)S = \left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}\Box\right)\frac{1}{\Box + 6H^2 - m^2_{\text{eff}}}S\,. \tag{2.4.18}$$

We erase the term proportional to $\nabla_\mu\nabla_\nu$ in Eq. (2.4.17) by a gauge transformation. Then, combining Eqs. (2.4.17) and (2.4.16), we obtain

$$(\Box - 2H^2 - m^2_{\text{eff}})h^{(m)}_{\mu\nu} = -2M_{\text{Pl}}^{-2}\left(T_{\mu\nu} - \left(1 + \frac{1}{3}\frac{\Box - 2H^2 - m^2_{\text{eff}}}{\Box + 6H^2 - m^2_{\text{eff}}}\frac{m^2_{\text{eff}} - 6H^2}{m^2_{\text{eff}} - 2H^2}\right)\frac{1}{4}\gamma_{\mu\nu}T\right)\,. \tag{2.4.19}$$

On the other hand, Eqs. (2.4.9) and (2.4.10) give the equation of motion for the massless mode as

$$\mathcal{E}^{\alpha\beta}{}_{\mu\nu}h^{(0)}_{\alpha\beta} + 3H^2h^{(0)}_{\mu\nu} = M_{\text{Pl}}^{-2}T_{\mu\nu}\,. \tag{2.4.20}$$

Imposing gauge fixing conditions $\nabla^\mu \left(h^{(0)}_{\mu\nu} - \frac{1}{2}h^{(0)}\gamma_{\mu\nu}\right) = 0$, Eq. (2.4.20) gives

$$\Box \left(h^{(0)}_{\mu\nu} - \frac{1}{2}h^{(0)}\gamma_{\mu\nu}\right) - H^2 \left(2h^{(0)}_{\mu\nu} + \gamma_{\mu\nu}h^{(0)}\right) = -2M_{\rm Pl}^{-2}T_{\mu\nu}\,, \tag{2.4.21}$$

whose trace part yields

$$(\Box + 6H^2)h^{(0)} = 2M_{\rm Pl}^{-2}T\,. \tag{2.4.22}$$

From above, the equations of motion for the massless mode $h^{(0)}_{\mu\nu}$ become

$$(\Box - 2H^2)h^{(0)}_{\mu\nu} = -2M_{\rm Pl}^{-2}\left[T_{\mu\nu} - \frac{1}{4}\left(1 + \frac{\Box - 2H^2}{\Box + 6H^2}\right)\gamma_{\mu\nu}T\right]\,. \tag{2.4.23}$$

Notice that the ghost free bigravity is invariant under the scale transformation $\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}$, $\kappa \to \kappa/\Omega^2$, $c_i \to c_i/\Omega^i$. The metric perturbation $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ are also invariant under this transformation, and hence the hidden metric $\tilde{g}_{\mu\nu}$ can be arbitrarily rescaled.

Next, we will investigate the ghost instabilities in the de Sitter solution of the ghost-free bigravity. For this purpose, we analyze the de Sitter background spacetime

$$ds^2 = a(t)^2(-dt^2 + d\vec{x}^2)\,, \tag{2.4.24}$$

$$\tilde{ds}^2 = \tilde{a}(t)^2(-dt^2 + d\vec{x}^2)\,, \tag{2.4.25}$$

where $\tilde{a}/a = \omega$. $\nabla^\mu B_{\mu\nu} = 0$ gives

$$3\Gamma(\omega)\left[aH - (\dot{\tilde{a}}/a)\right] = 0\,, \tag{2.4.26}$$

which yields two branches: one is $\Gamma = 0$ and the other one is $aH = \dot{\tilde{a}}/a$. However, it is known that in the branch $\Gamma = 0$ the helicity-1 mode becomes infinitely strongly coupled [12, 14]. Therefore, we choose the healthy branch $aH - \tilde{a}'/\tilde{a} = 0$ where the time evolution of $\tilde{a}$ is related to the one of a. Then, Eqs. (2.4.1) and (2.4.2) give the equation of motion for a as

$$3H^2 = M_{\rm Pl}^{-2}(\rho_m + \rho_V)\,, \tag{2.4.27}$$

where $H := \partial_t a/a^2$, ρ_m is the energy density of matter, and

$$\frac{\rho_V}{m^2M_{\rm Pl}^2} = \bar{f}(\omega)\,. \tag{2.4.28}$$

Eq. (2.4.2) yields the equation of motion for $\tilde{a}$ as

$$\frac{3}{a^2}\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = \frac{m^2}{\kappa}\left(\frac{c_1}{\omega} + 6c_2 + 18c_3\omega + 24c_4\omega^2\right)\,. \tag{2.4.29}$$

Eqs. (2.4.29) and (2.4.26) yields an algebraic relation between ρ_m and ω:

$$\begin{aligned}\frac{\rho_m}{m^2 M_{\rm Pl}^2} &= \frac{c_1}{\kappa\omega} + \left(\frac{6c_2}{\kappa} - c_0\right) + \left(\frac{18c_3}{\kappa} - 3c_1\right)\omega + \left(\frac{24c_4}{\kappa} - 6c_2\right)\omega^2 - 6c_3\omega^3 \\ &= -\bar{f}(\omega) + \frac{1}{4\kappa\omega} f'(\omega) =: F(\omega)\,. \end{aligned} \tag{2.4.30}$$

As is shown in Appendix. B, when the massive graviton's mass squared $m_{\rm eff}^2$ given in Eq. (2.4.15) is less than $2H^2$, the helicity-0 mode of the massive graviton becomes a ghost, so-called Higuchi ghost [62]. Eq. (2.4.15) shows that $m_{\rm eff}^2$ is positive when $\Gamma(\omega) > 0$ and that $m_{\rm eff}^2 - 2H^2$ is simplified using $F(\omega)$ in Eq. (2.4.30) as

$$m_{\rm eff}^2 - 2H^2 = -\frac{m^2\omega}{3} F'(\omega)\,. \tag{2.4.31}$$

This implies that the solution is free from Higuchi ghost when $F'(\omega)$ is negative. Let us tune the cosmological constant Λ so as to possess a vacuum Minkowski solution with $\omega = \omega_0$ and $\rho_m(\omega_0) = 0$, and consider a branch of the solution which connect to this Minkowski solution with $F'(\omega) < 0$. Increasing the energy density ρ_m from the Minkowski solution, de Sitter solution cease to exist at the point where $F'(\omega) = 0$. Up to this energy density, $F'(\omega)$ remains to be negative as long as $m_{\rm eff}^2 > 0$ is satisfied for $\omega = \omega_0$, and so Eq. (2.4.31) indicates that $m_{\rm eff}^2 - 2H^2$ is positive on the branch we consider. Therefore, the de Sitter solution constructed in this way is free from Higuchi ghost as long as the graviton's mass squared is positive in the Minkowski limit. On the branch where $F'(\omega) > 0$, on the other hand, the graviton's mass squared $m_{\rm eff}^2$ is less than $2H^2$ and the Higuchi ghost appears.

Chapter 3

Braneworld model

In this chapter, we will briefly review the braneworld model, *i.e.*, the higher-dimensional gravitational theory with 4-dimensional boundary(ies), so-called brane(s). Here we refer to Ref. [46]. The braneworld model is a higher-dimensional gravitational model inspired by string theory and its predictions: higher dimensional spacetime and D-brane. In this model, the matter fields described by the standard model are localized on the 4-dimensional brane(s) and only the graviton propagates to the spatial extra dimension. This leak of gravity to the extra dimension is expected to explain the hierarchy problem between the Planck mass and the other mass scales in the standard model consistently with the experimental support to the standard model [47]. As we do not currently observe the extra dimension, the extra dimension should be compactified on a so small scale that we cannot observe. Now we consider $4+d$-dimensional Einstein theory. In the weak field limit, the gravitational potential is given by solving the $4+d$-dimensional Poisson equation as

$$U(r) \propto \frac{1}{M_{4+d}^{2+d}\, r^{1+d}}\,, \tag{3.0.1}$$

where, M_{4+d} is the $4+d$-dimensional Planck mass. Setting the length scale of the compactified extra dimensions to be L, the gravitational potential becomes $4+d$-dimensional for $r \lesssim L$, *i.e.*, $U \simeq M_{4+d}^{-(2+d)} r^{-(1+d)}$, while for $r \gg L$ the gravitational potential becomes 4-dimensional as $U \simeq M_{4+d}^{-(2+d)} L^{-d} r^{-1}$ since the gradient in the extra dimensions does not contribute to the potential there. With this 4-dimensional gravitational potential for $r \gg L$, the effective 4-dimensional Planck mass is given as $M_{4+d}^{2+d} L^d$. This can explain that the 4-dimensional Planck mass is extremely larger than the other energy scales in standard model by tuning the scale of the extra dimension to be large, even when the fundamental Planck mass M^{4+d} is comparable to the other energy scales in the standard model.

Because the gravity propagates also into the extra dimensions, the graviton becomes massive in the 4-dimensional effective theory. For simplicity, we consider the perturbation around the flat 5-dimensional background spacetime with a flat brane. We compactify the extra dimension by assuming that the spacetime is identified under the transformation $y \to y + 2\pi n L$, $n = 0, 1, 2, \cdots$. Then, the amplitude of the perturbation $f(y, x^\mu)$ is

Fourier expanded as

$$f(y,\, x^\mu) = \sum_n \mathrm{e}^{iny/L} f_n(x^\mu)\,. \tag{3.0.2}$$

The 5-dimensional linearized Einstein equation gives ${}^5\Box f = 0$ with the d-dimensional d'Alembert operator ${}^d\Box$, and consequently

$${}^4\Box f_n = m_n^2 f_n\,, \qquad m_n = \frac{n}{L}\,, \tag{3.0.3}$$

which indicates that the 4-dimensional effective theory contains infinite number of massive gravitons whose mass spectrum becomes discretized as L^{-1}, $2L^{-1}$, $\cdots$. These massive gravitons are called Kaluza-Klein(KK) gravitons.

In the following, we derive the boundary condition on the brane, so-called the junction condition, assuming the Z_2 symmetry across the brane. In general, the action of 5-dimensional brane model is given as

$$S = \int d^5x\sqrt{-{}^5g}\left[\frac{M_5^3}{2}({}^5R + 12\ell_\Lambda^{-2}) + \delta(y - y_{\rm brane})L_s\right]\,, \tag{3.0.4}$$

where M_5 and ℓ_Λ^{-2} are the 5-dimensional Planck mass and the 5-dimensional cosmological constant, respectively. ${}^5g_{ab}$ and ${}^5R_{ab}$ are the 5-dimensional metric and its Ricci tensor, and L_s denotes the Lagrangian of the field localized on the brane, such as the Lagrangian of matter and the brane tension [1]. The 5-dimensional Einstein equation becomes

$${}^5R_{ab} - \frac{1}{2}{}^5R g_{ab} + \Lambda {}^5g_{ab} = \frac{2}{M_5^3}\delta(y - y_{\rm brane})S_{ab}\,, \tag{3.0.5}$$

where $y = y_{\rm brane}$ denotes the location of the brane in the bulk spacetime and S_{ab} is the contribution from the brane-localized term $S_s = \int d^5x\sqrt{-{}^5g}\,\delta(y - y_{\rm brane})L_s$ derived from the variation of S_s with respect to ${}^5g_{ab}$. In order to derive the junction condition, we choose the Gaussian normal coordinates in the vicinity of the brane where the metric is given as

$$ds^2 = dy^2 + g_{\mu\nu}dx^\mu dx^\nu\,. \tag{3.0.6}$$

Foliating the spacetime with $y =$ const. hypersurfaces in this coordinates system, the evolution equation Eq. (A.2.10) in Appendix. A gives

$${}^5R_{\mu\nu} = -\partial_y K_{\mu\nu} - KK_{\mu\nu} + 2K_\mu^\alpha K_{\alpha\nu} + R_{\mu\nu}\,, \tag{3.0.7}$$

by setting $\epsilon = 1$, $N = 1$, $N^i = 0$. Here $K_{\mu\nu}$ and $R_{\mu\nu}$ are the extrinsic curvature and the 4-dimensional Ricci tensor evaluated on each y-constant surface, respectively. We can set the brane position to be $y = 0$ without loss of generality, and then the integration of

[1] In the DGP brane model that we introduce later, L_s also contains the 4-dimensional Einstein-Hilbert term induced on the brane.

Eq. (3.0.7) along y-coordinate in the vicinity of the brane gives

$$\lim_{\delta\to 0}\int_{-\delta}^{\delta} {}^5R_{\mu\nu}dy = -\lim_{\delta\to 0}\int_{-\delta}^{\delta}\partial_y K_{\mu\nu}dy\,. \tag{3.0.8}$$

Here we assume that the metric is continuous over the whole spacetime. Using the 5-dimensional Einstein equation, Eq. (3.0.8) becomes

$$\lim_{\delta\to 0}\left(K_{\mu\nu}(\delta)-K_{\mu\nu}(-\delta)\right) = -\frac{2}{M_5^3}\left(S_{\mu\nu}-\frac{1}{3}g_{\mu\nu}S\right)\,. \tag{3.0.9}$$

Imposing the Z_2 symmetry across the brane, on the other hand, we obtain

$$\lim_{\delta\to 0}K_{\mu\nu}(\delta) := K^{(+)}_{\mu\nu} = -K^{(-)}_{\mu\nu} := -\lim_{\delta\to 0}K_{\mu\nu}(-\delta)\,, \tag{3.0.10}$$

with which Eq. (3.0.9) yields the junction condition:

$$K^{(+)}_{\mu\nu} = -\frac{1}{M_5^3}\left(S_{\mu\nu}-\frac{1}{3}g_{\mu\nu}S\right)\,. \tag{3.0.11}$$

3.1 Randall-Sundrum model

First, we review Randall-Sundrum(RS) model [22, 23], in which we introduce a negative 5-dimensional cosmological constant. Because of the negative cosmological constant, the extra dimension is effectively compactified with the AdS bulk asymptotics even without the compactification discussed above. Here we discuss RS 2-brane model (RS I model) that contains two branes with Z_2 symmetry across each brane. In RS 2-brane model, the bulk compactification is achieved by this Z_2 symmetry, but by taking the infinite brane-separation limit we can discuss RS 1-brane model (RS II model) in which the bulk is compactified by the bulk curvature effectively. The action of RS 2-brane model is given as

$$S = \int d^5x\sqrt{-{}^5g}\left[\frac{M_5^3}{2}({}^5R+12\ell_\Lambda^{-2})+\sum_{\pm}\delta(y-y_\pm)(L_m^{(\pm)}-2\tau^{(\pm)})\right]\,, \tag{3.1.1}$$

where $y_\pm$ with $y_+ < y_-$ are the positions of the respective $(\pm)$-branes, $L_m^{(\pm)}$ are the Lagrangians of matter localized on the $(\pm)$-branes, and $\tau^{(\pm)}$ are the brane tension on the $(\pm)$-branes. Assuming that both branes are Minkowski when we consider the vacuum branes with $L_m^{(\pm)} = 0$, we should add appropriate positive tension on $(+)$-brane and also appropriate negative tension on $(-)$-brane, so as to cancel out the contribution of the bulk negative cosmological constant. For simplicity, we choose the Gaussian normal coordinates and set the brane positions as $y_+ = 0$, $y_- = L$. We consider the spacetime defined in $-L < y < L$ compactified by the identification under the Z_2 transformation

$$y\to -y\,,\qquad y+L\to L-y\,. \tag{3.1.2}$$

Solving the 5-dimensional Einstein equation, we obtain the metric in $0 < y < L$ as

$$ds^2 = dy^2 + \mathrm{e}^{-2y/\ell_\Lambda} \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{3.1.3}$$

The spacetime in $-L < y < 0$ is given by the transformation $y \to -y$ of the one in $0 < y < L$. In RS 2-brane model, the junction conditions on the $(\pm)$-branes are given as

$$\mp \ell_\Lambda^{-1} = -\frac{1}{M_5^3} \left(T_{\mu\nu}^{(\pm)} - \frac{1}{3} g_{\mu\nu} (T^{(\pm)} - \tau^{(\pm)}) \right) \,, \tag{3.1.4}$$

which determine the brane tension that we should add to obtain the Minkowski branes as a vacuum solution. The effective 4-dimensional Planck mass is derived as follows: with Eq. (3.1.3), the gravitational sector of the action (3.1.1) becomes

$$\begin{aligned} \frac{M_5^3}{2} \int d^5x \sqrt{-{}^5 g} ({}^5 R + 12\ell^{-2}) &= M_5^3 \int_0^L dy \, \mathrm{e}^{-2y/\ell_\Lambda} \int d^4x \sqrt{-g} \, R \\ &= \frac{M_5^3 \ell_\Lambda}{2} \left(1 - \mathrm{e}^{-2L/\ell_\Lambda}\right) \int d^4x \sqrt{-g} \, R \,. \end{aligned} \tag{3.1.5}$$

This indicates that the effective 4-dimensional Planck mass becomes $\left(1 - \mathrm{e}^{-2L/\ell_\Lambda}\right) M_5^3 \ell_\Lambda$ on the (+)-brane, and $\left(\mathrm{e}^{2L/\ell_\Lambda} - 1\right) M_5^3 \ell_\Lambda$ on the $(-)$-brane. Hence, even when $M_5 \simeq \ell_\Lambda^{-1} \simeq$ TeV, the effective 4-dimensional Planck mass on the $(-)$-brane can recover 10^{16}TeV by tuning L to be appropriately large. In RS 1-brane mode, the effective 4-dimensional Planck mass becomes $M_5^3 \ell_\Lambda$ by taking $L \to \infty$ limit. Because the length scale of the extra dimension is given as $\mathcal{O}(\ell_\Lambda)$ in RS 1-brane model, the 4-dimensional effective theory of RS 1-brane model is shown to contain KK gravitons with the mass of $\mathcal{O}(\ell_\Lambda^{-1})$, as is discussed in Eq. (3.0.3). Therefore, RS model gives a UV modification of general relativity[2].

3.2 Dvali-Gabadadze-Porrati model

Next, we review the Dvali-Gabadadze-Porrati (DGP) model [25], which we will use to reproduce the ghost-free bigravity. The DGP model is the 5-dimensional braneworld model with the 4-dimensional gravity induced on the brane(s). In Ref. [25], they consider the DGP 1-brane model with infinitely large flat extra dimension: the action is given as

$$S = \frac{M_{\rm Pl}^2}{2} \left(\frac{1}{2r_c} \int d^5x \sqrt{-{}^5 g} \, {}^5 R + \int_{y = y_{\rm brane}} d^4x \sqrt{-g} \, R \right) + \int_{y = y_{\rm brane}} d^4x \sqrt{-g} \, \left(L_m - 2\tau \right) \,, \tag{3.2.1}$$

where we introduce the 4-dimensional Planck mass for the brane-induced gravity term $M_{\rm Pl}$ and a length parameter $r_c := M_{\rm Pl}^2 / 2M_5^3$. Again, L_m and τ are the matter Lagrangian localized on the brane and the brane tension. In the UV region with $r \lesssim r_c$, the 4-dimensional Einstein-Hilbert term dominates in Eq. (3.2.1) and the DGP model recovers

[2]Also in RS 2-brane model, there appears KK gravitons whose masses are similarly given using ℓ_Λ and L, which gives a UV modification of gravity.

general relativity. In the IR region with $r \gtrsim r_c$, on the other hand, the 5-dimensional Einstein-Hilbert term dominates and the gravitational potential becomes 5-dimensional. Therefore, the DGP model gives an IR modification of gravity, in contrast to RS model. As the gravity leaks into the bulk and becomes weakened at the length scale $r \gtrsim r_c$, the DGP model can explain the accelerating universe without introducing 4-dimensional cosmological constant by tuning r_c to the cosmological length scale, as is shown in the following: in the DGP model, the junction condition is given as

$$K_{\mu\nu}(y = y_{\rm brane}) = r_c \left[-\frac{1}{M_{\rm Pl}^2} \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu}(T - \tau) \right) + \left(G_{\mu\nu} - \frac{1}{3} G g_{\mu\nu} \right) \right], \tag{3.2.2}$$

where $G_{\mu\nu}$ is the Einstein tensor evaluated from the induced metric on the brane. Considering a vacuum de Sitter brane with the curvature H in flat 5-dim spacetime, Eq. (3.2.2) becomes

$$\pm H = r_c \left(H^2 - \frac{\tau}{3M_{\rm Pl}^2} \right). \tag{3.2.3}$$

The choice of sign $\pm$ in front of H in the l.h.s. corresponds to which part of bulk spacetime divided by the de Sitter brane we consider. The solution in the $+$ branch and the one in the $-$ branch describe the spacetime defined in $y_b < y < \infty$ and in $0 < y < y_b$, respectively, where y_b is the brane position. $+$ branch is called the self-accelerating branch, since the self-accelerating branch of Eq. (3.2.3) gives de Sitter solution $H = 1/r_c$ even when $\tau = 0$, while the other branch, so-called normal branch, does not [30]. In the self-accelerating branch, since the bulk spacetime is infinitely large and the massless mode cannot be normalized, the graviton becomes massive and is known to exhibit the Higuchi instability inevitably [31–33]. We will discuss the appearance of Higuchi instability in Chapter. 5, using the DGP 2-brane model that recovers the DGP 1-brane model in the large brane-separation limit.

If we want to explain the accelerating universe using the DGP 1-brane model with $r_c \sim H^{-1}$, it is known that the 4-dimensional effective theory on the brane becomes strongly coupled in the relatively low energy regime $\Lambda \sim (H^2 M_{\rm Pl})^{1/3}$ [49, 50]. Furthermore, the DGP 1-brane model without cosmological constant is denied by the observation of the large-scale structure [51–53].

3.3 Stabilization mechanism

Considering the braneworld model compactified by two branes, we can freely choose the distance between the branes. Therefore, the fluctuation of the brane separation behaves as a dynamical scalar degree of freedom, so-called radion. In this section, we review the stabilization mechanism with which the radion degree of freedom is erased and consequently the brane separation is stabilized.

As an explicit stabilization mechanism, we consider the scalar stabilization mechanism proposed by Goldberger and Wise [29]. This stabilization is driven by a 5-dimensional scalar field and its potential. First, we will explain how this scalar stabilization works qualitatively: in this mechanism, we need to introduce deep scalar potentials localized on the branes so as to pin down the scalar-field values on the branes. If the branes approach closer, the fixed field values yields larger gradient of the scalar field, which avoid the branes colliding because the effective energy diverges when the brane separation goes to zero. When the branes move away from each other, on the other hand, the negative tension of one of the branes becomes negligible because the bulk scale factor becomes small near the negative-tension brane. Therefore, the effective energy increases and the large brane separation is also disfavored. From the above, we can locate the branes keeping some finite distance using the scalar stabilization.

Now we will explicitly show that the scalar stabilization mechanism fixes the distance between the branes. For simplicity, we consider RS 2-brane model sandwiched by two vacuum Minkowski branes, whose metric is given as

$$ds^2 = dy^2 + a(y)^2\eta_{\mu\nu}dx^\mu dx^\nu \,. \tag{3.3.1}$$

We denote the brane position as $y = y_\pm$ for $y_+ < y_-$. The action with the stabilization scalar field ψ becomes

$$S = \frac{M_5^3}{2}\int dx^5\sqrt{-g}R + S_s, \tag{3.3.2}$$

$$S_s = \int d^5x\sqrt{-g}\left(-\frac{1}{2}g^{ab}\psi_{,a}\psi_{,b} - V_B(\psi) - \sum_{\sigma=\pm} V_{(\sigma)}(\psi)\delta(y-y_\sigma)\right), \tag{3.3.3}$$

where V_B is the bulk scalar potential and $V_{(\pm)}$ are the scalar potentials localized on the $(\pm)$-branes. Here, we consider the 5-dimensional cosmological constant and the brane tension as a part of the potential V_B, $V_{(\pm)}$. The bulk equation becomes

$$\mathcal{H}' = -\frac{1}{3M_5^3}\psi'^2 \,, \tag{3.3.4}$$

$$\mathcal{H}^2 = \frac{1}{6M_5^3}\left(\frac{1}{2}\psi'^2 - V_B\right) , \tag{3.3.5}$$

$$\psi'' + 4\mathcal{H}\psi' - \frac{\partial V_B}{\partial \psi} = 0 \,, \tag{3.3.6}$$

where " $'$ " is the partial differentiation with respect to y and we define $\mathcal{H} := a'/a$. The junction conditions give

$$\pm\mathcal{H}_\pm = -\frac{1}{6M_5^3}V_{(\pm)}(\psi_\pm) \,, \tag{3.3.7}$$

$$\psi'_\pm = \pm\frac{1}{2}\left.\frac{\partial V_{(\pm)}}{\partial\psi}\right|_{y=y_\pm} . \tag{3.3.8}$$

These equations show that, in the presence of the scalar field, the curvature $\mathcal{H}$ and the scalar field ψ are so involved in the equations that it is difficult to solve the equations explicitly. Hence, we assume that the bulk potential V_B is written in terms of a superpotential $W(\psi)$ [54] as

$$V_B(\psi) = \frac{1}{8}\left(\frac{\partial W(\psi)}{\partial \psi}\right)^2 - \frac{1}{6M_5^3}\,W(\psi)^2. \tag{3.3.9}$$

Under this assumption, Eqs. (3.3.4)-(3.3.8) become simple: the bulk equations become

$$\psi' = \frac{1}{2}\frac{\partial W(\psi)}{\partial \psi}\,, \quad \mathcal{H} = -\frac{1}{6M_5^3}\,W(\psi)\,, \tag{3.3.10}$$

and the junction conditions are given as

$$W(\psi_\pm) = \pm V_{(\pm)}(\psi_\pm)\,, \quad \left.\frac{\partial W(\psi)}{\partial \psi}\right|_{y=y_\pm} = \pm\left.\frac{\partial V_{(\pm)}(\psi)}{\partial \psi}\right|_{y=y_\pm}. \tag{3.3.11}$$

Then, Eqs. (3.3.9), (3.3.11) yield

$$\left[\frac{1}{8}\left(\frac{\partial V_{(\pm)}}{\partial \psi}\right)^2 - \frac{1}{6M_5^3}\,V_{(\pm)}^2\right]_{y=y_+} = V_B(\psi_+)\,, \tag{3.3.12}$$

$$\left[\frac{1}{8}\left(\frac{\partial V_{(\pm)}}{\partial \psi}\right)^2 - \frac{1}{6M_5^3}\,V_{(\pm)}^2\right]_{y=y_-} = V_B(\psi_-)\,, \tag{3.3.13}$$

which allow only a discrete set of solutions for $\psi_\pm$ for given W. These discrete values of $\psi_\pm$ correspond to the brane positions at which flat branes can consistently locate. Therefore, we can freely determine the brane position by tuning the potentials of the stabilization scalar field.

In order to obtain the consistent solution that satisfies all the junction conditions, on the other hand, some tuning for V_B, $V_{(\pm)}$ is necessary. The bulk equations Eqs. (3.3.5), (3.3.6) contain three free parameters: the two integration constants $\psi(y_+)$ and $\psi'(y_+)$, and the brane separation $L := y_- - y_+$. Whilst, the junction conditions (3.3.7) and (3.3.8) give four constraints. Therefore, the system is overdetermined and in need of one fine-tuning. Now, we discuss how to fine-tune using the model written in terms of the superpotential $W(\psi)$ as in Eq. (3.3.9). First, when V_B is given, Eq. (3.3.9) determines $W(\psi)$ including a integration constant. Let us determine this integration constant of $W(\psi)$ as $W(\psi_+) = V_{(+)}(\psi_+)$ so that $W(\psi)$ satisfies one of the junction conditions on the $(+)$-brane in Eq. (3.3.11). Determining ψ_+ by the other junction condition on the $(+)$-brane, we can completely obtain $\mathcal{H}$ and ψ from Eq. (3.3.10). Now, one free parameter L remains, while we should impose still two more junction conditions, *i.e.*, the ones on the $(-)$-brane in Eq. (3.3.11). This extra constraint requests that the potential $-V_{(-)}(\psi)$ should be tuned to contact with the superpotential $W(\psi)$ at a point, namely, we need to tune the brane tension.

Chapter 4

The mass spectrum of DGP 2-brane model

Before trying to obtain the ghost-free bigravity as a low-energy effective theory, we should confirm whether only two of the gravitons' masses are kept at low energies in the parameter region discussed in Chapter. 1 where the brane-separation scale y_0 is much smaller than the model parameters of the DGP 2-brane model $r_c^{(\pm)}$. We need to introduce a stabilization scalar field and to kill the radion degree of freedom to obtain the ghost-free bigravity as the low-energy effective theory of the DGP 2-brane model. In order to obtain the mass spectrum, in this chapter, we investigate the perturbation around the background spacetime with two de Sitter branes. This chapter is based on Ref. [26].

4.1 Model and basic equations

The action of the DGP 2-brane model with Goldberger-Wise stabilization mechanism is given as

$$S = S_b + S_+ + S_- + S_s, \tag{4.1.1}$$

where

$$\begin{aligned}
S_b &:= \frac{M_5^3}{2}\int dx^5 \sqrt{-{}^5g}\,{}^5R\,, \\
S_+ &:= \int d^4x\sqrt{-g_+}\left(\frac{M_{\rm Pl}^2}{2}\,R_{(+)} + L_m^{(+)}\right)\,, \\
S_- &:= \int d^4x\sqrt{-g_-}\left(\frac{\chi M_{\rm Pl}^2}{2}\,R_{(-)} + L_m^{(-)}\right)\,, \\
S_s &:= \int d^5x\sqrt{-{}^5g}\left(-\frac{1}{2}{}^5g^{ab}\psi_{,a}\psi_{,b} - V_B(\psi) - \sum_{\sigma=\pm} V_{(\sigma)}(\psi)\delta(y-y_\sigma)\right)\,,
\end{aligned} \tag{4.1.2}$$

where ${}^5g_{\mu\nu}$, ${}^5R$, $g_{\mu\nu}^{(\pm)}$, $R_{(\pm)}$ are the 5-dimensional metric, the 5-dimensional Ricci scalar evaluated from ${}^5g_{\mu\nu}$, the 4-dimensional metrics induced on the $(\pm)$-branes and their 4-dimensional Ricci scalar, respectively. M_5, $M_{\rm Pl}$ and $\sqrt{\chi}M_{\rm Pl}$ are the 5-dimensional Planck mass, the 4-dimensional Planck masses for the (+)-brane and for the (−)-brane, respectively. Also, we introduced $L_m^{(\pm)}$, the Lagrangians of the matter fields localized on the branes. We leave the potential form of the stabilization scalar ψ unspecified here and will discuss later.

As in the brane models discussed in Chapter. 3, we impose the Z_2 symmetry across each brane. Then, the respective junction conditions on the (±)-branes become

$$\pm K^{(\pm)}_{\mu\nu} = -\frac{1}{2M_5^3}\left(T^{(\pm)}_{\mu\nu} - \frac{1}{3}T^{(\pm)}g^{(\pm)}_{\mu\nu} - \frac{1}{3}V_{(\pm)}(\psi_\pm)g^{(\pm)}_{\mu\nu}\right) + r_c^{(\pm)}\left(G^{(\pm)}_{\mu\nu} - \frac{1}{3}G^{(\pm)}g^{(\pm)}_{\mu\nu}\right), \tag{4.1.3}$$

where $K^{(\pm)}_{\mu\nu}$, $G^{(\pm)}_{\mu\nu}$ and $T^{(\pm)}_{\mu\nu}$ are the extrinsic curvatures on the branes, Einstein tensors evaluated from $g^{(\pm}_{\mu\nu}$, and the energy momentum tensors of the localized matter fields, respectively, and we define $r_c^{(\pm)}$ as

$$r_c^{(+)} := \frac{M_{\rm Pl}^2}{2M_5^3}, \qquad r_c^{(-)} := \frac{\chi M_{\rm Pl}^2}{2M_5^3}. \tag{4.1.4}$$

4.1.1 Background spacetime

As the background spacetime, we consider a warped bulk spacetime sandwiched by two 4-dimensional de Sitter branes labeled by (±): we put a metric ansatz

$$ds^2 = dy^2 + a^2(y)\gamma_{\mu\nu}dx^\mu dx^\nu, \tag{4.1.5}$$

where the 4-dimensional metric $\gamma_{\mu\nu}$ describes de Sitter spacetime with its curvature scale H^{-1}. In the bulk, we obtain the equations of motion for the scale factor $a(y)$ and the scalar filed $\psi(y)$ as

$$\mathcal{H}^2 := \left(\frac{a'}{a}\right)^2 = \frac{1}{6M_5^3}\left(\frac{1}{2}\psi'^2 - V_B\right) + \frac{H^2}{a^2}, \tag{4.1.6}$$

$$\mathcal{H}' = -\frac{1}{3M_5^3}\psi'^2 - \frac{H^2}{a^2}, \tag{4.1.7}$$

$$\psi'' + 4H\psi' - \frac{\partial V_B}{\partial\psi} = 0, \tag{4.1.8}$$

where Ψ' for an arbitrary scalar Ψ denotes the partial differentiation of Ψ with respect to y. The background spacetime is obtained by solving Eqs. (4.1.6)-(4.1.8), whose boundary conditions on the (±)-branes are given by the following junction conditions. Setting the y-coordinate value where two branes locate as $y = y_\pm$ and $y_+ < y_-$, the juction conditions (4.1.3) yield

$$\pm\mathcal{H}_\pm = r_c^{(\pm)}\frac{H^2}{a^2} - \frac{1}{6M_5^3}V_{(\pm)}(\psi_\pm), \tag{4.1.9}$$

and also a junction condition for the y-derivative of the 5-dimensional scalar field is obtained in the same way as for the extrinsic curvature:

$$\psi'_{\pm} = \pm \frac{1}{2} \frac{\partial V_{(\pm)}}{\partial \psi}\bigg|_{y=y_{\pm}} . \tag{4.1.10}$$

As is discussed in Sec. 3.3, the equations of motion (4.1.6)-(4.1.8) with the junction conditions (4.1.9), (4.1.10) overdetermine $\mathcal{H}$ and ψ, and hence the tuning of V_B, $V_{(\pm)}$ is necessary to obtain the solution. We will discuss this issue and show how we tune the potentials later, in Sec. 4.2.3.

4.1.2 Perturbations

Next, we consider a linear perturbation around the background spacetime given in the previous subsection. Here, we follow the notations in Ref. [34]. We choose the Newton gauge where the longitudinal spin-0 mode and the shift vector are set to zero when we foliate the 5-dimensional spacetime along y-direction. Using the traceless part and $\{y\mu\}$-component of Einstein equation, the metric perturbation h_{ab} and the scalar-field perturbation $\delta\psi$ are written in terms of a transverse-traceless spin-2 mode $h_{\mu\nu}^{(TT)}$ and spin-0 mode ϕ as

$$h_{yy} = 2\phi , \tag{4.1.11}$$
$$h_{y\mu} = 0 , \tag{4.1.12}$$
$$h_{\mu\nu} = h_{\mu\nu}^{(TT)} - \phi\tilde{\gamma}_{\mu\nu} , \tag{4.1.13}$$
$$\delta\psi = \frac{3M_5^3}{2\psi'}[\partial_y + 2\mathcal{H}]\phi , \tag{4.1.14}$$

The $\{\mu\nu\}$-component of Einstein equation gives the bulk equation for the spin-2 mode as

$$\left[\hat{L}^{(TT)} + \frac{1}{a^2}({}^4\Box - 2H^2)\right] h_{\mu\nu}^{(TT)} = 0 , \tag{4.1.15}$$
$$\hat{L}^{(TT)} := \frac{1}{a^2}\partial_y a^4 \partial_y \frac{1}{a^2} , \tag{4.1.16}$$

where we define ${}^4\Box = \gamma^{\mu\nu}\nabla_\mu\nabla_\nu$ for ∇_μ as the covariant differentiation associated with $\tilde{\gamma}_{\mu\nu} := a^2(y)\gamma_{\mu\nu}$. The bulk equation for the spin-0 mode becomes

$$\left[\hat{L}^{(\phi)} + \frac{{}^4\Box + 4H^2}{\psi'^2}\right]\phi = 0 , \tag{4.1.17}$$

where

$$\hat{L}^{(\phi)} := a^2\partial_y \frac{1}{a^2\psi'^2}\partial_y a^2 - \frac{2}{3M_5^3}a^2 . \tag{4.1.18}$$

Next, we derive the junction conditions for the perturbed variables $h_{\mu\nu}^{(TT)}$, ϕ. For this purpose, as discussed in Chapter. 3, it is convenient to use the Gaussian normal coordinates in which the lapse function and shift vector are set to be unity and zero, respectively, and the locations of the branes are not perturbed. Here, we denotes the variables in the Gaussian normal coordinates as the ones with bar, $\bar{h}^{(\pm)}_{\mu\nu}$ and $\delta\bar{\psi}$. As there exist two branes, we have two Gaussian normal coordinates for each brane. Hence, we add the subscript $(\pm)$ to distinguish them. In these coordinates, the junction conditions (4.1.3) for the perturbed metric become

$$\pm(\partial_y - 2\mathcal{H})\bar{h}^{(\pm)}_{\mu\nu} = -\frac{1}{M_5^3}\left[T^{(\pm)}_{\mu\nu} - \frac{1}{3}T^{(\pm)}\tilde{\gamma}_{\mu\nu}\right] \mp \frac{2}{3M_5^3}\tilde{\gamma}_{\mu\nu}\psi'\delta\psi + 2r_c^{(\pm)}\left[X^{(\pm)}_{\mu\nu} - \frac{1}{3}X^{(\pm)}\tilde{\gamma}_{\mu\nu}\right], \tag{4.1.19}$$

where $X^{(\pm)}_{\mu\nu}$ are the linearized induced Einstein tensors $G^{(\pm)}_{\mu\nu}$ defined as

$$\begin{aligned} X^{(\pm)}_{\mu\nu} :=& -\frac{1}{2}\left(\frac{1}{a^2}{}^4\Box\bar{h}^{(\pm)}_{\mu\nu} - \nabla_\mu\nabla_\alpha\bar{h}^{(\pm)\alpha}_{\nu} - \nabla_\nu\nabla_\alpha\bar{h}^{(\pm)\alpha}_{\mu} + \nabla_\mu\nabla_\nu\bar{h}^{(\pm)}\right) \\ & -\frac{1}{2}\tilde{\gamma}_{\mu\nu}\left(\nabla_\alpha\nabla_\beta\bar{h}^{\alpha\beta}_{(\pm)} - \frac{1}{a^2}{}^4\Box\bar{h}^{(\pm)}\right) + \frac{H^2}{a^2}\left(\bar{h}^{(\pm)}_{\mu\nu} + \frac{1}{2}\tilde{\gamma}_{\mu\nu}\bar{h}^{(\pm)}\right), \end{aligned} \tag{4.1.20}$$

where we use $\tilde{\gamma}_{\mu\nu}$ to raise the tensor indices. The junction conditions for the bulk scalar field are obtained as

$$\pm 2\delta\bar{\psi}' = V''^{(\pm)}(\psi)\delta\bar{\psi}\,. \tag{4.1.21}$$

We will rewrite the junction conditions (4.1.19) and (4.1.21) in terms of the variables in the Newton gauge: $h^{(TT)}_{\mu\nu}$ and ϕ. We denote the perturbed brane positions in Newton gauge with the brane bending modes $\hat{\xi}^y_{(\pm)}$ as $y|_{\text{brane}} = y_\pm + \hat{\xi}^y_{(\pm)}(x^\mu)$. Then, the generators of the transformation form the Gaussian normal coordinates to the Newton gauge, $\xi^a_{(\pm)}$, are written as

$$\xi^y_{(\pm)} = \int_{y_\pm}^y \phi(y')dy' + \hat{\xi}^y_{(\pm)}(x^\mu)\,, \tag{4.1.22}$$

$$\xi^\nu_{(\pm)} = -\int_{y_\pm}^y \tilde{\gamma}^{\mu\nu}(y')\left[\int_{y_\pm}^{y'} \phi_{,\mu}(y'')dy'' + \hat{\xi}^y_{(\pm),\mu}(x^\rho)\right]dy' + \hat{\xi}^\nu_{(\pm)}(x^\rho)\,. \tag{4.1.23}$$

Using $\xi^a_{(\pm)}$, the perturbed variables are transformed as

$$\bar{h}^{(\pm)}_{\mu\nu} = h_{\mu\nu} - 2\nabla_{(\mu}\xi^{(\pm)}_{\nu)} - 2\mathcal{H}\tilde{\gamma}_{\mu\nu}\xi^y_{(\pm)}\,, \tag{4.1.24}$$

$$\delta\bar{\psi}^\pm = \delta\psi^\pm - \psi'\xi^y_{(\pm)}\,. \tag{4.1.25}$$

Using the above relations, we obtain the junction conditions for $h^{(TT)}_{\mu\nu}$ from the traceless part of (4.1.19):

$$\pm(\partial_y - 2\mathcal{H}_\pm)h^{(TT)}_{\mu\nu} = -\frac{1}{M_5^3}\Sigma^{(\pm)}_{\mu\nu} - r_c^{(\pm)}a_\pm^{-2}({}^4\Box - 2H^2)h^{(TT)}_{\mu\nu}\,, \tag{4.1.26}$$

where we define $\Sigma^{(\pm)}_{\mu\nu}$ and $Z_{(\pm)}$ as

$$\Sigma^{(\pm)}_{\mu\nu} := \left(T^{(\pm)}_{\mu\nu} - \frac{1}{4}T^{(\pm)}\tilde{\gamma}^{\pm}_{\mu\nu}\right) \pm 2M_5^3\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right)Z_{(\pm)}\,, \tag{4.1.27}$$

$$Z_{(\pm)} := (1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm)\hat{\xi}^y_\pm \mp r_c^{(\pm)}\phi(y_\pm)\,. \tag{4.1.28}$$

The trace part of Eq. (4.1.19) gives the equations that determine $Z_\pm$ as

$$a_\pm^{-2}({}^4\Box + 4H^2)Z_{(\pm)} = \pm\frac{1}{6M_5^3}\,T^{(\pm)}\,. \tag{4.1.29}$$

The junction conditions for the bulk scalar field Eq. (4.1.21) become

$$\mp\frac{2}{3M_5^3}\left(\delta\psi - \psi'(1 \mp 2r_c^{(\pm)}\mathcal{H})^{-1}(Z_{(\pm)} \pm r_c^{(\pm)}\phi)\right) = \frac{\epsilon^{(\pm)}}{a^2\psi'}({}^4\Box + 4H^2)\phi\,, \tag{4.1.30}$$

where we define

$$\epsilon^{(\pm)} := \frac{2}{V''^{(\pm)} \mp 2\psi''/\psi'}\,. \tag{4.1.31}$$

Finally, we combine the bulk equations and the junction conditions to derive the equation valid in the whole spacetime. Eqs. (4.1.15), (4.1.26) yield the equation for the spin-2 mode as

$$\left[\hat{L}^{(TT)} + \frac{{}^4\Box - 2H^2}{a^2}\right]h^{(TT)}_{\mu\nu} = \sum_{\sigma=\pm}\left(-\frac{2}{M_5^3}\Sigma^{(\sigma)}_{\mu\nu} - 2r_c^{(\sigma)}a_\sigma^{-2}({}^4\Box - 2H^2)h^{(TT)}_{\mu\nu}\right)\delta(y - y_\sigma)\,. \tag{4.1.32}$$

Eqs. (4.1.17), (4.1.30) are combined to the equation for the spin-0 mode:

$$\begin{aligned}&\left[\hat{L}^{(\phi)} + \frac{{}^4\Box + 4H^2}{\psi'^2}\right]\phi \\ &\quad = \sum_{\sigma=\pm}\left(\frac{4a^2}{3M_5^3(\sigma 1 - 2r_c^{(\sigma)}\mathcal{H}_\sigma)}(Z_{(\sigma)} + \sigma r_c^{(\sigma)}\phi) - \frac{2\epsilon^{(\sigma)}}{\psi'^2}({}^4\Box + 4H^2)\phi\right)\delta(y - y_\sigma)\,.\end{aligned} \tag{4.1.33}$$

4.2 Mass spectrum

In this section, we will confirm that it is possible to archive the hierarchy among the gravitons' masses where only two masses are kept at low energies. For simplicity, we set $y_+ = -y_0$, $y_- = y_0$, $a(y = 0) = 1$.

In order to investigate the mass spectrum in the 4-dimensional effective theory, we set the source terms $\Sigma^{(\pm)}_{\mu\nu}$ and $Z_{(\pm)}$ in Eqs. (4.1.32) and (4.1.33) to zero, and separate the variables $h_{\mu\nu}^{(TT)}$ and ϕ into the y-dependent part and the x^μ-dependent part for each:

$$h_{\mu\nu}^{(TT)} = \sum_j h_{\mu\nu}^{(j)}(x^\rho)u_j(y)\,, \tag{4.2.1}$$

$$\phi = \sum_j \phi^{(j)}(x^\mu)v_j(y)\,. \tag{4.2.2}$$

Let us start with the spin-2 mode. Replacing the operator ${}^4\Box - 2H^2$ with the mass eigenvalue m_i^2 in Eq. (4.1.32), we obtain the equation that the corresponding eigenfucntion $u_i(y)$ obeys as

$$\left[\frac{m_i^2}{a^2}\left(1 + 2\sum_{\sigma=\pm} r_c^{(\sigma)}\delta(y - y(\sigma))\right)\right] u_i(y) = -\hat{L}^{(TT)}u_i(y)\,. \tag{4.2.3}$$

We define the inner product of the eigenfuctions so that the eigenfunctions with different eigenvalues become orthogonal as

$$(u_i, u_j)^{(TT)} := \oint \frac{dy}{a^2}\left(1 + 2\sum_{\sigma=\pm} r_c^{(\sigma)}\delta(y - y_\sigma)\right) u_i(y)u_j(y) = \delta_{ij}\,, \tag{4.2.4}$$

where the integration is performed over the whole compactified extra dimension $-2y_0 < y \le 2y_0$. By definiton, the norm $(u_i, u_i)^{(TT)}$ is always positive. Assuming that only the $(+)$-brane has the source $\Sigma_{\mu\nu}^{(+)}$, we rewrite Eq. (4.1.32) using the eigenfunctions $u_i(y)$ as

$$a^{-2}\sum_j({}^4\Box - 2H^2 - m_j^2)\left(1 + \sum_{\sigma=\pm} 2r_c^{(\sigma)}\delta(y - y_\sigma)\right) h_{\mu\nu}^{(j)}(x^\mu)u_j(y) = -\frac{2}{M_5^3}\Sigma_{\mu\nu}^{(+)}\delta(y - y_+)\,. \tag{4.2.5}$$

This equation reduces to

$$({}^4\Box - 2H^2 - m_i^2)h_{\mu\nu}^{(i)}(x^\mu) = -\frac{2}{M_5^3}\Sigma_{\mu\nu}^{(+)}u_i(y_+)\,, \tag{4.2.6}$$

by integrating both sides of Eq. (4.2.5) multiplied by $u_i(y)$ over $-2y_0 < y \le 2y_0$. This gives the solution for $h_{\mu\nu}^{(TT)}$ as

$$h_{\mu\nu}^{(TT)}(y) = -\frac{2}{M_5^3}\sum_i \frac{u_i(y_+)u_i(y)}{{}^4\Box - 2H^2 - m_i^2}\Sigma_{\mu\nu}^{(+)}\,. \tag{4.2.7}$$

As in the case with spin-2 mode, we obtain the solution for ϕ from Eq. (4.1.33). The eigenvalue problem for the eigenfunction $v_i(y)$ becomes

$$\frac{\mu_i^2+4H^2}{\psi'^2}\left(1+\sum_{\sigma=\pm}2\epsilon^{(\sigma)}\delta(y-y(\sigma))\right)v_i(y)$$
$$=\left[-\hat{L}^{(\phi)}+\sum_{\sigma=\pm}\frac{4r_c^{(\sigma)}a^2}{3M_5^3(1-\sigma 2r_c^{(\sigma)}\mathcal{H}_\sigma)}\delta(y-y(\sigma))\right]v_i(y)\,, \tag{4.2.8}$$

by replacing the operator ${}^4\Box$ with the corresponding mass eigenvalue μ_i^2. Again, we define the inner product of v_i:

$$(v_i,v_j)^{(\phi)}:=\oint\frac{dy}{\psi'^2}\left(1+\sum_{\sigma=\pm}2\epsilon^{(\pm)}\delta(y-y_\sigma)\right)v_i(y)v_j(y)=\delta_{ij}\,. \tag{4.2.9}$$

In order to guarantee the positiveness of the norm of v_i, we assume $\epsilon^{(\pm)}>0$. This assumption is easily realized by tuning the bulk-scalar potentials so that both $V''_{(+)}$ and $V''_{(-)}$ take the sufficiently large value. Then, we can obtain the solution for ϕ of Eq. (4.1.33) using $v_i(y)$ as

$$\phi(y)=\frac{4a_+^2}{3M_5^3}(1-2r_c\mathcal{H}_+)^{-1}\sum_i\frac{v_i(y_+)v_i(y)}{{}^4\Box-\mu_i^2}Z_{(+)}\,, \tag{4.2.10}$$

where we assume that only the (+)-brane has the source term $Z_{(+)}$.

4.2.1 spin-2 mode

First, we study the mass spectrum of the spin-2 mode. In the bulk, the eigenfuction u_i obeys

$$\hat{L}^{(TT)}u_i=-a^{-2}m_i^2u_i\,, \tag{4.2.11}$$

with the boundary conditions

$$\pm(\partial_y-2\mathcal{H}_\pm)u_i=-r_c^{(\pm)}a^{-2}m_i^2u_i\,. \tag{4.2.12}$$

From Eqs. (4.2.11) and (4.2.12), we immediately obtain the solution for the massless mode as

$$u_0=C_0a^2\,, \tag{4.2.13}$$

where we introduce a normalization constant C_0 so that u_0 satisfies Eq. (4.2.4). As is discussed in Chapter. 1, we want to investigate the mass spectrum in the low-energy regime where the mass eigenvalues are much smaller than $\mathcal{O}(y_0^{-1})$. Hence, we consider the lowest massive KK mode with $m_1\ll y_0^{-1}$. Furthermore, we assume that $\mathcal{H}$ is small enough to satisfy $a(y)\sim\mathcal{O}(1)$. As we will show later in Sec. 4.2.2, this assumption is necessary to avoid the instability of the spin-0 mode, not just simplifying the analysis here. We introduce a dimensionless coordinate $Y = y/y_0$, then Eqs. (4.2.11) and (4.2.12) for u_1 become

$$\frac{1}{a^2}\partial_Y \left[a^4 \partial_Y \left(\frac{1}{a^2} u_1\right)\right] = -\frac{(m_1 y_0)^2}{a^2} u_1 \,, \tag{4.2.14}$$

$$\pm \left(\partial_Y - 2\frac{\partial_Y a}{a}\right) u_1 = -\frac{m_1^2 r_c^{(\pm)} y_0}{a^2} u_1 \,, \qquad \text{at} \;\; y = y_\pm \,, \tag{4.2.15}$$

Using $m_1^2 y_0^2 \ll 1$, we can ignore the right-hand side of Eq. (4.2.14) and obtain the approximate solution in the bulk as

$$u_1^{(0)} \propto a^2 \int^Y a^{-4} dY' \,. \tag{4.2.16}$$

The juction conditions (4.2.15) determine the mass eigenvalue and an integration constant in u_1: m_1 becomes

$$m_1^2 = \frac{1}{y_0 \int_{-1}^{1} a^{-4} dY} \left(\frac{1}{r_c^{(+)} a_+^{-2}} + \frac{1}{r_c^{(-)} a_-^{-2}}\right) \,, \tag{4.2.17}$$

and u_1 is

$$u_1^{(0)} = C_1 a^2 \left(1 - m_1^2 r_c^{(+)} y_0 \int_{-1}^{Y} a^{-4} dY\right) \,, \tag{4.2.18}$$

at the leading order of $m_1 y_0$. The normalization constant C_1 is to be determined by Eq. (4.2.4). Since the Wronskian of the solution for Eq. (4.2.3) is non-zero, the eigenvalues m_i do not degenerate and m_1 in Eq. (4.2.17) is the unique eigenvalue that satisfies $m_i^2 \ll y_0^{-2}$. $u_1(y)$ in Eq. (4.2.18) has one node, which confirms that this corresponds to the first KK mode, as is proved in general Sturm-Liouville eigenvalue problem (Sturm's theorem [55]). When both of $r_c^{(\pm)}$ are much larger than y_0, Eq. (4.2.17) yields

$$m_1^2 \simeq \frac{1}{2 r_c^{(\pm)} y_0} \ll \frac{1}{y_0^2} \,, \tag{4.2.19}$$

where we used $a(y) \simeq 1$. Therefore, the DGP 2-brane model with $y_0 \ll r_c^{(\pm)}$ realizes the hierarchy between the mass of the lowest massive KK mode m_1 and the ones of the other KK modes which is at least comparable to y_0^{-1}.

4.2.2 spin-0 mode

Next, we estimate the lowest mass eigenvalue for the spin-0 mode. Here, we assume $H \simeq 0$, for simplicity. If we do not introduce the stabilization mechanism, there should exist the radion, the massless scalar degree of freedom corresponing to the choice of the brane separation. In fact, Eq. (4.2.8) has a solution with zero eigenvalue for $\psi' = 0$.

The correpoding eigenfunction is given as $v_0 \propto a^{-2}$ and has no node, and hence Sturm's theorem guarantees that this mode has the lowest eigenvalue [1]. First, we will start with the case where the back reaction to the background spacetime by the stabilization scalar field is small, namely, the background spacetime satisfies

$$\frac{|\mathcal{H}'|}{\mathcal{H}^2} = \frac{\psi'^2}{3M_5^3\mathcal{H}^2} \ll 1\,, \tag{4.2.20}$$

so that we can obtain the lowest mass eigenvalue perturbatively. Under this weak back reaction approximation, we can treat the terms that are not enhanced by a factor ψ'^{-2} in the square brackets on the right-hand side of Eq. (4.2.8) as perturbation. Since the zeroth order of the weak back reaction approximation gives the zero eigenvalue, we obtain the leading order correction of the eigenvalue from Eq. (4.2.8) as

$$\mu^2 \approx \frac{2\displaystyle\int_{y_+}^{y_-}\frac{dy}{a^2} + \sum_\sigma \frac{2r_c^{(\sigma)}}{a_\sigma^2}\frac{1}{1-\sigma 2r_c^{(\sigma)}\mathcal{H}_\sigma}}{\displaystyle\int_{y_+}^{y_-}\frac{dy}{a^4(-\mathcal{H}')} + \sum_\sigma \frac{\epsilon^{(\sigma)}}{a_\sigma^4(-\mathcal{H}'_\sigma)}}\,. \tag{4.2.21}$$

In the following, we set $\epsilon^{(\pm)}$ to be zero for simplicity. Under the approximation (4.2.20), we can consider $\mathcal{H} \approx$ const. and $a \approx \mathrm{e}^{\mathcal{H}y}$ in Eq. (4.2.21), which simplifies μ^2 as

$$\mu^2 \approx \frac{1}{\mathcal{H}}\left[\frac{e^{2\mathcal{H}y_0}}{1-2r_c^{(+)}\mathcal{H}} - \frac{e^{-2\mathcal{H}y_0}}{1+2r_c^{(-)}\mathcal{H}}\right] \Big/ \int_{y_+}^{y_-}\frac{dy}{a^4(-\mathcal{H}')}\,. \tag{4.2.22}$$

The positiveness of μ^2 requires $1+2r_c^{(-)}\mathcal{H}_- > 0$ for negative $\mathcal{H}$ and $1-2r_c^{(+)}\mathcal{H}_+ > 0$ for positive $\mathcal{H}$. In both cases with $1\mp 2r_c^{(\pm)}\mathcal{H}_\pm < 0$, μ^2 becomes negative. When $1\mp 2r_c^{(\pm)}\mathcal{H}_\pm$ crosses zero, μ^2 diverges and the above perturbative treatment breaks down. We can understand the meaning of the critical situation $1\mp 2r_c^{(\pm)}\mathcal{H}_\pm = 0$ as follows. Combining the background bulk equation (4.1.7) evaluated at $y = y_\pm$ and the background junction conditions (4.1.9), we obtain a quadratic equation for $\mathcal{H}_\pm$ as

$$\mathcal{H}_\pm^2 \pm \frac{1}{r_c^{(\pm)}}\mathcal{H}_\pm + \frac{1}{6M_5^3}\left(-\frac{1}{2}\psi_\pm'^2 + V_B(\psi_\pm) + \frac{1}{r_c^{(\pm)}}V_{(\pm)}(\psi_\pm)\right) = 0\,. \tag{4.2.23}$$

This gives the solution for $\mathcal{H}_+$ as

$$2r_c^{(+)}\mathcal{H}_+ - 1 = \pm\sqrt{1-\frac{2r_c^{(+)}}{3M_5^3}\bar{V}_{(+)}}\,, \tag{4.2.24}$$

and the one for $\mathcal{H}_-$ as

$$2r_c^{(-)}\mathcal{H}_- + 1 = \pm\sqrt{1-\frac{2r_c^{(-)}}{3M_5^3}\bar{V}_{(-)}}\,, \tag{4.2.25}$$

[1] Sturm's theorem can be applied only for the case with positive $\epsilon^{(\pm)}$.

where we introduce $\bar{V}_{(\pm)} := -\frac{1}{2}\psi'^2_\pm + V_B(\psi_\pm) + \frac{1}{r_c^{(\pm)}} V_{(\pm)}(\psi_\pm)$. With $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm \neq 0$, each of $\mathcal{H}_\pm$ has two different solutions for given $\bar{V}_{(\pm)}$, which corresponds two branches: the self-accelerating branch and the normal branch. We can choose the self-accelerating or normal branch by taking the corresponding signatures on the right-hand side of Eqs. (4.2.24) and (4.2.25). However, two branches degenerate when $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm = 0$, which implies that the condition $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm = 0$ defines the boundary between two branches. Furthermore, Eq. (4.1.28) indicates that at the critical point $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm = 0$ the sign of brane bending $\hat{\xi}^y_{(\pm)}$ becomes indefinite. From Eq. (4.1.29), this perplexity about the branch choice for $1 - 2r_c^{(+)}\mathcal{H}_+ = 0$ $(1 + 2r_c^{(-)}\mathcal{H}_- = 0)$ yields the strong coupling of the spin-0 mode to the source on the (+)-brane ((−)-brane). Here, we define that the solution belongs to the normal branch when the both conditions $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$ are satisfied. In Sec. 5.2, we will show that the accelerating branch where either of $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm$ is negative inevitably suffers from ghost instabilities, while the normal branch is free from ghosts. This implies that $|\mathcal{H}|$ need to be small to avoid the instabilities, which is consistent with the assumption that we used to estimate m_1^2 in Sec. 4.2.1.

In order to obtain bigravity as an low-energy effective theory, we must tune the lowest mass of the spin-0 mode to be much larger than m_1 in Eq. (4.2.19). Eq. (4.2.22) suggests that, for sufficiently large $|\mathcal{H}'/\mathcal{H}^2|$, we can obtain large μ^2 keeping $r_c^{(\pm)}\mathcal{H}_\pm$ small so as to avoid instabilities in the self-accelerating branch. Using the condition $y_0/r_c^{(\pm)} \ll 1$ that comes from $m_1^2 \ll m_{i\geq 2}^2$, the first term of the numerator of μ^2 in Eq. (4.2.21) becomes ignorable compared with the other terms. Then, we expect that μ^2 can be as large as $O(y_0^{-2}) \gg m_1^2$ without violating the condition $|r_c^{(\pm)}\mathcal{H}| \lesssim 1$ by tuning $\mathcal{H}'$ as $|\mathcal{H}'| \sim 1/y_0 r_c^{(\pm)}$. However, the parameter region $|\mathcal{H}' y_0 r_c^{(\pm)}| \sim 1$ is outside the validity range of the perturbative analysis used to derive Eq. (4.2.21). In the next subsection, we will numerically show that we can realize the hierarchy between the masses of the spin-0 mode and the lowest KK graviton's mass.

4.2.3 Numerical construction of the model that realizes the hierarchy

Here, we numerically solve the eigenvalue problem Eqs. (4.2.3) and (4.2.8) to confirm that it is possible to realize the hierarchy which reproduces bigravity as a low-energy effective theory. Here, we assume $H = 0$ for simplicity, and consider the stabilization model whose bulk potential is written in terms of a superpotential as in Eq. (3.3.9). We choose the superpotential $W(\psi)$ and the induced potential on branes $V_\pm$ as [2]

$$W(\psi) = \frac{3}{L} - b\psi^2 \,, \tag{4.2.26}$$

$$V_{(+)}(\psi) = W(\psi_+) + W'(\psi_+)(\psi - \psi_+) + \gamma_{(+)}(\psi - \psi_+)^2 \,, \tag{4.2.27}$$

$$V_{(+)}(\psi) = -W(\psi_-) - W'(\psi_-)(\psi - \psi_-) + \gamma_{(-)}(\psi - \psi_-)^2 \,, \tag{4.2.28}$$

[2]The fine-tuning for the potentials discussed in the last of Sec. 3.3 is archived by this choice of the $V_{(\pm)}$ values and gradients at their minima.

where we introduce the model parameters L, b and $\gamma_{(\pm)}$. $\gamma_{(\pm)}$ must be sufficiently large so that $\epsilon^{(\pm)}$ becomes small and negligible. Then, we obtain the analytical solution for ψ and $\mathcal{H}$ as

$$\psi = \psi_0 \mathrm{e}^{-by}\,, \tag{4.2.29}$$

$$\mathcal{H} = -\frac{1}{6M_5^3}\left(\frac{3}{L} - b\psi_0^2 \mathrm{e}^{-2by}\right)\,. \tag{4.2.30}$$

When $by_0 \ll 1$ and $M_5^{-3/2}\psi_0 \lesssim 1$, $|\mathcal{H}'/\mathcal{H}^2|$ can be large in the whole spacetime by tuning L as $3/L \sim b\psi_0^2$. From Eqs. (4.2.29) and (4.2.30), the conditions $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$, which guarantee that the mass squared of the lowest spin-0 mode μ_0^2 is positive, imply a condition $(\kappa\psi_0 b y_0)^2 \lesssim y_0/r_c \ll 1$ where we used $m_1^2 \ll m_{i\geq 2}^2$ in the last equality. Therefore, we expect that μ_0^2 take a large positive value and simultaneously we realize the hierarchy among KK gravitons, by tuning L as $3/L \sim b\psi_0^2$ and setting $\kappa\psi_0 b y_0$ to be small. Then, we numerically confirm that the requested mass hierarchy is available with the above stabilization model. Here, we numerically solve Eqs. (4.2.3) and (4.2.8), setting the parameters as $M_5 = 1.00$, $y_0 = 0.50$, $r_c^{(\pm)} = 1.00\times 10^5$, $L = 3.00\times 10^3$, $b = 1.00\times 10^{-2}$, $\gamma_{(\pm)} = 1.00 \times 10^3$ and $\psi_0 = 0.316$. In Fig. 4.1, we present the result for the eigenfuctions $u_0(y)$, $u_1(y)$, and $v_0(y)$. The mass eigenvalues are given as $m_1^2 = 2.00 \times 10^{-5}$, $m_2^2 = 9.87$ and $\mu_0^2 = 1.78$: only the lowest KK graviton is kept at low energies. Hence, there exists a concrete model that realizes the requested hierarchy.

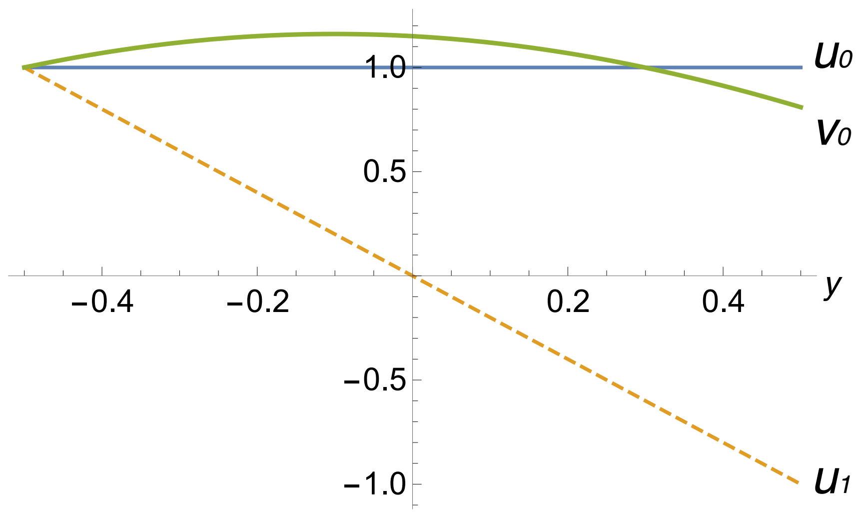

Figure 4.1: The eigenfuctions $u_0(y)$, $u_1(y)$, $v_0(y)$. The solid line, dotted line and thick line correspond to $u_0(y)$, $u_1(y)$ and $v_0(y)$, respectively.

Chapter 5

Equations of motion and instabilities

In Chapter. 5, we showed that we can keep only the massless graviton and the lowest KK gravitons in low energy regime when $y_0/r_c^{(\pm)} \ll 1$ in the DGP 2-brane model with a proper stabilization mechanism. Therefore, its low-energy effective theory will have 7 degrees of freedom: one massless graviton and one massive graviton, and no BD-ghost like mode, as the ghost-free bigravity does. If the ghost-free bigravity is the unique model that is free from BD ghost, the low-energy effective theory of the DGP 2-brane model should coincide with the ghost-free bigravity, when the massive KK modes are completely decoupled except for the lowest one. In this chapter, we show the coincidence of the equations of motion of these two models and how their model parameters correspond to each other, considering the linear perturbation around de Sitter spacetime. Also, we investigate the stability of the DGP 2-brane model and show how the correspondence between the DGP 2-brane model and the ghost-free bigravity breaks down by the appearance of instability. This chapter is also based on Ref. [26].

5.1 Equations of motion

We consider the linear perturbation around de Sitter brane solution of the DGP 2-brane model excited by the matter field localized on the (+)-brane, for simplicity. The perturbations of the metrics induced on the branes, $\bar{h}^{(\pm)}_{\mu\nu}(y = y_\pm)$, are

$$\bar{h}^{(\pm)}_{\mu\nu}(y_\pm) + \nabla_\mu \hat{\xi}_\nu + \nabla_\nu \hat{\xi}_\mu = h^{(TT)}_{\mu\nu}(y_\pm) - \tilde{\gamma}_{\mu\nu}\left(\phi(y_\pm) + 2\mathcal{H}_\pm \hat{\xi}^y_\pm\right) . \tag{5.1.1}$$

Using Eqs. (4.1.27), (4.1.28) and (4.1.29), Eqs. (4.2.7) and (4.2.10) yield

$$\begin{aligned} h^{(TT)}_{\mu\nu}(y_\pm) = &-\frac{2}{M_5^3}\sum_i \frac{u_i(y_+)u_i(y_\pm)}{{}^4\Box - 2H^2 - m_i^2}\left[T^{(+)}_{\mu\nu} - \frac{1}{4}\tilde{\gamma}_{\mu\nu}T^{(+)}\right. \\ &\left. + \frac{a_+^2}{3(m_i^2 - 2H^2)}\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right)T^{(+)}\right] \\ &+ \frac{2a_+^2}{3M_5^3}\left(\frac{u_i(y_+)u_i(y_\pm)}{m_i^2 - 2H^2}\right)\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right)\frac{1}{{}^4\Box + 4H^2}T^{(+)} , \end{aligned} \tag{5.1.2}$$

$$\begin{aligned}\phi_+ + 2\mathcal{H}_+\hat{\xi}^y_+ =& \frac{2a_+^4}{9M_5^6\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)^2}\sum_i \frac{v_i(y_+)^2}{\mu_i^2+4H^2}\frac{1}{{}^4\Box - \mu_i^2}T^{(+)} \\ &- \frac{a_+^2}{3M_5^3\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)}\left[\frac{2a_+^2}{3M_5^3\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)}\left(\sum_i \frac{v_i(y_+)^2}{\mu_i^2+4H^2}\right)\right. \\ &\left. + \mathcal{H}_\pm\right]\frac{1}{{}^4\Box + 4H^2}T^{(+)}\,, \end{aligned} \tag{5.1.3}$$

$$\begin{aligned}\phi_- + 2\mathcal{H}_-\hat{\xi}^y_- =& \frac{2a_+^4}{9M_5^6\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)\left(2r_c^{(-)}\mathcal{H}_- + 1\right)}\times \\ &\sum_i \frac{v_i(y_+)v_i(y_-)}{\mu_i^2+4H^2}\left[\frac{1}{{}^4\Box - \mu_i^2}T^{(+)} - \frac{1}{{}^4\Box + 4H^2}T^{(+)}\right]\,. \end{aligned} \tag{5.1.4}$$

Here, we use the relations

$$\frac{1}{({}^4\Box - 2H^2 - m_i^2)({}^4\Box - 4H^2)} = \frac{1}{m_i^2 - 2H^2}\left(\frac{1}{{}^4\Box - 2H^2 - m_i^2} - \frac{1}{{}^4\Box - 4H^2}\right)\,, \tag{5.1.5}$$

$$\frac{1}{({}^4\Box - \mu_i^2)({}^4\Box + 4H^2)} = \frac{1}{\mu_i^2 + 4H^2}\left(\frac{1}{{}^4\Box - \mu_i^2} - \frac{1}{{}^4\Box + 4H^2}\right)\,, \tag{5.1.6}$$

and the identity which holds for an arbitrary scalar S:

$$\frac{1}{{}^4\Box - 4H^2}\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right)S = \left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right)\frac{1}{{}^4\Box + 4H^2}S\,, \tag{5.1.7}$$

to extract the terms proportional to $({}^4\Box + 4H^2)^{-1}T^{(+)}$. Here, we notice that $Z_{(-)}$ can take some non-zero value that satisfies the homogeneous equation $({}^4\Box + 4H^2)Z_{(-)} = 0$. However, this mode can be removed by a residual gauge transformation discussed in Appendix. C. This reflects that there is no physical degree of freedom at the critical mass $\mu^2 = -4H^2$, which imposes that the terms proportional to $({}^4\Box + 4H^2)^{-1}T^{(+)}$ in $\bar{h}^{(\pm)}_{\mu\nu}(y_\pm)$ must be canceled with each other:

$$\begin{aligned}&2\left(\sum_i \frac{u_i^2(y_+)}{m_i^2 - 2H^2}\right) \\ &\quad + \frac{a_+^2}{H^2\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)}\left[\frac{2a_+^2}{3M_5^3\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)}\left(\sum_i \frac{v_i^2(y_+)}{\mu_i^2 + 4H^2}\right) + \mathcal{H}_+\right] = 0\,, \end{aligned} \tag{5.1.8}$$

$$\left(\sum_i \frac{u_i(y_+)u_i(y_-)}{m_i^2 - 2H^2}\right) + \frac{a_+^2a_-^2}{3M_5^3H^2\left(2r_c^{(+)}\mathcal{H}_+ - 1\right)\left(2r_c^{(-)}\mathcal{H}_- + 1\right)}\left(\sum_i \frac{v_i(y_+)v_i(y_-)}{\mu_i^2 + 4H^2}\right) = 0\,. \tag{5.1.9}$$

These identities simplify the expressions of the induced metrics on the branes. Moreover, we consider only massless spin-2 mode and the lowest KK mode, cutting the other modes off. Then, $\bar{h}_{\mu\nu}(y_\pm)$ becomes

$$\begin{aligned}\bar{h}^{(\pm)}_{\mu\nu}(y_\pm) = -\frac{2}{M_5^3}\sum_{i=0,1}\frac{u_i(y_+)u_i(y_\pm)}{{}^4\Box - 2H^2 - m_i^2}\times \\ \left(T^{(+)}_{\mu\nu} - \left(1+\frac{1}{3}\frac{{}^4\Box - 2H^2 - m_i^2}{{}^4\Box + 6H^2 - m_i^2}\frac{m_i^2 - 6H^2}{m_i^2 - 2H^2}\right)\frac{1}{4}\tilde{\gamma}_{\mu\nu}T^{(+)}\right),\end{aligned} \tag{5.1.10}$$

where we ignore the terms which can be removed by 4-dimensional coordinates transformation on the branes. The induced metrics on the branes are given as

$$\bar{h}^{(\pm)}_{\mu\nu}(y_\pm) = 2r_c^{(+)}h^{(0)}_{\mu\nu}u_0(y_+)u_0(y_\pm) + 2r_c^{(+)}h^{(1)}_{\mu\nu}u_1(y_+)u_1(y_\pm)\,, \tag{5.1.11}$$

$$\left(\Box - 2H^2\right)h^{(0)}_{\mu\nu} = -\frac{2}{M_{\rm Pl}^2}\left(T^{(+)}_{\mu\nu} - \left(1+\frac{{}^4\Box - 2H^2}{{}^4\Box + 6H^2}\right)\frac{1}{4}\tilde{\gamma}_{\mu\nu}T^{(+)}\right), \tag{5.1.12}$$

$$(\Box - 2H^2 - m_1^2)h^{(1)}_{\mu\nu} = -\frac{2}{M_{\rm Pl}^2}\left(T^{(+)}_{\mu\nu} - \left(1+\frac{1}{3}\frac{{}^4\Box - 2H^2 - m_1^2}{{}^4\Box + 6H^2 - m_1^2}\frac{m_1^2 - 6H^2}{m_1^2 - 2H^2}\right)\frac{1}{4}\tilde{\gamma}_{\mu\nu}T^{(+)}\right), \tag{5.1.13}$$

where we used Eq. (4.1.4). These equations are identical to the one in the ghost-free bigravity, Eqs. (2.4.19) and (2.4.23). Here, we take the limit where $y_0/r_c^{(\pm)} \to 0$ while m_1^2 is fixed so that the KK modes other than the the massless mode and the lowest KK mode decouple. Choosing the normal branch which satisfies $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$, $y_0/r_c^{(\pm)} \to 0$ implies $|\mathcal{H}|/m_1 \to 0$, and hence we can approximate $\mathcal{H} \simeq 0$, namely $a \simeq 1$, as is assumed in Sec. 4.2.1. In this limit, Eqs. (4.2.17) and (4.2.18) yield

$$m_1^2 = \frac{1}{2r_c^{(+)}y_0}\left(1+\frac{1}{\chi}\right), \tag{5.1.14}$$

and

$$u_0(Y) = C_0\,, \tag{5.1.15}$$

$$u_1(Y) = C_1\left(1-\left(1+\frac{1}{\chi}\right)Y\right), \tag{5.1.16}$$

where we used $\chi = r_c^{(+)}/r_c^{(-)}$ from Eq. (4.1.4). The normalization constants C_0 and C_1 are to be determined by Eq. (4.2.4). At the leading order of $y_0/r_c^{(\pm)}$, the integration in Eq. (4.2.4) is computed only by taking into account the boundary conditions on the branes, which gives C_0 and C_1 as

$$C_0^2 = \frac{1}{2(1+\chi)\,r_c^{(+)}}, \tag{5.1.17}$$

$$C_1^2 = \frac{1}{2\left(1+\chi^{-1}\right)\,r_c^{(+)}}. \tag{5.1.18}$$

Therefore, the metrics induced on the branes become

$$\bar{h}^{(+)}_{\mu\nu}(y_+) = \frac{1}{1+\chi}\, h^{(0)}_{\mu\nu} + \frac{1}{1+\chi^{-1}}\, h^{(1)}_{\mu\nu}\,,$$
$$\bar{h}^{(-)}_{\mu\nu}(y_-) = \frac{1}{1+\chi}\left(h^{(0)}_{\mu\nu} - h^{(1)}_{\mu\nu}\right)\,. \tag{5.1.19}$$

Comparing Eqs. (5.1.19) and (2.4.13), the bigravity obtained from the DGP 2-brane model is shown to be identical to the ghost-free bigravity when $\kappa\omega^2 = \chi$, where the metric on the (+)-brane and the one on the (−)-brane in the DGP 2-brane model corresponds to the physical metric and the hidden metric in the ghost-free bigravity.

5.2 Instabilities

As is seen in Appendix. B, even in the theory that contains massive gravitons without BD ghost instability, the helicity-0 mode of the massive graviton becomes a ghost mode around de Sitter background when the graviton's mass squared is less than the double of the background Hubble scale squared. In addition, the DGP 2-brane model contains a spin-0 mode which can also be a ghost mode [34]. This spin-0 ghost may yield the difference in the instabilities between the DGP 2-brane model and the ghost-free bigravity. In this section, we study the instabilities in the DGP 2-brane model and compare how the instabilities appear in the DGP 2-brane model with that in the ghost-free bigravity, focusing on the normal branch which satisfies the conditions $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$ and is connected to the solution with two Minkowski branes by continuous variation of the model parameters.

As is discussed in Appendix. B and Appendix. D, The KK mode with the mass m of $0 < m^2 < 2H^2$ and the spin-0 mode with the mass μ of $\mu^2 < -4H^2$ become Higuchi ghost and a scalar ghost, respectively. Let us consider the solution with two Minkowski brane first, and add the brane tension on the (+)-brane little by little. The identity (5.1.8) shows that the spacetime constructed in this way is ghost-free as long as it satisfies $1 \mp 2r_c^{(\pm)}\mathcal{H} > 0$ [1], as follows. When we add infinitesimal brane tension to the solution with Minkowski branes, the mass spectrum is essentially unchanged and there is no instability: all the KK gravitons' masses are larger than $2H^2$ and the spin-0 modes' masses are larger than $-4H^2$ for the Hubble scale on the branes H. Continuing to add the brane tension, one can obtain the solution with de Sitter branes of arbitrary Hubble scale. Under this process, the mass spectrum must change satisfying the identity (5.1.8). The assumption $\epsilon^{(\pm)} \geq 0$ guarantees v_i^2 is positive. With this assumption, one of the normal-branch conditions $1 - 2r_c^{(+)}\mathcal{H}_+ > 0$ requests that the first term on the left-hand

[1] In the self-accelerating branch where either of $1 \mp 2r_c^{(\pm)}\mathcal{H}$ is negative, the lowest spin-0 mode becomes a ghost mode with $\mu_0^2 < -4H^2$ when $H = 0$, as shown in Eq. (4.2.22). Let us try to make the spin-0 mode healthy by raising μ_0^2 larger than the critical value $-4H^2$. When μ_0^2 crosses the critical value, the first term in the square brackets on the left-hand side of the identity (5.1.8) diverges to negative infinity, with which the identity (5.1.8) requests that m_i^2 crosses the critical value $2H^2$, namely that the spin-2 mode becomes Higuchi ghost, at the same time. Similarly, making the spin-2 modes healthy yields spin-0 ghost mode. Therefore, the self-accelerating branch inevitably contains ghost [34].

side of the identity (5.1.8) should diverge to positive infinity when m_i^2 crosses the critical value $2H^2$ from above, and also that the first term in the square brackets on the left-hand side of the identity (5.1.8) should diverge to positive infinity when μ_i^2 crosses the critical value $-4H^2$ from above. Whilst, there is no term that diverges to negative infinity in the identity (5.1.8), and hence it is prohibited by the identity (5.1.8) that m_i^2 or μ_i^2 crosses the critical value. Therefore, the solution constructed in this manner is shown to be ghost-free.

However, Eq. (4.1.9) implies when we add the brane tension, $|\mathcal{H}_\pm|$ becomes as large as $r_c^{(\pm)}H^2$, unless the change of $V_{(\pm)}(\psi_\pm)$ cancels out $r_c^{(\pm)}H^2$. At least, as long as we consider the model where $\psi_\pm$ are pinned down by sufficiently large $V''_{(\pm)}$, such cancellation of $r_c^{(\pm)}H^2$ cannot happen. Therefore, when $r_c^{(\pm)}$ are large, $|\mathcal{H}_\pm|$ becomes larger than $\mathcal{O}(1/r_c^{(\pm)})$, namely, either of the normal-branch conditions $1 - 2r_c^{(\pm)}\mathcal{H}_\pm > 0$ is violated, just by considering a slightly high energy regime with $H \gtrsim 1/r_c^{(\pm)}$. When we add the brane tension little by little starting with Minkowski branes, μ^2 becomes negative and the spin-0 mode exhibits tachyonic instability before crossing $-4H^2$. At this point where the spin-0 tachyon appears, the spacetime structure with stably separated two branes becomes unsustainable and two branes will move away from each other. This destabilization disables the DGP 2-brane model from reproducing bigravity as a low-energy effective theory. In order to avoid this tachyonic instability in the slightly high energy regime, we should introduce some other mechanism that can stabilize the brane separation much more strongly.

The way how the ghost appears in the DGP 2-brane model is clearly different from the one in the ghost-free bigravity discussed in Sec. 2.4: before Higuchi ghost appears, the DGP 2-brane model exhibits tachyonic instability, while there appears no instability in the ghost-free bigravity,. This incoincidence of the appearance of instabilities is to be understood as caused by the truncation of the spin-0 mode of the DGP 2-brane model. In the DGP 2-brane model, we can neglect the spin-0 modes when all of the mass of the spin-0 modes are sufficiently massive. Once the masses of the spin-0 modes become light, however, we cannot neglect them, and eventually the spin-0 mode becomes tachyon when its mass squared becomes negative. In the ghost-free bigravity, on the other hand, the spin-0 mode does not exist from the beginning. Hence, the ghost-free bigravity does not exhibit the spin-0 tachyon that appears in the DGP 2-brane model, which lets the ghost-free bigravity "healthy" even in the energy regime higher than $1/r_c^{(\pm)}$.

Chapter 6

Reproducing bigravity action

In Chapter. 4, we analyzed the perturbation around the spacetime with two de Sitter branes in the DGP 2-brane model, where we estimated the mass spectrum and confirmed that the model coincides with the ghost-free bigravity at low energies. Although in Chapter. 5 the Goldberger-Wise stabilization mechanism was found too weak to sustain even a slightly curved spacetime, now we will put this problem aside and go to the next step, derive the effective bigravity action from the DGP 2-brane model and examine whether the obtained action coincides with the ghost-free bigravity action. Here, we do not introduce the radion stabilization for simplicity. Therefore, the effective action is expected to contain two gravitons and a scalar radion. Furthermore, the radion should couple to both gravitons without no BD-ghost like problem in this effective action. This means that the obtained action should solve the problem of doubly coupled matter in the ghost-free bigravity discussed in Appendix. E in some way. This chapter is based on Ref. [27].

6.1 Model and Strategy

We investigate the DGP 2-brane model without the radion stabilization $S-S_s$ in Eqs. (4.1.1) and (4.1.2). We consider two vacuum branes with the brane tension $\sigma_\pm$ on the respective branes, and also introduce the 5-dimensional cosmological constant $6/\ell_\Lambda^2$. Therefore, the whole action is given as

$$S = S_b + \sum_{\sigma=\pm} S_\sigma \,, \tag{6.1.1}$$

where

$$\begin{aligned} S_b &:= \frac{1}{2}\frac{M_{\rm Pl}^2}{2r_c^{(+)}}\int d^5x\sqrt{-{}^5g}\left({}^5R-\frac{12}{\ell_\Lambda^2}\right)\,, \\ S_+ &:= \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g_{(+)}}\left(R_{(+)}-2\sigma_+\right)\,, \\ S_- &:= \frac{\chi M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g_{(-)}}\left(R_{(-)}-2\sigma_-\right)\,. \end{aligned} \tag{6.1.2}$$

The task in this chapter is to derive the 4-dimensional effective action from Eqs. (6.1.1) and (6.1.2), by solving the bulk geometry for given boundary metrics on two branes and integrating out the bulk degrees of freedom. In order to solve the bulk equation, we consider the approximation where we assume that the brane separation is sufficiently small compared with the 5-dimensional curvature scale, and hence the metrics on the both branes are identical at the leading order of the small separation expansion. This approximation is called the gradient expansion. Here, we present the strategy to derive the effective action using the gradient expansion in detail.

We choose the coordinate system as follows:

$$ds^2 = N^2(x)dy^2 + g_{\mu\nu}dx^\mu dx^\nu \,, \tag{6.1.3}$$

where we use the gauge degrees of freedom to fix the lapse function N and the shift vector N^μ as $\partial_y N = 0$ and $N^\mu = 0$. Simultaneously with these gauge fixing conditions, at the lowest order of the expansion, the $(\pm)$-branes can be set at the constant y-coordinate value $y = y^\pm$, respectively. Decomposing the bulk equations with respect to the $y =$ constant hypersurfaces under the above gauge fixing conditions, as is discussed in Appendix. A, we obtain the evolution equations, the Hamiltonian constraint and the momentum constraints from Eqs. (A.2.10), (A.0.5), (A.2.2) and (A.2.3), respectively as

$$\begin{aligned}
\partial_{\tilde{y}} K_{\mu\nu} &= -2K^\rho_\mu K_{\rho\nu} + KK_{\mu\nu} + \frac{4}{\ell_\Lambda^2} g_{\mu\nu} - R_{\mu\nu} + \frac{N_{;\mu\nu}}{N} \,, && (6.1.4)\\
K_{\mu\nu} &= -\frac{1}{2}\partial_{\tilde{y}} g_{\mu\nu} \,, && (6.1.5)\\
K^2 - K^\mu_\nu K^\nu_\mu &= -\frac{12}{\ell_\Lambda^2} + R \,, && (6.1.6)\\
K_{;\mu} - K^\nu_{\mu;\nu} &= 0 \,, && (6.1.7)
\end{aligned}$$

where we define $\partial_{\tilde{y}} \equiv \frac{1}{N}\partial_y$. $K_{\mu\nu}$ and $R_{\mu\nu}$ are the extrinsic curvature and the 4-dimensional Ricci tensor evaluated on each y-constant surface (notice that not on the $\tilde{y} \equiv Ny =$ constant hypersurface), respectively. $f_{;\mu}$ is the covariant derivative of the field f associated with the 4-dimensional metric $g_{\mu\nu}$, and the indices are raised and lowered by $g_{\mu\nu}$.

We will solve these equations with the given boundary metrics. Let the boundary geometries specified by $g^{(+)}_{\mu\nu}(x)dx^\mu dx^\nu$ and $\tilde{g}_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu$ ($g^{(-)}_{\mu\nu}(x)$ is reserved for later use) on the respective branes. The bulk equations of motion should be able to give an appropriately interpolating bulk solution for these two given boundary geometries. There seems to be the arbitrariness in the way how to relate the 4-dimensional coordinates on one brane with those on the other, however, since we adopt the above metric ansatz (6.1.3), the way of connecting between two branes is completely specified by its gauge choice. In the coordinates on the (+)-brane, namely, the metric on the (−)-brane given as

$$g^{(-)}_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}(\tilde{x}) \,, \tag{6.1.8}$$

cannot be chosen arbitrarily. If the bulk equations Eqs. (6.1.4)-(6.1.6) (excluding (6.1.7)) are solved with the boundary metrics $g_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)}$, the coordinate transformation $\tilde{x}^{\alpha}(x)$ is determined by the momentum constraint (6.1.7). Now, we consider the gradient expansion, then we can expand $g_{\mu\nu}$ around the middle point $\bar{y}=0$ in the bulk as

$$g_{\mu\nu}^{(\pm)} = \bar{g}_{\mu\nu} + \overline{\partial_{\tilde{y}} g_{\mu\nu}} \tilde{y}^{\pm} + \frac{1}{2}\overline{\partial_{\tilde{y}}{}^{2} g_{\mu\nu}} (\tilde{y}^{\pm})^2 + \cdots , \tag{6.1.9}$$

where $\bar{g}_{\mu\nu} := g_{\mu\nu}(0)$, and $\tilde{y}^{\pm} \equiv N(x) y^{\pm}$. As we set the middle point of the bulk $\bar{y}=0$, the values of $\tilde{y}^{+}$ and $\tilde{y}^{-}$ are related with each other as $\tilde{y}^{-} = -\tilde{y}^{+}$. Since at the leading order of the gradient expansion the $(\pm)$-branes locate at the respective constant y-coordinate values, we denote the brane positions as $y^{\pm} = \mp y_0$ and obtain

$$\tilde{y}^{+} = -N y_0 < 0 , \qquad \tilde{y}^{-} = N y_0 > 0 . \tag{6.1.10}$$

Using the expression (6.1.9), we obtain $\bar{g}_{\mu\nu}$ and $\bar{K}_{\mu\nu}$ in terms of $g_{\mu\nu}^{(\pm)}$ and $\tilde{y}^{+}$ from the evolution equations (6.1.4) and (6.1.5), and then the Hamiltonian constraint (6.1.6) determines $\tilde{y}^{+}$, and equivalently N, in terms of $g_{\mu\nu}^{(\pm)}$. Finally, the effective action is given in terms of the boundary metrics $g_{\mu\nu}^{(\pm)}$ by substituting the obtained $\bar{g}_{\mu\nu}$ and $\bar{K}_{\mu\nu}$ into the bulk action expanded around $y=0$:

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2}\frac{1}{2r_c^{(+)}} \oint \sqrt{-g} d^4x N dy \left({}^5R - \frac{12}{\ell_\Lambda^2} \right) \\ &= \frac{M_{pl}^2}{2}\frac{1}{r_c^{(+)}} \int d^4x N \int_{-y_0}^{y_0} dy \left[\sqrt{-\bar{g}} \left({}^5\bar{R} - \frac{12}{\ell_\Lambda^2} \right) + y \overline{\frac{\delta \left(\sqrt{-g} \left({}^5R - \frac{12}{\ell_\Lambda^2} \right) \right)}{\delta g_{\mu\nu}} \partial_y g_{\mu\nu}} \right. \\ &\qquad \left. + \frac{y^2}{2} \overline{\partial_y \left(\frac{\delta \left(\sqrt{-g} \left({}^5R - \frac{12}{\ell_\Lambda^2} \right) \right)}{\delta g_{\mu\nu}} \partial_y g_{\mu\nu} \right)} + \cdots \right] , \end{aligned} \tag{6.1.11}$$

with the aid of the equation

$${}^5R - \frac{12}{\ell_\Lambda^2} = K^2 - K^{\mu}_{\nu} K^{\nu}_{\mu} + R - \frac{12}{\ell_\Lambda^2} , \tag{6.1.12}$$

which is derived in the Appendix. A as Eq. (A.2.13). Since no second derivative of the metric in the y-direction appears in the integrand of Eq. (6.1.11), we need not consider the boundary Gibbons-Hawking terms in this expression. We do not manifestly need to impose the momentum constraints, because they are automatically satisfied by the junction conditions as follows. In this model, the junction conditions Eqs. (4.1.3) are written as

$$K_{\mu\nu}^{(\pm)} - K^{(\pm)} g_{\mu\nu}^{(\pm)} = \pm r_c^{(\pm)} \left(G_{\mu\nu}^{(\pm)} + \sigma_{\pm} g_{\mu\nu}^{(\pm)} \right) , \tag{6.1.13}$$

where $r_c^{(\pm)}$ are defined in the same way in the previous chapters, as in Eq. (4.1.4). Then, it is obvious that the divergence of Eq. (6.1.13) yields the momentum constraints (6.1.7) using the Bianchi identity. As long as $K_{\mu\nu}^{(\pm)}$ for the bulk solution agree with the ones that are derived from the variation of the obtained effective action with respect to $g_{\mu\nu}^{(\pm)}$, as is expected, the variation of the effective action with respect to $g_{\mu\nu}^{(\pm)}$ gives the same equation as the junction conditions (6.1.13). Therefore, the correctly obtained solution automatically satisfies the momentum constraint.

6.2 Perturbation around de Sitter spacetime

First, we again consider the perturbation around a de Sitter brane solution with the comoving curvature H, the one discussed in Chapter. 4 and 5, and calculate the effective action quadratic in the perturbation, as a warm-up. The background spacetime is given as

$$ds^2 = dy^2 + a^2(y)\gamma_{\mu\nu}dx^\mu dx^\nu \,, \tag{6.2.1}$$

where we set $N = 1$, $a(y = 0) = 1$ and $\gamma_{\mu\nu}$ is the 4-dimensional de Sitter metric with the Hubble scale H. Similarly in Eqs. (4.1.6) and (4.1.9), $\mathcal{H} := \partial_y a/a$, which corresponds to the background value of $-K/4$, is determined by the background equations of motion as

$$\mathcal{H} = \pm\sqrt{-\frac{1}{\ell_\Lambda^2} + \frac{H^2}{a^2}} \,, \tag{6.2.2}$$

$$\mp\mathcal{H}_\pm = r_c^{(\pm)}\left(-\frac{\sigma_\pm}{3} + \frac{H^2}{a_\pm^2}\right) , \tag{6.2.3}$$

where $\mathcal{H}_\pm := \mathcal{H}\,(y = \mp y_0)$ and $a_\pm := a\,(y = \mp y_0)$. Here we choose the convention with $\mathcal{H}(y = 0) > 0$ so that the tension on the (+)-brane should be positive: $\sigma_+ > 0$. Eq. (6.2.2) is solved as

$$\mathcal{H} = -\frac{1}{\ell_\Lambda}\tan\left(\frac{y}{\ell_\Lambda} + A\right), \tag{6.2.4}$$

with the integration constant

$$A = -\arccos\left(\frac{1}{\ell_\Lambda H}\right)(< 0) \,. \tag{6.2.5}$$

Using $a(y = 0) = 1$, we obtain

$$a = \ell_\Lambda H\cos\left(\frac{y}{\ell_\Lambda} + A\right), \tag{6.2.6}$$

From the above expression, we can expand $\mathcal{H}$ around $y = 0$ as

$$\mathcal{H}_\pm := \mathcal{H}\,(\mp y_0) = \mathcal{H}_0 \pm H^2 y_0 + \mathcal{H}_0\mathcal{O}(H^2 y_0^2). \tag{6.2.7}$$

where we define $\mathcal{H}_0 := \mathcal{H}(y=0)$ as

$$\mathcal{H}_0 = \sqrt{-\ell_\Lambda^{-2} + H^2}\,. \tag{6.2.8}$$

We represent the perturbation around this background as

$$\begin{aligned} g_{\mu\nu}^{(\pm)} &= a^2(\mp y_0)\left(\gamma_{\mu\nu} + h_{\mu\nu}^{(\pm)}\right), & (6.2.9)\\ N &= 1 + x\,, \quad i.e.\,, \quad \tilde{y}^\pm = \mp y_0(1+x)\,. & (6.2.10)\end{aligned}$$

Here, we emphasize that we can consider the value of y-coordinate at the location of the perturbed ($\pm$)-brane as $y = \mp y_0$ in our coordinate system, as long as we work on the leading order of the gradient expansion. To avoid the complexity from the y-dependence related to the background warp factor $a(y)$, we introduce new variables as

$$\begin{aligned} \mathcal{G}_{\mu\nu}(y) &:= a^{-2}(y) g_{\mu\nu}(y)\,, & (6.2.11)\\ \mathcal{K}_{\mu\nu}(y) &:= a^{-2}(y)\left(K_{\mu\nu}(y) + \frac{\mathcal{H}(y)}{N} g_{\mu\nu}(y)\right). & (6.2.12)\end{aligned}$$

With these new variables, Eqs. (6.1.4)-(6.1.7) become

$$\begin{aligned} \partial_{\tilde{y}}\mathcal{K}_{\mu\nu} &= -2\mathcal{K}_\mu^\rho\mathcal{K}_{\rho\nu} + \mathcal{K}\mathcal{K}_{\mu\nu} - \frac{\mathcal{H}}{N}\mathcal{K}\mathcal{G}_{\mu\nu} - 4\frac{\mathcal{H}}{N}\mathcal{K}_{\mu\nu}\\ &\quad + \left(\frac{4}{\ell_\Lambda^2} + 4\frac{\mathcal{H}^2}{N^2} + \frac{\partial_{\tilde{y}}\mathcal{H}}{N}\right)\mathcal{G}_{\mu\nu} - a^{-2}R_{\mu\nu} + a^{-2}\frac{N_{;\mu\nu}}{N}\,, & (6.2.13)\\ \mathcal{K}_{\mu\nu} &= -\frac{1}{2}\partial_{\tilde{y}}\mathcal{G}_{\mu\nu}\,, & (6.2.14)\\ \mathcal{K}^2 - 6\frac{\mathcal{H}}{N}\mathcal{K} - \mathcal{K}_\nu^\mu\mathcal{K}_\mu^\nu &= R - 12\left(\frac{\mathcal{H}^2}{N^2} + \frac{1}{\ell_\Lambda^2}\right). & (6.2.15)\end{aligned}$$

Then, we will solve Eqs. (6.2.11)-(6.2.15) using the gradient expansion. We expand $\mathcal{G}_{\mu\nu}^{(\pm)}$ as

$$\begin{aligned} \mathcal{G}_{\mu\nu}^{(\pm)} &= \bar{\mathcal{G}}_{\mu\nu} + \overline{\partial_y\mathcal{G}_{\mu\nu}}\tilde{y}^\pm + \frac{1}{2}\overline{\partial_y^2\mathcal{G}_{\mu\nu}}(\tilde{y}^\pm)^2 + \cdots\\ &= \bar{\mathcal{G}}_{\mu\nu} - 2\bar{\mathcal{K}}_{\mu\nu}\tilde{y}^\pm - \overline{\partial_{\tilde{y}}\mathcal{K}_{\mu\nu}}(\tilde{y}^\pm)^2 + \cdots\,, & (6.2.16)\end{aligned}$$

where the overbar denotes the value evaluated at $y = 0$. In the following, we only work on the lowest order in the gradient expansion, *i.e.*, keep the leading order of

$$\epsilon := |K y_0|\,, \tag{6.2.17}$$

when we compute the effective action up to the quadratic order of the metric perturbation. We must choose the parameter region with $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$, in which both radion and the massive gravitons have no ghost instabilities, as is discussed in Sec. 5.2, since such ghost appearance will prohibit recovering the ghost-free bigravity interaction in the effective action. Under the gradient expansion $\epsilon \ll 1$, then, $\mathcal{O}(|K|)$ is estimated as

$$|K| \lesssim \mathcal{O}\left(1/r_c^{(\pm)}\right) . \tag{6.2.18}$$

Furthermore, we want to investigate the energy regime comparable to the mass of the lowest mode of massive gravitons. Using Eq. (5.1.14) as the mass squared of the lowest mode of massive gravitons at the lowest order of the gradient expansion, therefore, H^2 should be

$$H^2 \sim m_1^2 \sim \mathcal{O}\left(|K|^2/\epsilon\right) , \tag{6.2.19}$$

and the 4-dimensional momentum squared is estimated as well. Equation (6.2.19) implies that ℓ_Λ^{-2} should be tuned so as to cancel out with H^2 in Eq. (6.2.8) at the leading order of the gradient expansion, since $\mathcal{H}_0^2 \sim \mathcal{O}\left(K^2\right)$ is much smaller than $H^2 \sim \mathcal{O}\left(|K|^2/\epsilon\right)$. This requires that the background expansion rate should balance with the value determined by the bulk cosmological constant:

$$H \approx \ell_\Lambda^{-1} . \tag{6.2.20}$$

Among the perturbed geometrical variables, also, the Hamiltonian constraint (6.1.6) requires

$$R - 12\ell_\Lambda^{-2} \sim \mathcal{O}\left(K^2\right) . \tag{6.2.21}$$

Furthermore, the trace of the junction conditions (6.1.13) implies that the trace part of $R_{\mu\nu}^{(\pm)}$ should cancel with the brane tension $\sigma_\pm$ at the leading order of the gradient expansion in Eq. (6.1.13), while the traceless part of $R_{\mu\nu}$ is required to balance with $K_{\mu\nu}/r_c^{(\pm)} \sim \mathcal{O}\left(K^2\right)$ by the traceless part of Eq. (6.1.13). Therefore, Eqs. (6.1.6) and (6.1.13) determines $R_{\mu\nu}$ at the leading order of gradient expansion as

$$R_{\mu\nu} \sim 3\ell_\Lambda^{-2} g_{\mu\nu} + \mathcal{O}\left(K^2\right) . \tag{6.2.22}$$

The above fine-tuned conditions are easily violated if we consider large matter energy momentum tensors on the respective branes in addition, and hence one cannot use the present model to describe the system with the background energy scale varied by a large amount from the fine-tuned value.

Based on this understanding, $\mathcal{K}_{\mu\nu}$ at the middle point $y = 0$ is given from Eq. (6.2.16) as

$$\begin{aligned} \bar{\mathcal{K}}_{\mu\nu} &= -\frac{\mathcal{G}_{\mu\nu}^{(+)} - \mathcal{G}_{\mu\nu}^{(-)}}{4\tilde{y}^+} + \cdots \\ &= \left(x\mathcal{H}_0\gamma_{\mu\nu} + \frac{1}{4y_0}\Delta h_{\mu\nu}\right)(1-x) + \mathcal{O}\left(\epsilon h K\right) , \end{aligned} \tag{6.2.23}$$

where we assume that $x \sim \mathcal{O}(\epsilon^0 h)$, which will be confirmed later, and define

$$\Delta h_{\mu\nu} := h^{(+)}_{\mu\nu} - h^{(-)}_{\mu\nu} \sim \mathcal{O}(\epsilon h) \ . \tag{6.2.24}$$

The first term in the first parentheses in Eq. (6.2.23) comes from the factor $a^{-2}(y)$ in the definition of $\mathcal{G}_{\mu\nu}$ in Eq. (6.2.11). In order to solve Eqs. (6.2.13) and (6.2.15), we also calculate $\bar{\mathcal{G}}_{\mu\nu}$ to raise the subscript of $\bar{\mathcal{K}}_{\mu\nu}$ and to obtain $\bar{R}$. Eq. (6.2.16) determines $\bar{\mathcal{G}}_{\mu\nu}$ as

$$\begin{aligned}
\bar{\mathcal{G}}_{\mu\nu} &= \frac{1}{2}\left(\mathcal{G}^{(+)}_{\mu\nu} + \mathcal{G}^{(-)}_{\mu\nu}\right) + \overline{\partial_{\tilde{y}}\mathcal{K}_{\mu\nu}}(\tilde{y}^{+})^2 + \cdots \\
&= \gamma_{\mu\nu} + \tilde{h}_{\mu\nu} + 2H^2 x y_0^2 \gamma_{\mu\nu} + H^2 (x y_0)^2 \gamma_{\mu\nu} \\
&\qquad + (1+x)\left(\nabla_\mu \nabla_\nu x\right) y_0^2 + \mathcal{O}\left(\epsilon^2, \epsilon h\right) \ ,
\end{aligned} \tag{6.2.25}$$

where ∇_μ is the covariant differentiation associated with $\gamma_{\mu\nu}$ and we define

$$\tilde{h}_{\mu\nu} := \frac{1}{2}\left(h^{(+)}_{\mu\nu} + h^{(-)}_{\mu\nu}\right) \ . \tag{6.2.26}$$

In the second equality in Eq. (6.2.25), we used Eq. (6.2.13) but only its last three terms where with the aid of Eqs. (6.2.20) and (6.2.22) ℓ_Λ^{-2} and $\bar{R}_{\mu\nu}$ are replaced by H^2 and $3H^2\bar{g}_{\mu\nu}$, respectively, since the other contribution is subdominant in (6.2.25) at the order of the present approximation.

At first order of the metric perturbation h, the Hamiltonian constraint (6.2.15) determines x as

$$x^{(1)} = -\frac{\Delta h}{16 y_0 \mathcal{H}_0} - \frac{\bar{R}^{(1)}}{24\mathcal{H}_0^2} + \mathcal{O}(\epsilon h) =: -\frac{\Phi}{16 y_0 \mathcal{H}_0} + \mathcal{O}(\epsilon h) \ , \tag{6.2.27}$$

where x and $\bar{R}$ are expanded as $A = A^{(1)} + A^{(2)} + \cdots$ with respect to the order of the metric perturbation h. The last equality defines a new variable Φ, which denotes the perturbation of the brane separation rescaled by $\mathcal{H}_0$. $\bar{R}^{(1)}$ is computed using Eq. (6.2.25) as

$$\begin{aligned}
\bar{R}^{(1)} &= \mathcal{L}^{\mu\nu}\left(\tilde{h}_{\mu\nu} + 2H^2 y_0^2 x^{(1)} \gamma_{\mu\nu}\right) + \mathcal{O}\left(\epsilon h K^2\right) \\
&= \frac{1}{1 - \hat{H}^2 \hat{\Box}} \mathcal{L}^{\mu\nu}\left(\tilde{h}_{\mu\nu} - \frac{\hat{H}^2}{4}\Delta h \gamma_{\mu\nu}\right) + \mathcal{O}\left(\epsilon h K^2\right) \ ,
\end{aligned} \tag{6.2.28}$$

where we define

$$\mathcal{L}_{\mu\nu} := \nabla_\mu \nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu} - \frac{3}{4}\left(\Box + 4H^2\right)\gamma_{\mu\nu} \, , \qquad \Box := \nabla^\mu \nabla_\mu \, , \tag{6.2.29}$$

and

$$\alpha := \frac{y_0}{2\mathcal{H}_0} \, , \qquad \hat{H}^2 := \alpha H^2 \, , \qquad \hat{\Box} := \alpha\left(\Box + 4H^2\right) \ .$$

In the last expression of Eq. (6.2.28), we formally solved the differential equation after substituting Eq. (6.2.27) into $x^{(1)}$ in the first line, and we also use the identity

$$\mathcal{L}_{\mu\nu}\nabla^{\mu}A^{\nu} = 0\,, \tag{6.2.30}$$

which holds for an arbitrary vector A^{ν}. Substituting Eq. (6.2.28) into Eq. (6.2.27), we obtain

$$\Phi = -16y_0\mathcal{H}_0 x^{(1)} = \Delta h + \frac{4\alpha}{3}\frac{1}{1-\hat{H}^2\hat{\Box}}\mathcal{L}^{\mu\nu}\left(\tilde{h}_{\mu\nu} - \frac{\hat{H}^2}{4}\Delta h\gamma_{\mu\nu}\right)\,. \tag{6.2.31}$$

As $\bar{R}^{(1)}$ contains the non-local operator $(1-\hat{H}^2\hat{\Box})^{-1}$ in Eq. (6.2.28), one may concern that an extra degree of freedom appears corresponding to the pole at $\hat{\Box} = \hat{H}^{-2}$. However, we have already seen that there is no physical mode at $\hat{\Box} = \hat{H}^{-2}$ from the linear perturbation analysis in Sec. 5.1. In order to prevent such an unphysical degree of freedom from appearing, the non-local operator $(1-\hat{H}^2\hat{\Box})^{-1}$ should be expanded as $1+\hat{H}^2\hat{\Box}+\hat{H}^4\hat{\Box}^2+\cdots$ by setting the upper bound to the energy scale as $\hat{H}^2\hat{\Box} < 1$. Imposing that $H \approx m_1$ and that the normal branch is chosen: $1 \mp 2r_c^{(\pm)}\mathcal{H}_{\pm} > 0$, moreover, the allowed energy region becomes

$$\Box + 4H^2 \lesssim \mathcal{O}\left(K^2\frac{r_c^{(+)}}{y_0}\right) \lesssim \mathcal{O}\left(m_1^2\right)\,, \tag{6.2.32}$$

which is marginally consistent with the energy scale we want to investigate. Even if we keep this pole at $\hat{\Box} = \hat{H}^{-2}$, interestingly, the analysis of the evaluated effective action reveals that its contribution to the metric perturbation is not sourced by the matter energy momentum tensors localized on the branes, as we will find in the next section.

Similarly, we will apply the above sequence of calculation to the second order of the metric perturbation. With Eq. (6.2.23), the Hamiltonian constraint (6.2.15) at the 2nd order of h is given as

$$\begin{aligned}
&- \quad 6\mathcal{H}_0\left[4\mathcal{H}_0x^{(2)} - \left(\frac{\Delta h}{4y_0} + 4\mathcal{H}_0 x^{(1)}\right)x^{(1)}\right] \\
&\qquad -12\mathcal{H}_0^2 x^{(1)\,2} - \mathcal{H}_0\frac{x^{(1)}\Delta h}{2y_0} - \frac{\Delta h^{\mu\nu}\Delta h_{\mu\nu}}{16y_0^2} + \mathcal{O}\left(\epsilon h^2 K^2\right) = \bar{R}^{(2)}\,.
\end{aligned} \tag{6.2.33}$$

$\bar{R}^{(2)}$ is explicitly expressed using Eq. (6.2.25) as

$$\bar{R}^{(2)} = y_0^2\mathcal{L}^{\mu\nu}\left[x^{(1)}\nabla_\mu\nabla_\nu x^{(1)} + H^2\left(x^{(1)}\right)^2\gamma_{\mu\nu} + 2H^2x^{(2)}\gamma_{\mu\nu}\right] + \mathcal{O}\left(\epsilon h^2 K^2\right)\,, \tag{6.2.34}$$

and then, as in the case of $x^{(1)}$, $x^{(2)}$ is solved as

$$\begin{aligned}
x^{(2)} \;=\; & \frac{1}{1-\hat{H}^2\hat{\Box}}\frac{1}{384\mathcal{H}_0^2y_0^2}\left[\Delta h^2 - \Delta h^{\mu\nu}\Delta h_{\mu\nu} - \frac{3}{4}\Phi^2 + 4\alpha\Phi\bar{R}^{(1)}\right. \\
&\left. -\frac{\alpha^2}{4}\mathcal{L}^{\mu\nu}\left(\Phi\nabla_\mu\nabla_\nu\Phi + H^2\Phi^2\gamma_{\mu\nu}\right)\right] + \mathcal{O}\left(\epsilon h^2\right)\,.
\end{aligned} \tag{6.2.35}$$

Again, x contains a non-trivial non-local operator, but the pole locates at the same point as the one in $x^{(1)}$.

From the above calculation, we can integrate out the bulk degrees of freedom and evaluate the total second-order effective action from

$$\begin{aligned} S = \frac{M_{pl}^2}{2}\Bigg[& \frac{1}{2r_c^{(+)}}\oint d^5x\, 2\sqrt{-g}\left(R-\frac{12}{\ell_\Lambda^2}\right) \\ & + \int d^4x\sqrt{-g_{(+)}}\left(R_{(+)}-\frac{6H^2}{a_+^2}\right)+\chi\int d^4x\sqrt{-g_{(-)}}\left(R_{(-)}-\frac{6H^2}{a_-^2}\right)\Bigg], \end{aligned} \tag{6.2.36}$$

in terms of $g_{\mu\nu}^{(\pm)}$ as

$$\begin{aligned} & \frac{1}{2r_c^{(+)}}\oint d^5x 2\sqrt{-g}\left(R-\frac{12}{\ell_\Lambda^2}\right) \\ = & \frac{1}{r_c^{(+)}}\int_{y_0^+}^{-y_0^+} Ndy\int d^4x 2\sqrt{-\gamma}\left\{12\mathcal{H}_0^2+\mathcal{L}^{\mu\nu}\left(\bar{g}_{\mu\nu}-\tilde{y}^2\overline{\partial_{\tilde{y}}K_{\mu\nu}}\right)+\mathcal{O}\left(\epsilon h^2K^2\right)\right\} \\ = & \frac{1}{r_c^{(+)}}\int d^4x\, 2\sqrt{-\gamma}\,\Big\{2Ny_0\left(12\mathcal{H}_0^2+\bar{R}^{(1)}+\bar{R}^{(2)}\right) \\ & \qquad -\frac{2}{3}Ny_0^3\mathcal{L}^{\mu\nu}\left(N\nabla_\mu\nabla_\nu N+N^2H^2\gamma_{\mu\nu}\right)+\mathcal{O}\left(\epsilon^2h^2K\right)\Big\} \\ = & \int d^4x\sqrt{-\gamma}\,\frac{1}{8r_c^{(+)}y_0}\left(384\left(\frac{y_0}{\ell_\Lambda}\right)^2\left(1-4\hat{H}^4\right)x^{(2)}-4\alpha\Phi\bar{R}^{(1)}+\mathcal{O}\left(\epsilon^3h^2\right)\right) \\ = & \int d^4x\sqrt{-\gamma}\,m_*^2\left\{\Delta h^2-\Delta h^{\mu\nu}\Delta h_{\mu\nu}-\frac{3}{4}\Phi\left(1-\hat{H}^2\hat{\Box}\right)\Phi+\mathcal{O}\left(\epsilon^3h^2\right)\right\}, \end{aligned} \tag{6.2.37}$$

where we define

$$m_*^2 := \frac{1}{8r_c^{(+)}y_0}\,. \tag{6.2.38}$$

In the first equality in Eq. (6.2.37), $R(y)-12\ell_\Lambda^{-2}$ is expanded around $y=0$ as

$$\begin{aligned} R(y)-\frac{12}{\ell_\Lambda^2} &= 12H^2+\mathcal{L}^{\mu\nu}g_{\mu\nu}(y)-\frac{12}{\ell_\Lambda^2}+\mathcal{O}\left(\epsilon K^2\right) \\ &= 12\mathcal{H}_0^2+\mathcal{L}^{\mu\nu}\left(\bar{g}_{\mu\nu}-2\tilde{y}\bar{K}_{\mu\nu}-\tilde{y}^2\overline{\partial_{\tilde{y}}K_{\mu\nu}}+\cdots\right)+\mathcal{O}\left(\epsilon K^2\right), \end{aligned} \tag{6.2.39}$$

and the linear term in $\tilde{y}$ is integrated to be zero. In Eq. (6.2.37), also, we use Eq. (6.1.4) in the second equality and Eqs. (6.2.30) and (6.2.34) in the third equality where the total derivative terms are neglected. In the obtained action (6.2.37), a non-local operator $(1-\hat{H}^2\hat{\Box})^{-1}$ appears through Φ defined by Eq. (6.2.31). As mentioned above, we treat the apparently non-local operator $(1-\hat{H}^2\hat{\Box})^{-1}$ as a local one by restricting the energy scale as in Eq. (6.2.32).

In the limit $\alpha m_*^2\to 0$ which corresponds to the extreme limit of self-accelerating branch: $1-r_c^{(+)}\mathcal{H}_+\to-\infty$, Eq. (6.2.31) requires $\Phi\to\Delta h$ and Eq. (6.2.37) reduces the bulk contribution in the effective action to a non-derivative interaction between two metrics induced on the branes:

$$S_b = \frac{M_{pl}^2 m_*^2}{2} \int d^4x \sqrt{-\gamma} \left(\frac{1}{4} \Delta h^2 - \Delta h_{\mu\nu} \Delta h^{\mu\nu} \right) , \tag{6.2.40}$$

while the other part of the effective action only gives two Einstein-Hilbert actions for the induced metrics. The obtained interaction form differs from the one of the ghost-free dRGT mass terms, which means that a BD ghost appears; the system possesses an extra scalar mode in addition to the two gravitons. As discussed in Sec. 5.2 for the case with the radion stabilization field (and in Ref. [34] for the case without radion stabilization as well), the self-accelerating branch of the DGP 2-brane model inevitably exhibits either of the spin-0 ghost or the Higuchi ghost. This feature of the DGP model should explain the apparent existence of BD ghost in the extreme limit of self-accelerating branch as the appearance of the spin-0 ghost or the Higuchi ghost. When the mass squared of the massive graviton is smaller than $2H^2$, the ghost in the DGP model is the Higuchi ghost. In the specific bigravity model with the interaction Eq. (6.2.40), therefore, the BD ghost can be the Higuchi ghost, which may not be so harmful as a usual scalar ghost, as discussed in Ref. [61].

Now we return to the general case without taking the limit $\alpha m_*^2 \to 0$. In the rest of this section, we will show that the degrees of freedom in the total effective action (6.2.36) consists of one massless graviton, one massive graviton, and one scalar, by rewriting the action (6.2.36) so that the rescaled brane separation Φ, which will correspond to the radion, is manifestly treated as an independent degree of freedom. Instead of using the constraint $\Phi = \Delta h + \frac{4\alpha}{3}\bar{R}^{(1)}$ and deriving the effective action in terms of $h_{\mu\nu}^{(\pm)}$, therefore, we introduce a Lagrange multiplier λ to impose the constraint $\Phi = \Delta h + \frac{4\alpha}{3}\bar{R}^{(1)}$ in the effective action as

$$\begin{aligned} S = \frac{M_{pl}^2}{2} \Bigg[\int d^4x \sqrt{-\gamma}\, m_*^2 \Bigg\{ & \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} - \frac{3}{4} \Phi \left(1 - \hat{H}^2 \hat{\Box} \right) \Phi \\ & + \lambda \left(\Phi - \Delta h - \frac{4\alpha}{3} \bar{R}^{(1)} \right) \Bigg\} \\ + \int d^4x \sqrt{-g_{(+)}} \left(R_{(+)} - \frac{6H^2}{a_+^2} \right) & + \chi \int d^4x \sqrt{-g_{(-)}} \left(R_{(-)} - \frac{6H^2}{a_-^2} \right) \Bigg] . \end{aligned} \tag{6.2.41}$$

λ is determined by taking the variation with respect to Φ and takes a simple expression in term of Φ:

$$\lambda = \frac{3}{2} \left(1 - \hat{H}^2 \hat{\Box} \right) \Phi . \tag{6.2.42}$$

In this expression, λ seems to contain $1 - \hat{H}^2 \hat{\Box}$ that yields a higher derivative Lagrangian from $\lambda \bar{R}^{(1)}$ term, but this second-derivative operator cancels out the apparently non-local operator $(1 - \hat{H}^2 \hat{\Box})^{-1}$ in $\bar{R}^{(1)}$ after substituting this λ back into the action. In fact, the substitution Eq. (6.2.42) to Eq. (6.2.41) gives the effective action written in terms of the independent variables $h_{\mu\nu}^{(\pm)}$ and Φ up to their second derivatives as

$$S = \frac{M_{pl}^2}{2}\left[\int d^4x\sqrt{-\gamma}\, m_*^2\left\{\Delta h^2 - \Delta h^{\mu\nu}\Delta h_{\mu\nu} - \frac{3}{4}\alpha^2 H^2 \Phi\left(\Box + 4H^2\right)\Phi \right.\right.$$
$$\left. + \frac{3}{4}\Phi\left(\Phi - 2\Delta h\right) - 2\alpha\Phi\mathcal{L}^{\mu\nu}\tilde{h}_{\mu\nu}\right\}$$
$$\left. + \int d^4x\sqrt{-g_{(+)}}\left(R_{(+)} - \frac{6H^2}{a_+^2}\right) + \chi\int d^4x\sqrt{-g_{(-)}}\left(R_{(-)} - \frac{6H^2}{a_-^2}\right)\right] . \quad (6.2.43)$$

After separating the radion degrees of freedom Φ from $h_{\mu\nu}^{(\pm)}$ in this way, we can obtain the ghost-free Fierz-Pauli interaction even in the limit $\alpha m_*^2 \to 0$, by fixing the radion Φ to 0 by hand, assuming some steep stabilization potential for Φ . The last term in the curly brackets of Eq. (6.2.43) can be absorbed into the induced gravity terms by the conformal transformations $g_{\mu\nu}^{(+)} = \exp\left(m_*^2\alpha\Phi\right)\hat{g}_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)} = \exp\left([m_*^2\alpha/\chi]\Phi\right)\hat{g}_{\mu\nu}^{(-)}$: the action (6.2.43) transforms as

$$S = \frac{M_{pl}^2}{2}\left[\int d^4x\sqrt{-\gamma}\, m_*^2\left\{\Delta h^2 - \Delta h^{\mu\nu}\Delta h_{\mu\nu} + \frac{3}{4}\Phi\left(\Phi - 2\Delta h\right)\right.\right.$$
$$\left. + \frac{3}{2}\alpha^2\left(m_*^2\left(1 + \frac{1}{\chi}\right) - \frac{1}{2}H^2\right)\Phi\left(\Box + 4H^2\right)\Phi\right\}$$
$$\left. + \int d^4x\sqrt{-\hat{g}_{(+)}}\left(\hat{R}_{(+)} - \frac{6H^2}{a_+^2}\right) + \chi\int d^4x\sqrt{-\hat{g}_{(-)}}\left(\hat{R}_{(-)} - \frac{6H^2}{a_-^2}\right)\right] , \quad (6.2.44)$$

where $\hat{R}_{(+)}$ and $\hat{R}_{(-)}$ are the Ricci scalars for the respective conformal transformed metrics $\hat{g}_{\mu\nu}^{(+)}$ and $\hat{g}_{\mu\nu}^{(-)}$. Here, the radion field Φ, which is now an independent field, couples to both metrics but only through $\gamma_{\mu\nu}$. We cannot discriminate to which metric the radion field couples and two gravitons $h_{\mu\nu}^{(\pm)}$ interact with each other only through the Fierz-Pauli mass term, in our approximation where all the terms higher order in ϵ are neglected. Therefore, this system contains only two gravitons and one scalar radion, no extra degrees. On the other hand, the ghost-freeness cannot be immediately examined just by the action (6.2.44), since the constraints for $\Delta h_{\mu\nu}$ exist in the obtained action. In the next section, we will discuss whether the helicity-0 mode of the massive graviton and the scalar radion are healthy or not by writing down the effective action in terms of the matter energy-momentum tensor, integrating out the gravitational degrees of freedom.

6.3 Ghost-free condition

Now we confirm that the system contains only two gravitons and one scalar mode whose ghost-free condition is equivalent to the one discussed in Sec. 5.2. For this purpose, we analyze the equations of motion derived from the action (6.2.36) and (6.2.37) with additional matter fields localized on the branes and investigate the poles of the propagators and their coefficients. Taking the variations of the action (6.2.36) with (6.2.37) with respect to $h^{(\pm)}_{\mu\nu}$, we obtain the equations of motion as

$$\begin{aligned} \chi_\pm \mathcal{E}^{\alpha\beta}_{\mu\nu} h^{(\pm)}_{\alpha\beta} \pm 2m_*^2 \left(\Delta h_{\mu\nu} - \Delta h \gamma_{\mu\nu}\right) \\ +\frac{3}{2}m_*^2 \left[\pm\gamma_{\mu\nu} + \frac{2}{3}\alpha\mathcal{L}_{\mu\nu}\right]\left(\Delta h + \frac{4}{3}\alpha\bar{R}^{(1)}\right) = M_{pl}^{-2}T^{(\pm)}_{\mu\nu}\,, \end{aligned} \tag{6.3.1}$$

where we define $T^{(\pm)}_{\mu\nu}$ as the respective energy-momentum tensors of the localized matter fields, $\chi_+ = 1$, $\chi_- = \chi$, and

$$\begin{aligned} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \;:=\; -\frac{1}{2}\big(\Box h_{\mu\nu} + \nabla_\mu\nabla_\nu h - 2\nabla_{(\nu}\nabla_\sigma h^\sigma_{\mu)} - \gamma_{\mu\nu}\Box h + \gamma_{\mu\nu}\nabla_\alpha\nabla_\beta h^{\alpha\beta} \\ -2H^2 h_{\mu\nu} - H^2\gamma_{\mu\nu} h\big)\,, \end{aligned} \tag{6.3.2}$$

is the linearized Einstein tensor derived from the variation of the four-dimensional localized Einstein-Hilbert part of the action. Also, we approximated $a_\pm = 1$ in Eq. (6.3.1), since we need not to distinguish $a_\pm$ from unity within the approximation we work on. Here, we impose that the traceless part of $\tilde{h}_{\mu\nu}$, the averaged part of the metric perturbation (6.2.26), becomes transverse:

$$\nabla^\mu\left(\tilde{h}_{\mu\nu} - \frac{1}{4}\gamma_{\mu\nu}\tilde{h}\right) = 0\,, \tag{6.3.3}$$

by use of the gauge degrees of freedom. We decompose the difference of the metric perturbations (6.2.24) as

$$\Delta h_{\mu\nu} = \Delta h^{(TT)}_{\mu\nu} + \frac{1}{4}\phi\gamma_{\mu\nu} + \left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)\pi\,, \tag{6.3.4}$$

where $\Delta h^{(TT)}_{\mu\nu}$ is the transverse-traceless part of $\Delta h_{\mu\nu}$. The longitudinal mode π is determined by the divergence of Eq. (6.3.1): with the aid of the identities (6.2.30) and

$$\nabla^\mu \mathcal{E}^{\alpha\beta}_{\mu\nu} H_{\alpha\beta} = 0\,, \tag{6.3.5}$$

$$\nabla^\mu\left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)\Psi = \frac{3}{4}\nabla_\nu\left(\Box + 4H^2\right)\Psi\,, \tag{6.3.6}$$

$$\nabla^\mu\mathcal{L}_{\mu\nu}\Psi = 0\,, \tag{6.3.7}$$

where Ψ and $H_{\alpha\beta}$ are arbitrary scalar and tensor, respectively, the two equations derived from the divergence of Eq. (6.3.1) give the same equation as

$$\pi = \frac{\alpha}{2}\,\Xi\,, \tag{6.3.8}$$

where

$$\Xi := \left(1 - \hat{H}^2\hat{\Box}\right)^{-1}\left(2\tilde{h} - 2\hat{H}^2\phi\right)\,. \tag{6.3.9}$$

Here, the homogeneous solution that satisfies $(\Box + 4H^2)\,\pi = 0$ is ignored because the identity (6.3.6) indicates that such a solution degenerates with the transverse-traceless mode $\Delta h^{(TT)}_{\mu\nu}$. One may question whether π is uniquely determined because of the pole at $\hat{\Box} = \hat{H}^{-2}$. However, the restriction (6.2.32) has already excluded such ambiguity. Substituting Eq. (6.3.8) into Eq. (6.3.1), we obtain

$$\chi_\pm \mathcal{E}^{\alpha\beta}_{\mu\nu} h^{(\pm)}_{\alpha\beta} \pm 2m_*^2 \Delta h^{(TT)}_{\mu\nu} + m_*^2 \alpha \mathcal{L}_{\mu\nu} \left[\phi \pm \left(1 \mp \frac{\hat{\Box}}{2} \right) \Xi \right] = M_{pl}^{-2} T^{(\pm)}_{\mu\nu} \,. \tag{6.3.10}$$

The contribution from the scalar modes ϕ and Ξ in this equation of motion can be written in terms of the trace of the metric perturbation on each brane $h^{(\pm)}$. Hereafter, we eliminate the terms proportional to $\nabla_\mu \nabla_\nu \psi$ from Eq. (6.3.4) when we compute $h^{(\pm)}_{\mu\nu}$, because such terms can be erased by 4-dimensional coordinate transformations on the respective branes. Then, $h^{(\pm)}$ are found to be given by

$$\begin{aligned} 2h^{(\pm)} &:= 2\tilde{h} \pm (\phi - \Box\pi) \\ &= \pm \frac{\mathcal{H}_\pm}{\mathcal{H}_0} \left[\phi \pm \left(1 \mp \frac{\hat{\Box}}{2} \right) \Xi \right] , \end{aligned} \tag{6.3.11}$$

where in the second equality we use Eq. (6.3.9) to erase $\tilde{h}$ and Eq. (6.2.7) to rewrite $\hat{H}^2$ in terms of $\mathcal{H}_\pm$. This simplifies the equations of motion (6.3.10) as

$$\chi_\pm \mathcal{E}^{\alpha\beta}_{\mu\nu} h^{(\pm)}_{\alpha\beta} \pm 2m_*^2 \Delta h^{(TT)}_{\mu\nu} + \left(\pm 8 r_c^{(+)} \mathcal{H}_\pm \right)^{-1} \mathcal{L}_{\mu\nu} h^{(\pm)} = M_{pl}^{-2} T^{(\pm)}_{\mu\nu} \,. \tag{6.3.12}$$

The trace part of Eq. (6.3.12) gives

$$h^{(\pm)} = \frac{4M_{pl}^{-2}}{3} \left(\chi_\pm \mp \frac{1}{2 r_c^{(+)} \mathcal{H}_\pm} \right)^{-1} \frac{1}{\Box + 4H^2} T^{(\pm)} \,. \tag{6.3.13}$$

Diagonalizing the traceless part of Eq. (6.3.12) with respect to the mass eigenvalues[1], we obtain the equations of motion for the transverse-traceless part of the massless mode $h^{(0)}_{\mu\nu}$,

$$h^{(0)}_{\mu\nu} := h^{(+)}_{\mu\nu} + \chi h^{(-)}_{\mu\nu} \,, \tag{6.3.14}$$

and that of the massive mode $h^{(m)}_{\mu\nu}$

$$h^{(m)}_{\mu\nu} := \Delta h_{\mu\nu} \,, \tag{6.3.15}$$

[1]The eigenmodes coincide with the ones in Eq. (5.1.19), as is expected.

as

$$\left(\Box - 2H^2\right) h^{(0)\,TT}_{\mu\nu} = -2M_{pl}^{-2}\left[T^{(0)}_{\mu\nu} - \frac{1}{4}T^{(0)}\gamma_{\mu\nu} + \frac{1}{3}\left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)\frac{1}{\Box + 4H^2}T^{(0)}\right], \tag{6.3.16}$$

$$\left(\Box - 2H^2 - m_1^2\right) h^{(m)\,TT}_{\mu\nu} = -2M_{pl}^{-2}\left[T^{(m)}_{\mu\nu} - \frac{1}{4}T^{(m)}\gamma_{\mu\nu} + \frac{1}{3}\left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)\frac{1}{\Box + 4H^2}T^{(m)}\right], \tag{6.3.17}$$

where m_1^2 is defined in Eq. (6.2.19) and we define

$$T^{(0)}_{\mu\nu} := T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\,, \tag{6.3.18}$$

$$T^{(m)}_{\mu\nu} := T^{(+)}_{\mu\nu} - \frac{T^{(-)}_{\mu\nu}}{\chi}\,. \tag{6.3.19}$$

Using the identity (5.1.7) and eliminating some terms which can be erased by a 4-dimensional gauge transformation, just in the same way as deriving Eq. (5.1.10), we obtain $h^{(0)}_{\mu\nu}$ and $h^{(m)}_{\mu\nu}$ as

$$\begin{aligned} h^{(0)\,TT}_{\mu\nu} &= -2M_{pl}^{-2}\left[\frac{1}{\Box - 2H^2}\left\{T^{(0)}_{\mu\nu} - \frac{1}{4}T^{(0)}\gamma_{\mu\nu} + \frac{1}{3\left(-2H^2\right)}\left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)T^{(0)}\right\}\right. \\ &\quad \left. - \frac{1}{3\left(-2H^2\right)}\left(-\frac{\Box}{4}\gamma_{\mu\nu}\right)\frac{1}{\Box + 4H^2}T^{(0)}\right], \end{aligned} \tag{6.3.20}$$

$$\begin{aligned} h^{(m)\,TT}_{\mu\nu} &= -2M_{pl}^{-2}\left[\frac{1}{\Box - 2H^2 - m_1^2}\left\{T^{(m)}_{\mu\nu} - \frac{1}{4}T^{(m)}\gamma_{\mu\nu}\right.\right. \\ &\quad \left. + \frac{1}{3\left(m_1^2 - 2H^2\right)}\left(\nabla_\mu\nabla_\nu - \frac{\Box}{4}\gamma_{\mu\nu}\right)T^{(m)}\right\} \\ &\quad \left. - \frac{1}{3\left(m_1^2 - 2H^2\right)}\left(-\frac{\Box}{4}\gamma_{\mu\nu}\right)\frac{1}{\Box + 4H^2}T^{(m)}\right]. \end{aligned} \tag{6.3.21}$$

Now, we will examine the sign of the coupling of the scalar modes to the localized matter fields. The nature of the helicity-0 mode in the massive graviton is already known: it becomes Higuchi ghost when $m_1^2 < 2H^2$ and healthy otherwise. Therefore, we focus on the spin-0 mode in the following. In order to investigate the coupling to $T^{(+)}_{\mu\nu}$, first, we set $T^{(-)}_{\mu\nu} = 0$ for simplicity. In $h^{(+)}_{\mu\nu} = h^{(0)}_{\mu\nu}/(1+\chi) + h^{(m)}_{\mu\nu}/(1+\chi^{-1})$, the coefficient of the pole $(\Box + 4H^2)^{-1}$, which corresponds to the spin-0 mode, is extracted as

$$h^{(+)}_{\mu\nu} \supset \frac{M_{pl}^{-2}}{3}\left[\frac{2r_c^{(+)}\mathcal{H}_+}{2r_c^{(+)}\mathcal{H}_+ - 1} + \frac{1}{1+\chi}\left(-1 + \frac{2\chi H^2}{m_1^2 - 2H^2}\right)\right]\frac{1}{\Box + 4H^2}T^{(+)}\gamma_{\mu\nu}\,. \tag{6.3.22}$$

Integrating out the spin-0 degree of freedom, we obtain the contribution of the pole at

$\Box = -4H^2$ to the effective action in terms of $T^{(+)}_{\mu\nu}$ as

$$-\int d^4x\sqrt{-\gamma}\, h^{(+)}_{\mu\nu}T^{\mu\nu}_{(+)} \supset \frac{M_{pl}^{-2}}{3}\beta_+\int d^4x\sqrt{-\gamma}\, T^{(+)}\frac{1}{\Box+4H^2}T^{(+)}\,, \tag{6.3.23}$$

where

$$\beta_+ \;:=\; \frac{1}{1-2r_c^{(+)}\mathcal{H}_+} - \frac{\chi}{1+\chi}\frac{m_1^2}{m_1^2-2H^2}\,, \tag{6.3.24}$$

is the minus of the part embraced by the square brackets in Eq. (6.3.22). In order to obtain the healthy spin-0 mode, $\beta_+ > 0$ must be imposed. Otherwise, the spin-0 degree of freedom becomes ghost. Here, the sign of $1-2r_c^{(+)}\mathcal{H}_+$ plays an important role, as is expected from the discussion in Sec. 5.2. When the condition $1-2r_c^{(+)}\mathcal{H}_+>0$ is satisfied, the first term in Eq. (6.3.24) is positive and greater than unity. Considering sufficiently large m_1^2, the second term becomes negative and its absolute value is less than unity, and then β_+ becomes positive in total. In case with $1-2r_c^{(+)}\mathcal{H}_+<0$, on the other hand, the positiveness of β_+ imposes $m_1^2-2H^2<0$, namely, the helicity-0 mode of the massive graviton becomes Higuchi ghost.

Next, we consider $T^{(+)}_{\mu\nu}=0$ instead of $T^{(-)}_{\mu\nu}=0$ and investigate the coupling to $T^{(-)}_{\mu\nu}$ similarly. the coefficient of the pole $(\Box+4H^2)^{-1}$ in $h^{(-)}_{\mu\nu}$ is given as

$$-\int d^4x\sqrt{-\gamma}\, h^{(-)}_{\mu\nu}T^{\mu\nu}_{(-)} \supset \frac{M_{pl}^{-2}}{3}\beta_-\int d^4x\sqrt{-\gamma}\, T^{(-)}\frac{1}{\Box+4H^2}T^{(-)}\,, \tag{6.3.25}$$

where

$$\beta_- := \frac{1}{\chi}\left[\frac{1}{1+2r_c^{(-)}\mathcal{H}_-} - \frac{1}{1+\chi}\frac{m_1^2}{m_1^2-2H^2}\right]. \tag{6.3.26}$$

Again, the signature of $1+2r_c^{(-)}\mathcal{H}_-$, the other quantity that determines the branches in the DGP model derived in Sec. 5.2, becomes essential: β_- is positive and the graviton sector is free from Higuchi ghost when $1+2r_c^{(-)}\mathcal{H}_->0$ and the first term is larger than the second one in magnitude.

Suppose that m_1^2 is sufficiently large, the ghost-free conditions $\beta_\pm>0$ are satisfied when $1\mp 2r_c^{(\pm)}\mathcal{H}_\pm>0$, respectively. In the background energy scale we consider in Eq. (6.2.19), however, both β_+ and β_- can be negative when m_1^2 is larger than but close to $2H^2$. We need to derive the critical values of m_1^2 at which β_+ or β_- crosess zero. Since the spin-0 mode couples to both traces of the matter energy momentum tensors, $T_{(+)}$ and $T_{(-)}$, we expect that the β_+ and β_- cross zero simultaneously. In fact, at the leading order of the gradient expansion, $\beta_\pm$ are rewritten by use of Eqs. (6.2.7) and (6.2.19) as

$$\beta_\pm = \frac{1}{1+\chi}\frac{m_1^2}{m_1^2-2H^2}\frac{1\pm 2r_c^{(\mp)}\mathcal{H}_\mp}{1\mp 2r_c^{(\pm)}\mathcal{H}_\pm}\,. \tag{6.3.27}$$

Then, one can find that the conditions $\beta_\pm>0$ imply $1\pm 2r_c^{(\mp)}\mathcal{H}_\mp>0$, when $m_1^2>2H^2$

and $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$. When both $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm > 0$ are satisfied, the ghost-free normal branch is chosen, and otherwise is the self-accelerating branch, as in the case with the radion stabilization. This simultaneous sign flip of $\beta_\pm$ occurs at the transition from the ghost-free normal branch to the self-accelerating branch, and at the same time either β_+ or β_- diverges, which means the divergence of metric perturbation and the breakdown of the perturbative approach. These features are exactly the same ones that we have already shown in the presence of the radion stabilization at the end of Sec. 4.2.2 [2].

Finally, we examine how one can construct the background spacetime that satisfies the conditions for the ghost-free normal branch. From the normal-branch conditions $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm \geq 0$ with the aid of Eqs. (6.2.7) and (6.2.19), we obtain a constraint for $\mathcal{H}_0$ as

$$-1 + (1+\chi)\,\frac{H^2}{m_1^2} < 2r_c^{(-)}\mathcal{H}_0 < \chi - (1+\chi)\,\frac{H^2}{m_1^2}\,. \tag{6.3.28}$$

This constraint requires $-1 + (1+\chi)\,H^2/m_1^2 < \chi - (1+\chi)\,H^2/m_1^2$, equivalently $m_1^2 > 2H^2$. Hence, the normal-branch conditions automatically impose that the helicity-0 mode of the massive graviton is healthy, which can be easily satisfied by tuning y_0^+. Furthermore, Eq. (6.3.28) also requires tuning of $\mathcal{H}_0$ depending on H/m_1 and $r_c^{(\pm)}$.

6.4 Nonlinear generalization

We extend the above method to the nonlinear in the metric perturbation, while we do not extend the gradient expansion to the higher order. Here, we again assume that

$$\epsilon := |K\tilde{y}^+|\,, \tag{6.4.1}$$

is small with $g_{\mu\nu}^{(\pm)} \sim \mathcal{O}(1)$, $\Delta g_{\mu\nu} := g_{\mu\nu}^{(+)} - g_{\mu\nu}^{(-)} \sim \mathcal{O}(\epsilon)$, and $\nabla^2 \sim \ell_\Lambda^{-2} \sim \mathcal{O}(|K|^2/\epsilon)$, as in the previous sections. Then, the Hamiltonian constraint (6.1.6) and the junction conditions (6.1.13) imply Eq. (6.2.22), as is discussed in Sec. 6.2. The following procedure to derive the effective action is similar to the one in Sec. 6.2. From Eq. (6.1.9), $\bar{K}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ becomes

$$\begin{aligned}\bar{K}_{\mu\nu} &= \frac{\Delta g_{\mu\nu}}{4\Phi}\,, &(6.4.2)\\ \bar{g}_{\mu\nu} &= \tilde{g}_{\mu\nu} + \Phi\nabla_\mu\nabla_\nu\Phi + \frac{\Phi^2}{\ell_\Lambda^2}g_{\mu\nu} + \mathcal{O}\left(\epsilon^2\right)\,, &(6.4.3)\end{aligned}$$

where

$$\begin{aligned}\Phi &:= Ny_0\,, &(6.4.4)\\ \tilde{g}_{\mu\nu} &:= \frac{g_{\mu\nu}^{(+)} + g_{\mu\nu}^{(-)}}{2}\,. &(6.4.5)\end{aligned}$$

[2]These features are also shown in Ref. [34] using the original DGP 2-brane model action without radion stabilization, not the effective one we analyzed here.

The Hamiltonian constraint (6.1.6) evaluated at $y = 0$ imposes

$$\bar{R} - \frac{12}{\ell_\Lambda^2} = \frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{16\Phi^2} \,. \tag{6.4.6}$$

Here and hereafter, the tensor indices are raised by $\tilde{g}_{\mu\nu}$. From Eqs. (6.1.11) and (6.1.12), the bulk action is written as

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2}\frac{1}{r_c^{(+)}}\int d^4xN\Bigg[2y_0\sqrt{-\bar{g}}\left(\bar{R} - \frac{12}{\ell_\Lambda^2}\right) \\ &\qquad -\frac{2y_0^3}{3}\overline{\frac{\delta\left(\sqrt{-g}\left(R - \frac{12}{\ell_\Lambda^2}\right)\right)}{\delta g_{\mu\nu}}}N^2\left(\frac{1}{\Phi}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Phi + \frac{1}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right) \\ &\qquad +\mathcal{O}\left(|K|\epsilon^2\right)\Bigg] \,. \end{aligned} \tag{6.4.7}$$

Here, the linear term in y in the integrand is integrated to be zero, the term in which $\delta\left(\sqrt{-g}\left(R - 12\ell_\Lambda^{-2}\right)\right)/\delta g_{\mu\nu}$ is differentiated with respect to y turns out to be higher order of the gradient expansion, and Eq. (6.1.4) with Eq. (6.2.22) is used. The operator $\overline{\delta\left(\sqrt{-g}\left(R - 12\ell_\Lambda^{-2}\right)\right)/\delta g_{\mu\nu}}$ is derived as

$$\int d^4x\overline{\frac{\delta\left(\sqrt{-g}\left(R - \frac{12}{\ell_\Lambda^2}\right)\right)}{\delta g_{\mu\nu}}}h_{\mu\nu} = \int d^4x\left(\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{\mu\nu}h_{\mu\nu} + \mathcal{O}\left(|K|^2\right)\right) \,, \tag{6.4.8}$$

for an arbitrary tensor $h_{\mu\nu}$. Here we defined $\tilde{\mathcal{L}}^{\mu\nu}$ as

$$\tilde{\mathcal{L}}^{\mu\nu} := \tilde{\nabla}^\mu\tilde{\nabla}^\nu - \left(\tilde{\Box} + \frac{3}{\ell_\Lambda^2}\right)\tilde{g}^{\mu\nu} \,, \tag{6.4.9}$$

where $\tilde{\nabla}$ is the covariant differentiation associated with $\tilde{g}_{\mu\nu}$. Using Eqs. (6.4.6) and (6.4.8), the bulk action becomes

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2}\frac{2}{r_c^{(+)}}\int d^4x\sqrt{-\tilde{g}}\Bigg[\frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{16\Phi} + \frac{\Phi}{3}\tilde{\mathcal{L}}^{\mu\nu}\left(\left(\tilde{\nabla}_\mu\Phi\right)\left(\tilde{\nabla}_\nu\Phi\right) - \frac{\Phi^2}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right) \\ &\qquad +\mathcal{O}\left(|K|\epsilon^2\right)\Bigg] \,. \end{aligned} \tag{6.4.10}$$

Here we use an identity valid at the leading order of the expansion in ϵ: $\tilde{\mathcal{L}}^{\mu\nu}\tilde{\nabla}_\mu A_\nu = 0$ for an arbitrary vector A_ν. The analysis in the linear metric perturbation suggests that treating the radion Φ as an independent variable should make it easier to investigate the structure of the interaction between $g_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)}$. Therefore, we impose the Hamiltonian constraint (6.4.6) with a Lagrange multiplier λ in the obtained effective action as

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2}\frac{2}{r_c^{(+)}}\int d^4x\sqrt{-\tilde{g}}\Bigg[\frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{16\Phi} + \frac{\Phi}{3}\tilde{\mathcal{L}}^{\mu\nu}\left(\left(\tilde{\nabla}_\mu\Phi\right)\left(\tilde{\nabla}_\nu\Phi\right) - \frac{\Phi^2}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right) \\ &\qquad +\lambda\left(\bar{R} - \frac{12}{\ell_\Lambda^2} - \frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{16\Phi^2}\right)\Bigg] \,, \end{aligned} \tag{6.4.11}$$

and take the variation with respect to Φ to eliminate λ using the equation of motion for Φ. To do this, we need to compute $\bar{R} - 12\ell_\Lambda^{-2}$:

$$\bar{R} - \frac{12}{\ell_\Lambda^2} = \tilde{R} - \frac{12}{\ell_\Lambda^2} + \tilde{\mathcal{L}}^{\mu\nu}\left(\Phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Phi + \frac{\Phi^2}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right) + \mathcal{O}\left(|K|\epsilon\right) . \tag{6.4.12}$$

By taking the variation of Eq. (6.4.11) with respect to Φ, we obtain the equation of motion for Φ. This equation of motion is cosiderably simplified with the aid of the identity $\tilde{\mathcal{L}}_{\mu\nu}\tilde{\nabla}^\mu\tilde{\nabla}^\nu\Psi = \tilde{\nabla}^\mu\tilde{\nabla}^\nu\tilde{\mathcal{L}}_{\mu\nu}\Psi = 0$ for an arbitrary scalar Ψ, which holds at the leading order of the expansion in ϵ, as

$$2\mathcal{L}_\lambda\lambda = \mathcal{L}_\lambda\Phi \, , \tag{6.4.13}$$

where

$$\mathcal{L}_\lambda := \left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Phi + \frac{\Phi}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right)\tilde{\mathcal{L}}^{\mu\nu} + \frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{16\Phi^3} \, . \tag{6.4.14}$$

Eq. (6.4.13) yields a extremely simple relation $\lambda = \frac{\Phi}{2}$, if one ignore the homogeneous solution $\mathcal{L}_\lambda\lambda = 0$. This can be achieved as in the case with $(1 - \hat{H}^2\hat{\Box})^{-1}$ in Sec. 6.2 by expanding $\mathcal{L}_\lambda^{-1}$ as a local operator. It is difficult to derive the explicit energy regime where such expansion is valid, because the condition for the energy scale depends on Φ. Here, we just assume that the energy scale is low enough to satisfy this condition. Then, we can substitute $\lambda = \frac{\Phi}{2}$ into Eq. (6.4.11), obtaining the 4-dimensional effective action from the bulk part of the action as

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2}\frac{2}{r_c^{(+)}}\int d^4x\sqrt{-\tilde{g}}\left[\frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{32\Phi} + \frac{\Phi}{2}\left(\tilde{R} - \frac{12}{\ell_\Lambda^2}\right)\right. \\ &\qquad \left. - \frac{\Phi}{6}\tilde{\mathcal{L}}^{\mu\nu}\left(\left(\tilde{\nabla}_\mu\Phi\right)\left(\tilde{\nabla}_\nu\Phi\right) - \frac{\Phi^2}{\ell_\Lambda^2}\tilde{g}_{\mu\nu}\right)\right] + \mathcal{O}(\epsilon^2) \\ &= \frac{M_{pl}^2}{2}\frac{2}{r_c^{(+)}}\int d^4x\sqrt{-g}\left[\frac{\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}}{32\Phi} - \frac{1}{2\ell_\Lambda^2}\Phi^2\left(\Box + \frac{4}{\ell_\Lambda^2}\right)\Phi + \frac{\Phi}{2}\left(\tilde{R} - \frac{12}{\ell_\Lambda^2}\right)\right. \\ &\qquad \left. - \frac{1}{6}\left(\nabla_\mu\Phi\right)\left(\nabla_\nu\Phi\right)\left(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box - R^{\mu\nu}\right)\Phi\right] + \mathcal{O}(\epsilon^2) \, , \end{aligned} \tag{6.4.15}$$

where, in the second equality, we introduce a metric $g_{\mu\nu}$, which is indistinguishable from $g_{\mu\nu}^{(\pm)}$ and $\tilde{g}_{\mu\nu}$ at the leading order of gradient expansion, and its covariant differentiation ∇ and we use $R_{\mu\nu} = 3\ell_\Lambda^{-2}g_{\mu\nu} + \mathcal{O}\left(|K|^2\right)$. The first term in the effective action (6.4.15) gives a non-derivative interaction term between two metrics that takes the Fierz-Pauli form $\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu}$. This Fierz-Pauli interaction can be written in terms of dRGT mass terms, $V_n := \epsilon^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_n}\mathcal{U}^{\mu_1}_{\nu_1}\cdots\mathcal{U}^{\mu_n}_{\nu_n}$ with $\mathcal{U}^\mu_\nu := \sqrt{g^{\mu\rho}_{(+)}g_{(-)\,\rho\nu}}$, as

$$\Delta g^2 - \Delta g_{\mu\nu}\Delta g^{\mu\nu} = -\frac{8}{c_1 + 4c_2 + 6c_3}\sum_n c_n V_n \, , \tag{6.4.16}$$

using the condition where both $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$ identically recover Minkowski geometry (and hence $\Delta g_{\mu\nu} \to 0$) in the limit where the energy densities of matter on the branes and the brane tensions are sent to zero, which restricts the coefficients c_n as

$$\begin{aligned} c_4 &= -\frac{1}{24}\left(c_1 + 6c_2 + 18c_3\right) , \\ c_0 &= -3\left(c_1 + 2c_2 + 2c_3\right) . \end{aligned} \tag{6.4.17}$$

Although we wrote the interaction between two gravitons in terms of dRGT ones, we cannot investigate the form of mass interactions at higher order of $\mathcal{U}^\mu_\nu - \delta^\mu_\nu \sim \mathcal{O}(\epsilon)$ at the order of the present approximation. On the other hand, the action for Φ in the second line in Eq. (6.4.15) is a cubic Galileon [56, 57]. As in Sec. 6.2, $\Phi\tilde{R}$ term can be absorbed into the induced gravity terms by conformal transformations for $g^{(\pm)}_{\mu\nu}$ at the leading order of the gradient expansion, which produces a kinetic term for Φ. Here, the radion Φ again couples only to one metric $g_{\mu\nu}$, not to the distinguishable two. Therefore, the whole action $S_b + S_+ + S_-$ describes a well-known ghost-free system which contains two interacting spin-2 degrees of freedom and one spin-0 degree of freedom. If we want to investigate the coupling of radion as a doubly coupled matter, we have to extend the above analysis to the higher order of the gradient expansion. However, such extension will be contaminated by the other massive KK gravitons. At the higher order of the gradient expansion, in fact, there will appear higher derivative terms, which may yield extra degrees of freedom. Therefore, our method appears to be ineligible to obtain a doubly coupled matter model or the structure of the mass interaction non-linear in $\mathcal{U}^\mu_\nu - \delta^\mu_\nu \sim \mathcal{O}(\epsilon)$.

Chapter 7

Summary

In our work, we attempted to embed the ghost-free bigravity, whose interaction terms between two metrics are strictly constrained by the ghost-free condition, into higher dimensional gravity. As the higher dimensional theory, we considered the DGP 2-brane model in which the branes accompany localized gravity terms on them.

First, we showed that at the low energies the graviton's mass spectrum of the DGP 2-brane model is identical to the one of the ghost-free bigravity, namely, that the mass of the lowest massive KK graviton is much smaller than the ones of the other KK gravitons in the DGP 2-brane model, when the brane separation is much shorter than the length scale r_c. Furthermore, we introduced Goldberger-Wise stabilization mechanism so that the radion degree of freedom is erased and the mass spectrum of the whole system contains only one massless graviton and one massive graviton at low energies. In order to remove the radion degrees of freedom, however, we should avoid ghost instability in the radion sector. The solution of the DGP 2-brane model has two branches: the normal branch in which the system is free from ghost and self-accelerating branch in which either of the radion or helicity-0 modes of KK gravitons becomes ghost. Therefore, in order to obtain bigravity as the low-energy effective theory, we should choose the normal branch. We derived the normal-branch conditions under which we showed that the system is free from ghost instabilities.

In the normal branch, the DGP 2-brane model has no scalar ghost, and hence the bigravity obtained as its low-energy effective theory is expected to be identical to the ghost-free bigravity that is free from BD ghost and contains only two gravitons. In Chapter. 5, we explicitly showed that their equation of motions are identical at the linear order. Furthermore, we analyzed the appearance of instabilities such as Higuchi ghost and spin-0 tachyon in the bigravity from the DGP 2-brane model and the ghost-free bigravity. We showed that in both models the de Sitter solution remains ghost-free as long as the ghost-free branch is chosen. However, we found that it is difficult to satisfy the ghost-free conditions in the DGP 2-brane model if the energy density on the branes becomes slightly high. Consequently, the spin-0 mode becomes a tachyon and one cannot stabilize the brane separation with Goldberger-Wise scalar stabilization mechanism. On the other hand, the ghost-free bigravity does not have such a scalar mode corresponding to the brane separation in the DGP 2-brane model, the tachyonic instability in the DGP 2-brane model does not exist in the ghost-free bigravity. Therefore, the DGP 2-brane model with a scalar stabilization coincides with the ghost-free bigravity only in the very limited low-energy regime, not even in slightly high energy regime.

In Chapter. 6, we intended to derive the ghost-free bigravity action with a single scalar field from the DGP 2-brane model without stabilization, by solving the bulk equations for given boundary metrics $g_{\mu\nu}^{(\pm)}$ on the branes and integrating out the bulk degrees of freedom at the leading order of the gradient expansion where the brane separation is so close that the difference between $g_{\mu\nu}^{(\pm)}$ is sufficiently small. Then, we succeeded in deriving an action of a healthy bigravity written in terms of $g_{\mu\nu}^{(\pm)}$ with a scalar field, as the one of a 4-dimensional effective theory of the DGP 2-brane model: the interaction between $g_{\mu\nu}^{(\pm)}$ is reduced to the Fierz-Pauli mass term, which is equivalent to the dRGT mass terms at the leading order of the gradient expansion. The scalar field effectively couples to only one of the metrics $g_{\mu\nu}^{(\pm)}$ at the leading order of the gradient expansion and its equation of motion does not contain higher-order time derivative. To satisfy the normal-branch condition with this setup, we need to tune the brane tension so that $|K|$ is sufficiently small. This tuning is easily broken by additional matter fields, and so the energy density of the matter fields on the branes must be small enough.

We found that one can obtain the Fierz-Pauli mass interaction naturally from the DGP 2-brane model at the lowest order of the gradient expansion. However, it is difficult to obtain the nonlinear dRGT mass terms by extending this method to the higher order of the gradient expansion, because the higher-order terms will be contaminated by other bulk degrees of freedom. The terms higher in the gradient expansion will includes complex and higher-derivative interaction terms which will lead extra degrees of freedom in addition to two gravitons and one scalar radion, corresponding to other bulk degrees of freedom. It will be possible to investigate the nonlinear mass interactions between two metrics by taking the limit $\alpha m_*^2 \to 0$, although in this case the obtained mass interaction will not take the dRGT form. This is because the self-accelerating branch is chosen and radion or Higuchi ghost appears. If ghost appears, the interaction between two metrics $g_{\mu\nu}^{(\pm)}$ should be different from the dRGT one, as is shown in Eq. (6.2.40) at the perturbative level. However, it might be suggestive that the Fierz-Pauli mass term was recovered by fixing the radion by hand in Eq. (6.2.43).

We concluded that the DGP 2-brane model can reproduce the ghost-free bigravity as a low-energy effective theory when the energy densities on the branes are limited to be sufficiently small and the two metrics on the branes are almost identical. As future work, we will investigate stabilization models that enable to sustain the brane separation even with high energy densities on the branes, and also the nonlinear mass interactions by considering more extended models such as Gauss-Bonnet gravity.

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Appendix A

ADM decomposition

In this section, we review ADM(Arnowitt, Deser and Misner) decomposition [58] where the D-dimensional spacetime is foliated by $D-1$-dimensional hypersurfaces and its normal vector, and then derive the evolution equations and the constraints from the Einstein equation. Here, we refer Refs. [59, 60]. In this section, we use the abstract index notation for tensor indices, using Latin alphabet.

We consider to foliate D-dimensional spacetime $(M,\, g)$ using $D-1$-dimensional hypersurfaces Σ. We define a unit normal vector n^a with respect to Σ which satisfies $g_{ab}n^a n^b = \epsilon$ where ϵ is given as -1 when n^a is timelike, or $+1$ when n^a is spacelike. The necessary and sufficient condition for which a normal vector field n^a exists with respect to a family of hypersurfaces is given as

$$\nabla_{[a} n_{b]} = 0\,, \tag{A.0.1}$$

where ∇_a is the covariant differentiation defined on the spacetime $(M,\, g)$, which is known as Frobenius's theorem. The metric induced on the hypersurfaces Σ is given as

$$\gamma_{ab} = g_{ab} - \epsilon n_a n_b\,. \tag{A.0.2}$$

The induced metric γ_{ab} is by definition orthogonal to n^a: $\gamma_{ab}n^a = 0$, namely, γ_{ab} is the projection operator from the spacetime $(M,\, g)$ into the hypersurfaces Σ. The covariant differentiation D_a defined on Σ is obtained as the projection of ∇_a:

$$D_a T^{b_1, b_2, \cdots}_{a_1, a_2, \cdots} := \gamma^a_c \gamma^{c_1}_{a_1} \gamma^{c_2}_{a_2} \cdots \gamma^{b_1}_{d_1} \gamma^{b_2}_{d_2} \cdots \nabla_c T^{d_1, d_2, \cdots}_{c_1, c_2, \cdots}\,. \tag{A.0.3}$$

This definition satisfies $D_a \gamma_{bc} = 0$, indicating that D_a is in fact the covariant differentiation with respect to γ_{ab}.

Furthermore, we define a extrinsic curvature K_{ab} as

$$K_{ab} := \gamma^c_a \gamma^d_b \nabla_c n_d\,. \tag{A.0.4}$$

This present how the hypersurfaces Σ are embedded on the spacetime $(M,\, g)$. K_{ab} is a symmetric tensor when $\nabla_{[a} n_{b]} = 0$ is satisfied. We can show that the extrinsic curvature K_{ab} is equal to the half of the Lie derivative of the metric induced on the hypersurfaces

γ_{ab} with respect to n^a: operating the Lie derivative with respect to n^a on γ_{ab}, we obtain

$$\begin{aligned}\mathcal{L}_n\gamma_{ab} &= \mathcal{L}_n\left(g_{ab} - \epsilon n_a n_b\right)\\ &= \nabla_a n_b + \nabla_b n_a - \epsilon\left((n^c\nabla_c n_a)n_b + (n^c\nabla_c n_b)n_a\right)\\ &= \left(\delta_a^c - \epsilon n^c n_a\right)\nabla_c n_b + \left(\delta_b^c - \epsilon n^c n_b\right)\nabla_c n_a\\ &= K_{ab} + K_{ba} = 2K_{ab}\,. \end{aligned} \tag{A.0.5}$$

Therefore, we obtain $K_{ab} = \frac{1}{2}\mathcal{L}_n\gamma_{ab}$, which implies that the extrinsic curvature corresponds to the "velocity" of γ_{ab}.

A.1 Gauss-Codazzi Equations

In this subsection, we derive the equations which relate the geometrical variables in the D-dimensional spacetime with the ones on the $D-1$-dimensional hypersurfaces, so-called Gauss-Codazzi equations.

First, we derive the Gauss equation. By definition of Riemann tensor, the $D-1$-dimensional Riemann tensor evaluated from γ_{ab} is given using an arbitrary vector V^a on Σ as

$${}^{(D-1)}R_{abcd}V^d = (D_aD_b - D_bD_a)V_c\,. \tag{A.1.1}$$

Rewriting the right-hand side of this equation in terms of the D-dimensional variables as

$$\begin{aligned}D_{[a}D_{b]}V_c &= \gamma^f_{[a}\gamma^g_{b]}\gamma^h_c\nabla_f\left(\gamma^d_g\gamma^e_h\nabla_dV_e\right)\\ &= \gamma^f_{[a}\gamma^g_{b]}\gamma^h_c\left[\gamma^d_g\gamma^e_h\nabla_f\nabla_dV_e - \epsilon\left(\nabla_fn_g\right)n^d\gamma^e_h\nabla_dV_e - \epsilon\left(\nabla_fn_h\right)n^e\gamma^d_g\nabla_dV_e\right]\\ &= \gamma^f_{[a}\gamma^d_{b]}\gamma^e_c\nabla_f\nabla_dV_e - \epsilon K_{[ab]}n^d\gamma^e_c\nabla_dV_e - \epsilon K_{c[a}n^e\gamma^d_{b]}\nabla_dV_e\\ &= \gamma^f_a\gamma^d_b\gamma^e_c\,{}^{(D)}R_{fdeg}V^g + \epsilon K_{c[a}K^e_{b]}V_e\,, \end{aligned} \tag{A.1.2}$$

where we use $\gamma_{ab} = g_{ab} - \epsilon n^a n^b$, $\gamma_{ab}n^a = 0$, $n^aV_a = 0$, and that tensor K_{ab} is symmetric, we obtain the Gauss equation:

$${}^{(D-1)}R_{abcd} = \gamma^e_a\gamma^f_b\gamma^g_c\gamma^h_d\,{}^{(D)}R_{efgh} + \epsilon\left(K_{ac}K_{bd} - K_{bc}K_{ad}\right)\,. \tag{A.1.3}$$

Next, we derive the Codazzi equation. The definition of the D-dimensional Ricci tensor gives

$$\gamma^b_a\,{}^{(D)}R^d{}_{bdc}n^c = \gamma^b_a\left(\nabla^d\nabla_b - \nabla_b\nabla^d\right)n_d\,. \tag{A.1.4}$$

Now, we rewrite the right-hand side of this equation in terms of the $D-1$-dimensional variables. The first term in the right-hand side of Eq. (A.1.4) becomes

$$\begin{aligned}\gamma_a^b \nabla^d \nabla_b n_d &= \gamma_a^b \nabla^d \left[\left(\gamma_b^c + \epsilon n^c n_b\right) \left(\gamma_d^e + \epsilon n^e n_d\right) \nabla_c n_e \right] \\ &= \gamma_a^b \nabla^d K_{bd} + \epsilon K_{ae} n^c \nabla_c n^e \\ &= \gamma_a^b \left(\gamma^{cd} + \epsilon n^c n^d\right) \nabla_d K_{bc} + \epsilon K_{ae} n^c \nabla_c n^e \\ &= D^b K_{ab} + \epsilon \gamma_a^b n^c \nabla_c \left(K_{bd} n^d\right) \\ &= D^b K_{ab} \,, \end{aligned} \tag{A.1.5}$$

while the second term is

$$\begin{aligned}\gamma_a^b \nabla_b \nabla^d n_d &= D_a \left[\left(\gamma^{de} + \epsilon n^d n^e\right) \nabla_d n_e \right] \\ &= D_a K \,. \end{aligned} \tag{A.1.6}$$

Then, we obtain the Codazzi equation from Eq. (A.1.4) as

$$\gamma_a^b \, {}^{(D)}R_{bc} n^c = D_b K_a^b - D_a K \,. \tag{A.1.7}$$

Notice that the Gauss equation and the Codazzi equation are purely geometrical relations independent of what the gravitational model is.

A.2 Constraints and Evolution Equation

In this subsection, we will rewrite the D-dimensional Einstein equation in terms of the $D-1$-dimensional geometrical variables.

First, we rewrite the pure n^a component of the D-dimensional Einstein equation $G_{ab} n^a n^b = M_D^{-(D-3)} T_{ab} n^a n^b$, where M_D is the D-dimensional Planck mass. From the Gauss equation Eq. (A.1.3), the $D-1$-dimensional Ricci scalar is written as

$$\begin{aligned}{}^{(D-1)}R &= {}^{(D)}R_{abcd} \gamma^{ac} \gamma^{bd} + \epsilon \left(K^2 - K_{ab} K^{ab}\right) \\ &= {}^{(D)}R_{abcd} \left(g^{ac} - \epsilon n^a n^c\right) \left(g^{bd} - \epsilon n^b n^d\right) + \epsilon \left(K^2 - K_{ab} K^{ab}\right) \\ &= -2\epsilon \left({}^{(D)}R_{ab} - \frac{1}{2} \, {}^{(D)}R g_{ab}\right) n^a n^b + \epsilon \left(K^2 - K_{ab} K^{ab}\right) \,. \end{aligned} \tag{A.2.1}$$

Then, we obtain the pure n^a component of the D-dimensional Einstein equation written in terms of the $D-1$-dimensional intrinsic curvature and the extrinsic curvature as

$$\frac{1}{2} \left(-\epsilon \, {}^{(D-1)}R + K^2 - K_{ab} K^{ab}\right) = G_{ab} n^a n^b = M_D^{-(D-3)} T_{ab} n^a n^b \,. \tag{A.2.2}$$

Next, we rewrite the mixed components of the D-dimensional Einstein equation $G_{ab} n^a \gamma_c^b = M_D^{-(D-3)} T_{ab} n^a \gamma_c^b$ using the Codazzi equation. Using the Einstein equation in the left-hand side of the Codazzi equation Eq. (A.1.7), we obtain

$$D_b K_a^b - D_a K = {}^{(D)}R_{bc} n^b \gamma_a^c = M_D^{-(D-3)} T_{bc} n^b \gamma_a^c \,. \tag{A.2.3}$$

The extrinisic curvature defined in Eq. (A.0.4) contains only first derivative. Therefore, the pure n^a component and the mixed components of the Einstein equation (A.2.2) and (A.2.3) do not contain the second derivative of γ_{ab} in the direction of n^a: they are constraints when we consider the evolution of γ_{ab} along the n^a direction. The evolution equation which contains second derivative in the direction of n^a is derived by the remaining component of the Einstein equation. For this purpose, we introduce a coordinates system in which we choose the coordinate t so that Σ are defines as $t =$ const. hypersurfaces, and decompose the metric as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \epsilon N^2 dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) . \tag{A.2.4}$$

Here, $n_a = \epsilon N \nabla_a t = (\epsilon N, 0)$, $n^a = N^{-1}\left(1, -N^i\right)$ satisfies $D_{[a}n_{b]} = 0$.

Now, we rewrite the remaining components of the D-dimensional Einstein equation $G_{ab}\gamma^a_c\gamma^b_d = M_D^{-(D-3)}T_{ab}\gamma^a_c\gamma^b_d$ in terms of $D-1$-dimensional geometrical variables. Multiplying γ^{bd} to the Gauss equation Eq. (A.1.3) gives

$$\begin{aligned} {}^{(D-1)}R_{ac} &= \gamma^e_a\gamma^{fh}\gamma^g_c\,{}^{(D)}R_{efgh} + \epsilon(K_{ac}K - K_{ba}K^b_c) \\ &= \gamma^e_a\gamma^g_c\,{}^{(D)}R_{eg} - \epsilon\gamma^e_a\gamma^g_c n^f n^h\,{}^{(D)}R_{efgh} + \epsilon(K_{ac}K - K_{ba}K^b_c)\,, \end{aligned} \tag{A.2.5}$$

The second term in the right-hand side of Eq. (A.2.5) becomes

$$\gamma^e_a\gamma^g_c n^f n^h\,{}^{(D)}R_{efgh} = \gamma^e_a\gamma^g_c n^b(\nabla_e\nabla_b - \nabla_b\nabla_e)n_g\,. \tag{A.2.6}$$

Using

$$\begin{aligned} \nabla_a n_b &= \left(\gamma^c_a + \epsilon n^c n_a\right)\nabla_c n_b \\ &= K_{ab} + \epsilon n_a n^c\nabla_c(\epsilon N\nabla_b t) \\ &= K_{ab} + n_a n^c(\nabla_c N)\epsilon N^{-1}n_b + N n_a n^c\nabla_b(\epsilon N^{-1}n_c) \\ &= K_{ab} + \epsilon n_a n^c n_b(\nabla_c \ln N) - n_a\nabla_b \ln N \\ &= K_{ab} - n_a D_b \ln N\,, \end{aligned} \tag{A.2.7}$$

we obtain

$$\begin{aligned} \gamma^e_a\gamma^g_c n^f n^h\,{}^{(D)}R_{efgh} &= \gamma^e_a\gamma^g_c n^h(\nabla_e K_{hg} - \nabla_h K_{eg}) - \epsilon\frac{1}{N}D_a D_c N \\ &= -K_{cd}K^d_a + \gamma^e_a\gamma^g_c\left(\mathcal{L}_n K_{eg} - K_{eb}\nabla_g n^b - K_{bg}\nabla_e n^b\right) - \epsilon\frac{1}{N}D_a D_c N \\ &= -\mathcal{L}_n K_{ac} + K^b_a K_{bc} - \epsilon\frac{1}{N}D_a D_c N\,, \end{aligned} \tag{A.2.8}$$

with which Eq. (A.2.5) is reduced the evolution equation as

$$\mathcal{L}_n K_{ac} = 2K_{ab}K^b_c - KK_{ac} - \epsilon\frac{1}{N}D_a D_c N + \epsilon\,{}^{(D-1)}R_{ac} - \epsilon\gamma^b_a\gamma^d_c\,{}^{(D)}R_{bd}\,. \tag{A.2.9}$$

With the aid of the Einstein equation, we obtain

$$\mathcal{L}_n K_{ac} = 2K_{ab}K^b_c - KK_{ac} - \epsilon\frac{1}{N}D_aD_cN + \epsilon\,{}^{(D-1)}R_{ac} - \epsilon M_D^{-(D-3)}\left(\gamma^b_a\gamma^d_c T_{bd} - \frac{1}{D-2}\gamma_{ac}T\right). \tag{A.2.10}$$

From the above, the Einstein equation is shown to be decomposed into Eqs. (A.2.2), (A.2.3), (A.2.10). For convenience, in this paper we denote the constraints Eqs. (A.2.2) and (A.2.3) as the Hamiltonian and momentum constraints, respectively, although the constraints are not exactly "Hamiltonian" and "momentum" constraints when n^a is a spacelike vector.

Furthermore, we derive the expression of the D-dimensional Ricci scalar in terms of $D-1$-dimensional geometrical variables, for the later use. Eq. (A.2.2) gives

$${}^{(D)}R - 2\epsilon\,{}^{(D)}R_{ab}n^an^b = {}^{(D-1)}R + \epsilon\left(K_{ab}K^{ab} - K^2\right), \tag{A.2.11}$$

where the second term in the left-hand side yields

$$\begin{aligned} {}^{(D)}R_{ab}n^an^b &= {}^{(D)}R^c{}_{acb}n^an^b \\ &= n^a(\nabla^c\nabla_a - \nabla_a\nabla^c)n_c \\ &= \nabla^c(n_a\nabla^an_c - n_c\nabla^an_a) - \nabla^bn_a\nabla^an_b + (\nabla^an_a)^2 \\ &= \nabla^c(n_a\nabla^an_c - n_c\nabla^an_a) - K_{ab}K^{ab} + K^2\,. \end{aligned} \tag{A.2.12}$$

Therefore, we obtain

$${}^{(D)}R = {}^{(D-1)}R - \epsilon\left(K_{ab}K^{ab} - K^2\right) + 2\epsilon\nabla^c(n_a\nabla^an_c - n_c\nabla^an_a)\,. \tag{A.2.13}$$

Appendix B

Higuchi ghost

In the theory with massive graviton such as massive gravity and bigravity, the massive spin-2 degrees of freedom can exhibit ghost instabilities even when BD ghost is eliminated. One famous example is Higuchi ghost, the scalar ghost in de Sitter background [62].

We consider de Sitter background spacetime $\gamma_{\mu\nu}$ with Hubble scale H:

$$\gamma_{\mu\nu}dx^\mu dx^\nu = (-H\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)\,. \tag{B.0.1}$$

Since we discuss the linear instability of the massive graviton, it is sufficient to analyze the Fierz-Pauli massive gravity. Up to the quadratic order in the perturbation around the de Sitter background, the action is given as

$$\begin{aligned} S = M_{\rm Pl}^2 \int d^4x\sqrt{\gamma}\bigg[&-\frac{1}{4}\nabla_\alpha h_{\beta\gamma}\nabla^\alpha h^{\beta\gamma} + \frac{1}{8}\nabla_\alpha h\nabla^\alpha h - \frac{H^2}{2}\left(h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{2}h^2\right) \\ &-\frac{m^2}{4}\left(h^{\alpha\beta}h_{\alpha\beta} - h^2\right)\bigg]\,, \end{aligned} \tag{B.0.2}$$

where ∇_μ is the covariant differentiation associated with $\gamma_{\mu\nu}$. In order to discuss Higuchi ghost, we will focus on the helicity-0 mode. It does not affect the kinetic terms of the other modes to change the background spacetime from Minkowski to de Sitter, and hence they remain healthy as shown in Sec. 2.2.1.

Focusing only on the helicity-0 mode, $h_{\mu\nu}$ is decomposed to four variables as

$$h_{00} = \psi\,,\ \ h_{0i} = \partial_i\phi\,,\ \ h_{ij} = \partial_i\partial_j f - \delta_{ij}\chi\,. \tag{B.0.3}$$

These four variables are to be reduced to one variable, since the action (B.0.2) is free from BD ghost and so contains only one helicity-0 degree of freedom.

Now, we perform the Fourier transformation with respect to x^i and rewrite ϕ, f, χ in terms of $\psi(\mathbf{k}, \eta)$. Variation of the action (B.0.2) with respect to $h^{\mu\nu}$ gives equations of motion for $h_{\mu\nu}$ equivalent to Eqs. (2.4.12)-(2.4.19) for $h^{(m)}_{\mu\nu}$ with $T_{\mu\nu} \to 0$ and $m^2_{\rm eff} \to m^2$ in Sec. 2.4. From Eqs. (2.4.12) and (2.4.16), $h_{\mu\nu}$ satisfies the transverse-traceless condition:

$$h = 0\,, \tag{B.0.4}$$

$$\nabla_\mu h^\mu_\nu = 0\,, \tag{B.0.5}$$

with which h_{0i} and h_{ij} are rewritten in terms of h_{00} as

$$\sum_i h_{ii} = h_{00}\,, \tag{B.0.6}$$

$$\partial_i h_{i0} = \dot{h}_{00} - \frac{2}{\eta} h_{00}\,, \tag{B.0.7}$$

$$\partial_i h_{ij} = \dot{h}_{0j} - \frac{2}{\eta} h_{0j}\,. \tag{B.0.8}$$

Eqs. (B.0.6)-(B.0.8) with the aid of the equations of motion for h_{00}:

$$\left[-\partial_\eta^2 + \partial_i\partial^i + \frac{2}{\eta}\partial_\eta - \frac{m^2}{H^2\eta^2}\right] h_{00} = 0\,, \tag{B.0.9}$$

determine h_{0i} and h_{ij} in terms of $h_{00} = \psi$ and $\dot{\psi}$ [61] as

$$h_{0i}(\mathbf{x},\,\eta) = -\frac{\partial_i}{k^2}\left[\dot{\psi}(\mathbf{k}) - \frac{2}{\eta}\psi(\mathbf{k})\right]\,, \tag{B.0.10}$$

$$\begin{aligned} h_{ij}(\mathbf{x},\,\eta) = {} & \frac{\partial_i\partial_j}{k^4}\left[-k^2\psi(\mathbf{k}) - \frac{3}{2}\left(\frac{2}{\eta}\dot{\psi}(\mathbf{k}) + \frac{m^2-6H^2}{(H\eta)^2}\psi(\mathbf{k})\right)\right] \\ & - \frac{1}{2k^2}\delta_{ij}\left[\frac{2}{\eta}\dot{\psi}(\mathbf{k}) + \frac{m^2-6H^2}{(H\eta)^2}\psi(\mathbf{k})\right]\,. \end{aligned} \tag{B.0.11}$$

Then, the action (B.0.2) is written in terms of h_{00} as

$$S = \int d\mathbf{k} \int d\eta \frac{M_{\rm Pl}^2\, m^2(m^2-2H^2)}{k^4}\psi\left[-\partial_\eta^2 + \partial_i\partial^i + \frac{2}{\eta}\partial_\eta + \frac{m^2}{H^2\eta^2}\right]\psi\,. \tag{B.0.12}$$

This expression indicates that the sign of ψ's kinetic term flips at $m^2 = 2H^2$ and the helicity-0 mode becomes ghost when $m^2 < 2H^2$. This helicity-0 ghost is called Higuchi ghost [62].

Appendix C

Residual gauge transformation

In this section, we show that the mode which corresponds the pole at ${}^4\Box = -4H^2$ is unphysical and has no degree of freedom in the perturbative solution Eqs. (5.1.2)-(5.1.4) in the DGP 2-brane model with a scalar stabilization, because the terms proportional to $({}^4\Box + 4H^2)^{-1}T^{(+)}$ are totally canceled out each other by a residual gauge transformation [34].

In Chapter. 4 and 5, we adopt the Newton gauge (4.1.11)-(4.1.14). As is shown in the following, however, these gauge fixing conditions do not completely fix the gauge degrees of freedom: when spin-0 mode's mass becomes ${}^4\Box = -4H^2$ and equivalently spin-2 mode's mass becomes ${}^4\Box = 4H^2$, there exists a residual gauge transformation ξ as

$$\xi^y = \frac{1}{2}a^2\partial_y\left(\frac{X}{a^2}\right)\,, \quad \xi^\mu = -\frac{1}{2}\tilde{\gamma}^{\mu\nu}X_{,\nu}\,, \tag{C.0.1}$$

where the scalar X is the generator of this transformation and the Newton gauge condition $h_{y\mu} = 0$ is preserved under this transformation. By this residual gauge transformation, the trace part of the metric perturbation ϕ varies by

$$\delta\phi = -a'a\partial_y\left(\frac{X}{a^2}\right) - \frac{X}{a^2}\,, \tag{C.0.2}$$

where we used ${}^4\Box X = -4H^2X$. From $h_{yy} = 2\phi$, we can also calculate $\delta\phi$ by computing the variation of h_{yy} by the residual gauge transformation as

$$\delta\phi = \frac{1}{2}\partial_y\left(a^2\partial_y\left(\frac{X}{a^2}\right)\right)\,. \tag{C.0.3}$$

Equating these two expression for $\delta\phi$, we obtain an equation for X:

$$\hat{L}^{(TT)}X = -\frac{2}{a^2}X\,. \tag{C.0.4}$$

which is identical to the equation for the spin-2 mode (4.1.15) when ${}^4\Box = 4H^2$. On the other hand, the residual gauge transformation X changes the spin-2 mode $\delta h^{(TT)}_{\mu\nu}$ by

$$\delta h^{(TT)}_{\mu\nu} = -\left(\nabla_\mu \nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right) X \,. \tag{C.0.5}$$

Eqs. (C.0.4) and (C.0.5) show that the spin-2 mode generated by the residual gauge transformation X consistently satisfies the original equation for the spin-2 mode (4.1.15). Also, we can show that $\delta\phi$ given by Eq. (C.0.2), or equivalently by Eq. (C.0.3), satisfies the spin-0 mode's equation (4.1.17) when ${}^4\Box = -4H^2$. Hence, there certainly exists the residual gauge transformation given by Eq. (C.0.1).

Next, we study the boundary conditions for the modes generated by this residual gauge transformation. Focusing on the gravitational sector, we ignore the matter field assuming $T_{\mu\nu} = 0$. Then, both the spin-2 mode and the spin-0 mode are sourced by $Z_{(\pm)}$ which satisfies $({}^4\Box + 4H^2)Z_{(\pm)} = 0$ from Eq. (4.1.29). Also, the solution of the spin-2 mode $h^{(TT)}_{\mu\nu}$ becomes proportional to $\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right) Z_{(\pm)}$ from Eqs. (4.1.27) and (4.2.7). Such spin-2 mode can be erased by a residual gauge transformation as in Eq. (C.0.5) to yield $Z_{(\pm)} = 0$ by the junction conditions for the spin-2 mode (4.1.26). When $Z_{(\pm)} = 0$, $\phi := 0$ is the only consistent solution of Eqs. (4.1.17), (4.1.30), unless we fine-tune the background solution. From the above discussion, it is shown that there exist critical poles ${}^4\Box = -4H^2$ for the spin-0 mode and ${}^4\Box = 4H^2$ for the spin-2 mode, in which both spin-0 and spin-2 modes are considered just as gauge modes generated by a residual gauge transformation. These gauge modes are not physical modes and in fact we can show that these modes are canceled out each other, namely, the identities (5.1.8) and (5.1.9) hold, as follows: first, we set the spin-2 and spin-0 modes at their critical masses to zero by an appropriate gauge fixing, and then we consider pure gauge modes $h^{(TT)}_{c\ \mu\nu}$ and ϕ_c generated by a residual gauge transformation with a generator X that yields $Z_{(+)} = Z_0$. From Eqs. (C.0.2) and (C.0.5), the gauge modes are written as

$$h^{(TT)}_{c\ \mu\nu} = -\left(\nabla_\mu \nabla_\nu - \frac{1}{4}\gamma_{\mu\nu}{}^4\Box\right) X \,, \tag{C.0.6}$$

$$\phi_c = -a' a \partial_y \left(\frac{X}{a^2}\right) - \frac{X}{a^2} \,. \tag{C.0.7}$$

Imposing that these modes satisfy the junction conditions, we obtain the boundary condition for X as

$$a^2 \partial_y \left(\frac{X}{a^2}\right)\bigg|_{y=y_+} = 2Z_0 - 2r_c^{(+)} a_+^2 H^2 \left. X\right|_{y=y_+} \,, \tag{C.0.8}$$

at $y = y_+$ by Eq. (4.1.26). Substituting Eq. (C.0.8) into Eq. (C.0.7) gives

$$\phi_c(y_+) = -2\mathcal{H}_+ Z_0 - a_+^2 H^2 (1 - 2r_c^{(+)}\mathcal{H}_+) X \,. \tag{C.0.9}$$

On the other hand, X and ϕ_c can be written in terms of Z_0 using Eqs. (C.0.6), (4.2.7), and (4.2.10) as

$$X = 4\sum_i \frac{u_i^2(y_+)}{2H^2 - m_i^2} Z_0\,, \qquad \phi_c = \frac{4}{3M_5^3 a_+^2 H^2 (2r_c^{(+)}\mathcal{H}_+ - 1)} \sum_i \frac{v_i^2(y_+)}{4H^2 + \mu_i^2} Z_0\,. \tag{C.0.10}$$

Combining Eqs. (C.0.9) and (C.0.10), we obtain the identity (5.1.8). We obtain the identity (5.1.9) similarly from Eqs. (C.0.8)-(C.0.10) by considering the junction conditions at $y = y_-$ instead of the ones at $y = y_+$.

Appendix D

ghost stabilization scalar field

Here, we present the condition in which the spin-0 mode becomes ghost in the DGP 2-brane model with a scalar stabilization [34]. We decompose the induced-metric perturbation $\bar{h}_{\mu\nu}$ into the mode-expanded transverse-traceless part and the mode-expanded trace part:

$$\bar{h}_{\mu\nu} = \sum_i \bar{h}^{(i)}_{\mu\nu} + \frac{1}{4}\sum_j \bar{h}^{(j)} . \tag{D.0.1}$$

Rewriting the effective action on the (+)-brane in terms of $\bar{h}^{(i)}_{\mu\nu}$ and $\bar{h}^{(j)}$, its kinetic terms should take the form as

$$\begin{aligned} S_{kin} = \sum_i \alpha_i \int d^4x \sqrt{-\tilde{\gamma}_+}\bar{h}^{(i)\mu\nu}({}^4\Box - 2H^2 - m_i^2)\bar{h}^{(i)}_{\mu\nu} \\ + \sum_j \beta_j \int d^4x \sqrt{-\tilde{\gamma}_+}\bar{h}^{(j)}({}^4\Box - \mu_j^2)\bar{h}^{(j)} , \end{aligned} \tag{D.0.2}$$

with some constants α_i and β_i. When we assume that the matter field localizes only on the (+)-brane, the matter action that couples to the metric perturbations is given as

$$S_{matter} = \sum_i \frac{1}{2}\int d^4x \sqrt{-\tilde{\gamma}_+}T^{\mu\nu}\bar{h}^{(i)}_{\mu\nu} + \sum_j \frac{1}{8}\int d^4x\sqrt{-\tilde{\gamma}_+}T\bar{h}^{(j)} . \tag{D.0.3}$$

Comparing the equations of motion given by the variation of the effective action $S_{kin} + S_{matter}$ with Eqs. (5.1.2) and (5.1.3), we obtain the expression of α_i and β_i:

$$\alpha_i = \frac{M_5^3}{8u_i^2(y_+)} , \tag{D.0.4}$$

$$\beta_j = \frac{9M_5^6 a_+^4 (2r_c^{(+)}\mathcal{H} - 1)^2(\mu_j^2 + 4H^2)}{128 v_j^2(y_+)} . \tag{D.0.5}$$

Eq. (D.0.5) implies that the sign of the kinetic term of the spin-0 mode flips at $\mu_j^2 = -4H^2$, and hence the spin-0 mode becomes ghost when $\mu_j^2 < -4H^2$. On the other hand, the signature in front of the spin-2 mode's kinetic term stays positive and healthy. However, as is shown in Appendix. B, the helicity-0 mode in the spin-2 mode becomes ghost when $0 < m_i^2 < 2H^2$.

Appendix E

Doubly coupled matter in bigravity

In the presence of a matter field that couples to both metrics in the ghost-free bigravity, so-called doubly coupled matter, BD ghost reappears in general, since the coupling between two metrics through the matter field breaks the specific ghost-free interaction form.

The ghost-freeness proof of the ghost-free bigravity discussed in Sec. 2.3 can be easily extended to the case in which there are matter fields that couple to either metric. In this case, the contributions from the matter fields can be totally absorbed by the redefinition of R_0 and R_i or $\tilde{R}_0$ and $\tilde{R}_i$ in all the constraints, which makes the fundamental Poisson brackets (2.1.7)-(2.1.9) totally unchanged. However, a matter field that couples to both metric in general breaks the ghost freeness. Basically, this is because two kinetic terms of the doubly coupled matter field that couples to the respective metrics make the conjugate momentum of the matter field to be nonlinear in the lapse functions. The doubly coupled matter disable to define such a new shift-like vector that makes the Lagrangian linear in the both lapse functions. Then, the constraint that corresponds to $\mathcal{C}$ disappears and BD ghost reappears. Therefore, the doubly coupled matter model considerably restricted because of the BD-ghost reappearance.

One simple solution of this problem is not to introduce the kinetic term of the doubly coupled matter [35]. Another particular example of a ghost-free theory with a doubly coupled matter ϕ is

$$
\begin{aligned}
S \;=\; & \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2 R}{2} + 2m^2 M_{\rm Pl}^2 \sum_n c_n e_n\!\left(\sqrt{g^{\mu\nu}(\tilde{g}_{\mu\nu} + \alpha\partial_\mu\phi\,\partial_\nu\phi)}\right)\right] \\
& + \int d^4x\sqrt{-\tilde{g}}\;\left[\frac{\chi M_{\rm Pl}^2 \tilde{R}}{2} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right], \qquad \text{(E.0.1)}
\end{aligned}
$$

where α is a constant. In this model the BD ghost is absent, because the action is rewritten into the form with no coupling to the metric $g_{\mu\nu}$, after a field redefinition as

$$
\tilde{g}_{\mu\nu} \to \tilde{g}'_{\mu\nu} - \alpha\partial_\mu\phi\,\partial_\nu\phi\,. \qquad \text{(E.0.2)}
$$

The remaining kinetic terms contain no higher temporal derivative terms after the redefinition (E.0.2), since the temporal derivatives of the lapse and shift functions are removed by integrating by parts as in the case of the Einstein-Hilbert action. In addition, Refs. [36, 38] show that a model is ghost free under the strong coupling scale of the ghost-free bigravity, when a matter field canonically couples to a composite metric $g^{(c)}_{\mu\nu}$, which is given as

$$g^{(c)}_{\mu\nu} := \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\alpha(\mu} Y^{\alpha}_{\nu)} + \beta^2 \tilde{g}_{\mu\nu} \,, \tag{E.0.3}$$

where Y^{μ}_{ν} is defined in Eq. (2.3.5).

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